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TRIGONOMETRIC APPROXIMATION IN WEIGHTED ORLICZ SPACES

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Abstract. In the present article, we prove some trigonometric approximation theorems in weighted Orlicz spaces, having generating Young functions not necessary to be convex.

1. Introduction

The problems of approximation by trigonometric polynomials in classical Orlicz spaces were investigated by several mathematicians. In [16], Tsyganok obtained the Jackson type inequality of trigonometric approximation. In [13], Ponomarenko proved some direct theorems of trigonometric approximation by summation means of Fourier series. In [17], the results of Ponomarenko were generalized to weighted Orlicz spaces with Muckenhoupt weights. In [18], similar results were obtained for fractional modulus of smoothness. In these results, the generating Young function of Orlicz spaces are convex or quasiconvex. When the generating Young function is quasiconvex, similar problems were investigated in [1, 3, 4, 5, 6]. In [9],Chen generalized the definition of Orlicz spaces saving almost all known properties of them. In this definition, the generating Young function of Orlicz spaces is not necessary to be convex .

In this work we will develop this approach with Muckenhoupt weights and investigate some direct problems of approximation theory. We generalize the results obtained in the papers [13, 17, 18] to the weighted Orlicz spaces having generating Young functions not necessary to be convex.

First of all we give basic definitions and notations.

A nonnegative function ω defined on $\mathbf{T} := [0, 2\pi]$ will be called a weight function if ω is measurable and positive almost everywhere (a.e.). A convex function φ said to satisfy Δ_2 condition if there is a constant $C > 0$ and $u_0 > 0$ such that $\varphi(2u) \leq C\varphi(u)$ for all $u \geq u_0$.Let $\varphi(u)$ non-negative, convex, vanishing at the origin and satisfying Δ_2 condition, such that $\varphi(u)/u \to \infty$ as $u \to \infty$. Let $\psi(u)$ denote the complementary function of $\varphi(u)$ in the sense of Young. Consider the measurable functions $f(t)$, $t \in \mathbf{T}$, such that the product $f(t)\, g(t)$ is integrable

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over $\mathbf{T}$ for every measurable function $g(t)$ and set

$$\|f\|_{L^*_{\varphi,\omega}} := \sup\left\{ \int_{\mathbf{T}} |f(t)g(t)|\,\omega(t)\,dt : \int_{\mathbf{T}} \psi(|g(t)|)\,dt \le 1 \right\}. \tag{1.1}$$

The class of such f is denoted by $L^*_{\varphi,\omega}$ which is usually called the weighted Orlicz space. Furthermore we define the following norm which is equivalent to the norm (1.1)

$$\|f\|_{L^*_{(\varphi,\omega)}} := \inf\left\{ k>0 : \int_{\mathbf{T}} \varphi\left(k^{-1}|f(x)|\right)\omega(x)\,dx \le 1 \right\}. \tag{1.2}$$

This definition was generalized by Yung-Ming Chen in the following way [9]. Let $-\infty < p \le q < \infty$. By $\phi \in N[p,q]$ we mean that the function ϕ is increasing, even, nonnegative function in $[0,\infty)$ such that $\phi(\infty) = \infty$, $\phi(x)x^{-p}$ is non-decreasing and $\phi(x)x^{-q}$ is non-increasing as x increases in $(0,\infty)$. If $p<q$, the class of functions ϕ belonging to $N[p+\varepsilon, q-\varepsilon]$ for some small number $\varepsilon>0$ will be denote by $N\langle p,q\rangle$. By Φ it will be denote the class of functions M belonging to the class $N\langle p,q\rangle$ for some $1<p\le q<\infty$. Every function $M\in\Phi$ is continuous and satisfies the condition $M(0)=0$ and $M\in\Delta_2$. The functions $M\in\Phi$ are not necessary to be convex. (See the example given in [9, p. 67-68]). Let $M\in\Phi$ and ω be weight on $\mathbf{T}$. We denote $\varphi_M(t) = M(t)/t$. Since $1<p<q<\infty$, we have $\varphi_M(t)\to\infty$ as $t\to\infty$. Let $\psi_M(t)$ be the inverse function of positive nondecreasing continuous function $\varphi_M(t)$. We set

$$\Phi_M(x) = \int_0^x \varphi_M(t)dt$$

and

$$\Psi_M(x) = \int_0^x \psi_M(t)dt.$$

The function Φ_M is convex and hence Φ_M and Ψ_M are complementary functions in the sense of Young. We denote by $L_{M,\omega}(\mathbf{T})$, the class of Lebesgue measurable functions $f:\mathbf{T}\to\mathbb{R}$ satisfying the condition

$$\int_{\mathbf{T}} \Phi_M(c|f(x)|)\,\omega(x)\,dx < \infty$$

for some positive real constant c. On the space $L_{M,\omega}(\mathbf{T})$, we define the Orlicz norm

$$\|f\|_{M,\omega} := \sup_g \left\{ \int_{\mathbf{T}} |f(x)g(x)|\,\omega(x)\,dx : \int_{\mathbf{T}} \Psi_M(|g(x)|)\,\omega(x)dx \le 1 \right\}$$

and the Luxemburg norm

$$\|f\|_{(M),\omega} := \inf\left\{ k>0 : \int_{\mathbf{T}} \Phi_M\left(\left|k^{-1}f(x)\right|\right)\omega(x)\,dx \le 1 \right\}.$$

In this case, we have the equivalence

$$\|f\|_{M,\omega} \sim \|f\|_{(M),\omega} .$$

It can be easily seen that $L_{M,\omega}(\mathbf{T}) \subset L^1(\mathbf{T})$ and $L_{M,\omega}(\mathbf{T})$ becomes a Banach space with the above norms. For a weight ω, taking $M(x,p) := x^p$, $1 < p < \infty$, we denote $L^p(\mathbf{T},\omega) := L_{M(.,p),\omega}(\mathbf{T})$ the weighted Lebesgue space.

A weight function ω belongs to the Muckenhoupt class $A_p[0,2\pi]$ if

$$\left(\frac{1}{|I|}\int_I \omega^p(x)dx\right)^{1/p}\left(\frac{1}{|I|}\int_I \omega^{-q}(x)dx\right)^{1/q} \leq C \quad \text{for } 1 < p < \infty,$$

with a finite constant C independent of I, where I is any subinterval of $[0,2\pi]$ and $|I|$ denotes the length of I.

For a periodic function f, the Hardy-Littlewood Maximal operator is defined as

$$f^*(x) := \sup_{0<h\leq\pi} \frac{1}{2h}\int_{-h}^{h} f(x+t)dt.$$

In [2], it was proved that

$$\|f^*\|_{(M),\omega} \preceq \|f\|_{(M),\omega}$$

when $M \in \Phi$ and $\omega \in A_p$.

Taking $f \in L_{M,\omega}(\mathbf{T})$ we define the well-known Steklov's mean operator

$$A_h f(x) := \frac{1}{h}\int_{x-h/2}^{x+h/2} f(t)dt.$$

Using the boundedness of the Hardy-Littlewood Maximal operator in $L_{M,\omega}(\mathbf{T})$ for $M \in \Phi$ and $\omega \in A_p$, we get

$$\|A_h f\|_{(M),\omega} \leq C(M,\omega)\|f\|_{(M),\omega} < \infty.$$

Using this fact and setting $x, h \in \mathbf{T}$, $r \in \mathbb{R}^+$ we define via Binomial expansion that for $f \in L_{M,\omega}(\mathbf{T})$, $x \in \mathbf{T}$

$$\begin{aligned} \sigma_h^r f(x) \quad &: \quad = (I - A_h)^r f(x) \\ &= \sum_{k=0}^{\infty} (-1)^k [C_k^r] \frac{1}{h^k} \int_{-h/2}^{h/2} \ldots \int_{-h/2}^{h/2} f(x + u_1 + \ldots u_k) du_1 \ldots du_k \quad , \end{aligned}$$

where $[C_k^r] := \frac{r(r-1)\ldots(r-k+1)}{k!}$ for $k \geq 1$ and $[C_0^r] := 1$. Since [14, p.14, (1.51)] $|[C_k^r]| \leq \frac{C(r)}{k^{r+1}}$, $k \in \mathbb{N}$, we have $\sum\limits_{k=0}^{\infty} |[C_k^r]| < \infty$. Then we have

$$\|\sigma_h^r f\|_{(M),\omega} \leq C(r,M,\omega)\|f\|_{(M),\omega} < \infty.$$

Now, we define the weighted fractional modulus of smoothness of index r for a function $f \in L_{M,\omega}(\mathbf{T})$ and $r \in \mathbb{R}^+$, $M \in \Phi$, $\omega \in A_p$

$$\Omega_r(f,\delta)_{M,\omega} := \sup_{0<h_i,t\le\delta} \left\| \prod_{i=1}^{[r]} (I - A_{h_i})\, \sigma_t^{r-[r]} f \right\|_{(M),\omega}, \delta \ge 0,$$

where $[r]$ denotes the integer part of the real number r.

In this case

$$\Omega_r(f,\delta)_{M,\omega} \le C(r,M,\omega)\, \|f\|_{(M),\omega}\,.$$

The concept of fractional modulus of smoothness is not new (see for example [7] and [15]).

We set

$$E_n(f)_{M,\omega} := \inf \left\{ \|f-T\|_{(M),\omega} : T \in T_n \right\}$$

for $f \in L_{M,\omega}(\mathbf{T})$, where T_n is the class of trigonometric polynomials of degree not greater than n. For a given $f \in L_1(\mathbf{T})$, let

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{n} (a_k(f)\cos kx + b_k(f)\sin kx) := \sum_{k=0}^{\infty} A_k(x,f)$$

and

$$\tilde{f}(x) \sim \frac{a_0}{2} + \sum_{k=1}^{n} (a_k(f)\sin kx - b_k(f)\cos kx) := \sum_{k=0}^{\infty} A_k(x,\tilde{f})$$

be the Fourier series and the conjugate Fourier series of f. We define the partial sum of Fourier series of f as

$$S_n(x,f) := S_n(f) := \sum_{k=-0}^{n} A_k(x,f)$$

for $n = 0,1,2,....$

In [2], it was proved that the operators $S_n : L_{M,\omega}(\mathbf{T}) \to L_{M,\omega}(\mathbf{T})$ and $\tilde{f} : L_{M,\omega}(\mathbf{T}) \to L_{M,\omega}(\mathbf{T})$ are bounded in $L_{M,\omega}(\mathbf{T})$ for $M \in \Phi$, $\omega \in A_p$, $f \in L_{M,\omega}(\mathbf{T})$. Hence we have

$$\left\|\tilde{f}\right\|_{(M),\omega} \le C\,\|f\|_{(M),\omega}\,,\ \|S_n(f)\|_{(M),\omega} \le C\,\|f\|_{(M),\omega}\,, \qquad n = 0,1,2,...$$

and

$$\|f - S_n(f)\|_{(M),\omega} \le CE_n(f)_{M,\omega}, \qquad n = 0,1,2,.... \tag{1.3}$$

Furthermore, note that the set of trigonometric polynomials is a dense subset of $L_{M,\omega}(\mathbf{T})$ since the hypothesis of Lemma 3 of [11] are fulfilled for $M \in \Phi$, $\omega \in A_p$. Then the approximation problems make sense in $L_{M,\omega}(\mathbf{T})$. Hence $E_n(f)_{M,\omega} \to 0$ as $n \to \infty$ and therefore the Fourier Series of f converges to f in norm in $L_{M,\omega}(\mathbf{T})$, namely,

$$f(x) = \sum_{k=1}^{\infty} A_k(x,f).$$

Throughout this paper, the constant c denotes a generic constant, i.e. a constant whose values can change even between different occurrences in a chain of inequalities. The relation $\preceq$ is defined as "$A \preceq B$ $\Leftrightarrow$there exists a constant C such that $A \leq CB$"

Our main results are the following.

Theorem 1.1. *Let $M \in \Phi$, $\omega \in A_p$ and $f \in L_{M,\omega}(T)$. For the system of numbers* $\lambda_\nu^{(n)} = 1 - \left(\frac{\nu}{n}\right)^{2r}$ $(\nu \leq n, r > 0)$

$$R_n\left(f,\lambda\right)_{M,\omega} := \left\| f(x) - \left[\frac{a_0}{2} + \sum_{\nu=1}^{n} \lambda_\nu^{(n)} (a_\nu \cos \nu x + b_\nu \sin \nu x)\right] \right\|_{(M),\omega} \preceq \Omega_r(f, \frac{1}{n})_{M,\omega}.$$

Theorem 1.2. *Let $M \in \Phi$, $\omega \in A_p$ and $f \in L_{M,\omega}(T)$. For a sequence of functions*

$$\lambda_\nu\left(r\right) = 1 - \left(\nu\left|r - r_0\right|\right)^{2k}, \quad \left(\nu \leq \left[\frac{1}{|r - r_0|}\right]\right), k > 0, E \subset \mathbb{R}, r, r_0 \in E,$$

if the series $\frac{a_0}{2} + \sum\limits_{\nu=1}^{\infty} \lambda_\nu\left(r\right)\left(a_\nu \cos \nu x + b_\nu \sin \nu x\right)$ *convergences to a metric of the space* $L_{M,\omega}\left(\mathbf{T}\right)$*, then*

$$R_r\left(f,\lambda\right)_{M,\omega} := \left\| f(x) - \left[\frac{a_0}{2} + \sum_{\nu=1}^{\infty} \lambda_\nu\left(r\right)\left(a_\nu \cos \nu x + b_\nu \sin \nu x\right)\right] \right\|_{(M),\omega} \leq C\Omega_k(f, |r - r_0|)_{M,\omega}.$$

Theorem 1.3. *Let $M \in \Phi$, $\omega \in A_p$ and $f \in L_{M,\omega}(T)$ such that*

$$\Phi_M(uv) \leq c\Phi_M(u)\Phi_M(v) \tag{1.4}$$

with a constant $c > 0$. Then for an arbitrary triangular matrix of the numbers

$$\left\{\lambda_\nu^{(n)}\right\} \ \left(\lambda_0^{(n)} = 1, \lambda_\nu^{(n)} = 0, \ \nu > n, \ n = 0, 1, 2, ...\right)$$

$$R_n\left(f,\lambda\right)_{M,\omega} \preceq \left\{ \left[\sum_{\nu=0}^{m} E_{2^\nu-1}^2(f)_{M,\omega} \delta_{2^\nu-1}^2\right]^{1/2} + E_n(f)_{M,\omega} \right\}$$

if $\Phi_M\left(\sqrt{u}\right)$ *is convex and*

$$R_n\left(f,\lambda\right)_{M,\omega} \leq \inf_{k>0} \frac{1}{k} \left\{1 + \sum_{\nu=0}^{m} \varphi\left[ckE_{2^\nu-1}(f)_{M,\omega}\delta_{2^\nu-1}\right]\right\} + CE_n(f)_{M,\omega}$$

if $\Phi_M\left(\sqrt{u}\right)$ *is concave, where*

$$\delta_{2^r,n} \ : \ = \sum_{l=2^r}^{2^{r+1}-1} \left|\lambda_{\nu+1}^{(n)} - \lambda_\nu^{(n)}\right| + \left|1 - \lambda_{2^r}^{(n)}\right|,$$

$$2^m \ \leq \ n < 2^{m+1}.$$

2. Auxiliary results

We give the multiplier theorem and the Littlewood-Paley theorem in $L_{M,\omega}(\mathbf{T})$ which are very important for the proof of the main results.

Lemma 2.1. [2] *Let a sequence* χ_k *satisfy the conditions*

$$|\chi_k| \leq A, \sum_{k=2^{j-1}}^{2^j-1} |\chi_k - \chi_{k+1}| \leq A$$

where $A > 0$ *does not depend on* k *and* j. *If* $M \in \Phi$, $\omega \in A_p$ *and* $f \in L_{M,\omega}(\mathbf{T})$ *then there is a function* $F \in L_{M,\omega}(\mathbf{T})$ *such that the series*

$$\frac{\lambda_0 a_0}{2} + \sum_{k=0}^{\infty} \lambda_k (a_k \cos kx + b_k \sin kx)$$

is Fourier series for F *and*

$$\|F\|_{(M),\omega} \leq CA \|f\|_{(M),\omega},$$

holds with a positive constant C *independent of* f.

Lemma 2.2. [2] $M \in \Phi$, $\omega \in A_p$ *and* $f \in L_{M,\omega}(\mathbf{T})$ *there exist constants* $C > 0$ *and* $c > 0$ *depending only on* M *and* ω *such that*

$$c \|f\|_{(M),\omega} \leq \left\| \left(\sum_{j=0}^{\infty} \left| \sum_{k=2^{j-1}}^{2^j-1} A_k(x,f) \right|^2 \right) \right\|_{(M),\omega}^{\frac{1}{2}} \leq C \|f\|_{(M),\omega}.$$

The following Lemma can be proved by using [8, Th. 3.1].

Lemma 2.3. Let $M \in \Phi$, $\omega \in A_p$, $f_n(x)$ $(n = 1, 2, ...)$ *be a sequence of* 2π *periodic functions in* $L_{M,\omega}(\mathbf{T})$ *and let* $S_{n,k_n}(x)$ *be the* k*-th partial sum of Fourier series of the function* $f_n(x)$, $k = k_n$ *is a function of* n. *Then there exists a positive constant* C *such that*

$$\int_{\mathbf{T}} M\left(\left(\sum_{n=0}^{\infty} |S_{n,k_n}(x)|^2 \right)^{1/2} \right) \omega(x)\, dx \leq C \int_{\mathbf{T}} M\left(\left(\sum_{n=0}^{\infty} |f_n(x)|^2 \right)^{1/2} \right) \omega(x)\, dx$$

with a constant C *is independent of* $f_n(x)$.

The following direct approximation theorem can be proved by the same methods given in [4, prop. 1] and [10, theorems 2 and 4]:

Theorem 2.4. Let $M \in \Phi$, $\omega \in A_p$ and $f \in L_{M,\omega}(\mathbf{T})$. *Then the estimate*

$$E_n(f)_{M,\omega} \preceq \Omega_r(f, \frac{1}{n})_{M,\omega} \tag{2.1}$$

holds for $r \in \mathbb{R}^+$ and $n = 1, 2, ...$

3. Proof of main results

Proof of Theorem 1.1. Let $2^m \le n < 2^{m+1}$. By virtue of the property of the norm

$$\left\| f(x) - \left[\sum_{\nu=0}^{n} \lambda_\nu^{(n)} A_\nu(x) \right] \right\|_{(M),\omega} \le \left\| \sum_{\nu=0}^{n} \left(1 - \lambda_\nu^{(n)}\right) A_\nu(x) \right\|_{(M),\omega} + \left\| \sum_{\nu=n+1}^{\infty} A_\nu(x) \right\|_{(M),\omega} = I_1 + I_2,$$

where $A_\nu(x) := a_\nu \cos \nu x + b_\nu \sin \nu x$. From (1.3) and (2.1) we obtain

$$I_2 = \left\| \sum_{\nu=n+1}^{\infty} A_\nu(x) \right\|_{(M),\omega} \preceq E_n(f)_{M,\omega} \preceq \Omega_r(f, \frac{1}{n})_{M,\omega}.$$

Now, we estimate

$$I_1 = \left\| \sum_{\nu=0}^{n} \frac{1 - \lambda_\nu^{(n)}}{\left(1 - \frac{\sin \frac{\nu}{n}}{\frac{\nu}{n}}\right)^r} A_\nu(x) \left(1 - \frac{\sin \frac{\nu}{n}}{\frac{\nu}{n}}\right)^r \right\|_{(M),\omega}.$$

We define the numbers $\mu_{\nu,r}^{(n)}$, $n = 1, 2, ...$ as

$$\mu_{\nu,r}^{(n)} = \begin{cases} \frac{1-\lambda_\nu^{(n)}}{\left(1-\frac{\sin \frac{\nu}{n}}{\frac{\nu}{n}}\right)^r}, & \text{for } \nu \le n, \\ 0, & \text{for } \nu > n \end{cases}.$$

The sequence $\left(\mu_{\nu,r}^{(n)}\right)$ satisfies the conditions of lemma 2.1 [4]. Applying lemma 2.1, we get

$$\begin{aligned} I_1 &= \left\| \sum_{\nu=0}^{n} \mu_{\nu,r}^{(n)} A_\nu(x) \left(1 - \frac{\sin \frac{\nu}{n}}{\frac{\nu}{n}}\right)^r \right\|_{(M),\omega} \preceq \left\| \left(I - \sigma_{1/n}\right)^r f \right\|_{(M),\omega} = \\ &= \left\| \left(I - \sigma_{1/n}\right)^{[r]} \left(I - \sigma_{1/n}\right)^{r-[r]} f \right\|_{(M),\omega} \preceq \\ &\preceq \sup_{0<h_i,t\le\frac{1}{n}} \left\| \prod_{i=1}^{[r]} \left(I - \sigma_{h_i}\right) \left(I - \sigma_t\right)^{r-[r]} f \right\|_{(M),\omega} \preceq \Omega_r(f, \frac{1}{n})_{M,\omega} \end{aligned}$$

and theorem 1.1 is proved.

Proof of Theorem 1.2. Using the properties of the norm we have

$$R_r(f,\lambda)_{M,\omega} \le \left\| \sum_{\nu=1}^{\left[\frac{1}{|r-r_0|}\right]} (1-\lambda_\nu(r)) A_\nu(x) \right\|_{(M),\omega} + \left\| \sum_{\nu=\left[\frac{1}{|r-r_0|}\right]+1}^{\infty} (1-\lambda_\nu(r)) A_\nu(x) \right\|_{(M),\omega} = I_1' + I_2'.$$

We estimate the norm

$$I_1' = \left\| \sum_{\nu=1}^{\left[\frac{1}{|r-r_0|}\right]} \frac{1-\lambda_\nu(r)}{\left(1-\frac{\sin\nu|r-r_0|}{\nu|r-r_0|}\right)^k} A_\nu(x) \left(1-\frac{\sin\nu|r-r_0|}{\nu|r-r_0|}\right)^k \right\|_{(M),\omega}, \qquad k>0.$$

Let us assume

$$\mu_{\nu,r} = \begin{cases} \frac{1-\lambda_\nu(r)}{\left(1-\frac{\sin\nu|r-r_0|}{\nu|r-r_0|}\right)^k}, & \text{for } \nu \le \left[\frac{1}{|r-r_0|}\right], \\ 0, & \text{for } \nu > n \end{cases}.$$

The sequence $\left\{\mu_{\nu,r}^{(n)}\right\}$ satisfies the conditions of lemma 2.1. Using lemma 2.1, we get

$$\begin{aligned} I_1' &\le \left\| \sum_{\nu=1}^{2^{m+1}} \mu_{\nu,r} A_\nu(x) \left(1-\frac{\sin\nu|r-r_0|}{\nu|r-r_0|}\right)^k \right\|_{(M),\omega} \\ &\preceq \left\| \sum_{\nu=1}^{\infty} A_\nu(x) \left(1-\frac{\sin\nu|r-r_0|}{\nu|r-r_0|}\right)^k \right\|_{(M),\omega} \preceq \Omega_k(f,|r-r_0|)_{M,\omega}. \end{aligned}$$

for $2^m \le \left[\frac{1}{|r-r_0|}\right] < 2^{m+1}$. Let us now estimate I_2'. It is easily seen that the conditions of Lemma 2.1 are fulfilled for the system of the numbers $\{1-\lambda_\nu(r)\}$. Then, according to lemma 2.1

$$I_2' = \left\| \sum_{\nu=\left[\frac{1}{|r-r_0|}\right]+1}^{\infty} (1-\lambda_\nu(r)) A_\nu(x) \right\|_{(M),\omega} \preceq \left\| \sum_{\nu=\left[\frac{1}{|r-r_0|}\right]+1}^{\infty} A_\nu(x) \right\|_{(M),\omega}.$$

Hence, as the theorem 1.1, it follows

$$I_2' \preceq \Omega_k(f,|r-r_0|)_{M,\omega}.$$

This completes the proof.

Proof of Theorem 1.3. Let $2^m \leq n < 2^{m+1}$. From the property of the norm and (1.3)

$$\begin{aligned} R_n(f,\lambda)_{M,\omega} &= \left\| f(x) - \sum_{\nu=0}^{n} \lambda_\nu^{(n)} A_\nu(x) \right\|_{(M),\omega} \\ &\leq \left\| \sum_{\nu=1}^{n} \left(1-\lambda_\nu^{(n)}\right) A_\nu(x) \right\|_{(M),\omega} + \left\| \sum_{\nu=n+1}^{\infty} A_\nu(x) \right\|_{(M),\omega} \\ &\preceq \left\| \sum_{\nu=1}^{n} \left(1-\lambda_\nu^{(n)}\right) A_\nu(x) \right\|_{(M),\omega} + E_n(f)_{M,\omega}. \end{aligned} \tag{3.1}$$

Let $\Phi_M(\sqrt{u})$ be a convex function. Using lemma 2.2,

$$\left\| \sum_{\nu=1}^{n} \left(1-\lambda_\nu^{(n)}\right) A_\nu(x) \right\|_{(M),\omega}$$

$$= \inf\left(k>0 : \int_{\mathbf{T}} \Phi_M\left(k^{-1} \left| \sum_{\nu=1}^{n} \left(1-\lambda_\nu^{(n)}\right) A_\nu(x) \right| \right) \omega(x)dx \leq 1 \right)$$

$$\preceq \inf\left(k>0 : c_M \int_{\mathbf{T}} \Phi_M\left(k^{-1} \left(\sum_{\mu=0}^{m} \left| \sum_{\nu=2^\mu}^{2^{\mu+1}-1} \left(1-\lambda_\nu^{(n)}\right) A_\nu(x) \right|^2 \right)^{1/2} \right) \omega(x)dx \leq 1 \right).$$

Moreover, due to (1.4), the constant D_M may be chosen such that

$$c_M \Phi_M(u) \leq \Phi_M(D_M u). \tag{3.2}$$

Then

$$\left\| \sum_{\nu=1}^{n} \left(1-\lambda_\nu^{(n)}\right) A_\nu(x) \right\|_{(M),\omega}$$

$$\preceq \inf\left(k>0 : \int_{0}^{2\pi} \Phi_M\left(D_M k^{-1} \left(\sum_{\mu=0}^{m} \sigma_{n,\mu}^2(x) \right)^{1/2} \right) \omega(x)dx \leq 1 \right),$$

where

$$\sigma_{n,\mu}(x) = \sum_{\nu=2^\mu}^{2^{\mu+1}-1} \left(1-\lambda_\nu^{(n)}\right) A_\nu(x).$$

Let $\xi(u) = \Phi_M(\sqrt{u})$. Then

$$\left\| \sum_{\nu=1}^{n} \left(1-\lambda_\nu^{(n)}\right) A_\nu(x) \right\|_{(M),\omega}$$

$$\leq \inf\left(k>0 : \int_{\mathbf{T}} \xi\left(D_M^2 k^{-2} \sum_{\mu=0}^{m} \sigma_{n,\mu}^2(x) \right) \omega(x)dx \leq 1 \right)$$

$$
\begin{aligned}
&= \left[\inf\left(k>0:\int_{\mathbf{T}}\xi\left(D_M^2k^{-1}\sum_{\mu=0}^{m}\sigma_{n,\mu}^2(x)\right)\omega(x)dx\le 1\right)\right]^{1/2}\\
&= D_M\left\|\sum_{\mu=0}^{m}\sigma_{n,\mu}^2(x)\right\|_{L^*_{(\xi,\omega)}}^{1/2}\le D_M\left[\sum_{\mu=0}^{m}\left\|\sigma_{n,\mu}^2(x)\right\|_{L^*_{(\xi,\omega)}}\right]^{1/2}\\
&= D_M\left[\sum_{\mu=0}^{m}\|\sigma_{n,\mu}(x)\|_{(M),\omega}^2\right]^{1/2},
\end{aligned}
$$

where

$$
\begin{aligned}
\left\|\sigma_{n,\mu}^2(x)\right\|_{L^*_{(\xi,\omega)}} &= \inf\left(k>0:\int_{\mathbf{T}}\xi\left(k^{-1}\sigma_{n,\mu}^2(x)\right)\omega(x)dx\le 1\right)\\
&= \inf\left(k>0:\int_{\mathbf{T}}\Phi_M\left(k^{-1/2}\sigma_{n,\mu}(x)\right)\omega(x)dx\le 1\right)\\
&= \inf\left(t^2>0:\int_{\mathbf{T}}\Phi_M\left(t^{-1}\sigma_{n,\mu}(x)\right)\omega(x)dx\le 1\right)\\
&= \|\sigma_{n,\mu}(x)\|_{(M),\omega}^2 .
\end{aligned}
$$

Applying the Abel transform to $\sigma_{n,\mu}$, we get

$$
\begin{aligned}
\sigma_{n,\mu}(x) &= \sum_{\nu=2^\mu}^{2^{\mu+1}-1}\left(1-\lambda_\nu^{(n)}\right)A_\nu(x)\\
&= \sum_{\nu=2^\mu}^{2^{\mu+1}-1}\left(S_\nu(f,x)-S_{2^{\mu+1}-1}(f,x)\right)\left(\lambda_{\nu+1}^{(n)}-\lambda_\nu^{(n)}\right)\\
&\qquad + \left(S_{2^{\mu+1}-1}(f,x)-S_{2^\mu-1}(f,x)\right)\left(1-\lambda_{2^\mu}^{(n)}\right).
\end{aligned}
$$

From the inequality (1.3) and the monotonicity of the sequence of best approximations,

$$
\begin{aligned}
\|\sigma_{n,\mu}(x)\|_{(M),\omega} &\le \sum_{\nu=2^\mu}^{2^{\mu+1}-1}\left\|S_\nu(f,x)-S_{2^{\mu+1}-1}(f,x)\right\|_{(\varphi,\omega)}\left|\lambda_{\nu+1}^{(n)}-\lambda_\nu^{(n)}\right|+\\
&\quad +\left\|S_{2^{\mu+1}-1}(f,x)-S_{2^\mu-1}(f,x)\right\|_{(\varphi,\omega)}\left|1-\lambda_{2^\mu}^{(n)}\right|\\
&\preceq E_{2^\mu-1}(f)_{\varphi,\omega}\delta_{2^\mu,n}.
\end{aligned}
$$

Then

$$
\left\|\sum_{\nu=1}^{n}\left(1-\lambda_\nu^{(n)}\right)A_\nu(x)\right\|_{(M),\omega}\le D_M\left(\sum_{\mu=0}^{m}E_{2^\mu-1}^2(f)_{\varphi,\omega}\delta_{2^\mu,n}^2\right)^{1/2}.
$$

Assume that $\Phi_M(\sqrt{u})$ is a concave function. We use the well-known formula for calculation of the norm [12, p. 92]:

$$\left\|\sum_{\nu=1}^{n}\left(1-\lambda_\nu^{(n)}\right)A_\nu(x)\right\|_{(M),\omega} = \inf_{k>0} k^{-1}\left(1+\int_{\mathbf{T}}\Phi_M\left(k\sum_{\nu=1}^{n}\left(1-\lambda_\nu^{(n)}\right)A_\nu(x)\right)\omega(x)dx\right)$$

Applying lemma 2.2 and (3.2), we get

$$\left\|\sum_{\nu=1}^{n}\left(1-\lambda_\nu^{(n)}\right)A_\nu(x)\right\|_{(M),\omega} = \inf_{k>0} k^{-1}\left(1+\int_{\mathbf{T}}\Phi_M\left(D_M^2k^2\sum_{\mu=0}^{m}\sigma_{n,\mu}^2(x)\right)^{1/2}\omega(x)dx\right).$$

Since $\Phi_M(\sqrt{u})$ is a concave function

$$\left\|\sum_{\nu=1}^{n}\left(1-\lambda_\nu^{(n)}\right)A_\nu(x)\right\|_{(M),\omega} = \inf_{k>0} k^{-1}\left(1+\sum_{\mu=0}^{m}\int_{\mathbf{T}}\Phi_M\left(D_M k\sigma_{n,\mu}(x)\right)\omega(x)dx\right).$$

Using the proof of lemma 9.2 in [12, p. 74], it is easily seen that

$$\int_{\mathbf{T}}\Phi_M\left[\frac{u(x)}{\|u(x)\|_{(M),\omega}}\right]\omega(x)dx \le 1.$$

From this inequality, (1.4) and (3.2)

$$\int_{\mathbf{T}}\Phi_M\left(D_M k\sigma_{n,\mu}(x)\right)\omega(x)dx = C\int_{\mathbf{T}}\Phi_M\left(\frac{\sigma_{n,\mu}(x)}{\|\sigma_{n,\mu}(x)\|_{(M),\omega}}\right)\Phi_M\left(D_M k\left\|\sigma_{n,\mu}(x)\right\|_{(M),\omega}\right)\omega(x)dx \le \Phi_M\left(D_M' k\left\|\sigma_{n,\mu}(x)\right\|_{\varphi,\omega}\right).$$

Consequently, we get

$$\left\|\sum_{\nu=1}^{n}\left(1-\lambda_{\nu}^{(n)}\right)A_{\nu}\left(x\right)\right\|_{(M),\omega}$$
$$\leq \inf_{k>0}k^{-1}\left(1+\sum_{\mu=0}^{m}\Phi_{M}\left(D_{M}^{\prime}k\left\|\sigma_{n,\mu}(x)\right\|_{(M),\omega}\right)\right)$$
$$\leq \inf_{k>0}k^{-1}\left(1+\sum_{\mu=0}^{m}\Phi_{M}\left(D_{M}^{\prime}kE_{2^{\mu}-1}(f)_{\varphi,\omega}\delta_{2^{\mu},n}\right)\right).$$

This completes the proof.

References

[1] R. Akgun, D. M. Israfilov, *Approximation of weighted Orlicz spaces,* Math. Slovaca, 61 (2011), No. 4, 601-618.
[2] R. Akgun, *Some inequalities of trigonometric approximation in weighted Orlicz spaces*, Math. Slovaca, to appear.
[3] R. Akgun, D. M. Israfilov, *Approximation and moduli of fractional order in Smirnov-Orlicz classes*, Glasnik Matematicki, 43 (2008), 121-136.
[4] R. Akgun, *Sharp Jackson and converse theorems of trigonometric approximation in weighted Lebesgue spaces*, Proc. A. Razmadze Math. Inst., 152 (2010), 1-18.
[5] R. Akgun, *Approximating polynomials for functions of weighted Smirnov-Orlicz spaces*, J. Funct. Spaces Appl. Vol. 2012, Article ID 982360, 41 pp.
[6] R. Akgun, *Inequalities for one sided approximation in Orlicz spaces*, Hacettepe journal of mathematics and statistics, 40 (2011), no. 2, 231-240.
[7] P. L. Butzer, H. Dyckhoff, E. Görlich and R. L. Stens, *Best trigonometric approximation, fractional order derivatives and Lipschitz classes*, Canad. J. Math., 29 (1977), no. 4, 781-793.
[8] G. P. Curbera, J. García-Cuerva, J. M. Martell and C. Pérez, *Extrapolation with weights, rearrangement-invariant function spaces, modular inequalities and applications to singular integrals*, Adv. Math., 203 (2006), no. 1, 256-318.
[9] Y. M. Chen, *On two-functional spaces*, Studia Math. 24 (1964), 61-88.
[10] D. M. Israfilov and A. Guven, *Approximation by trigonometric polynomials in weighted Orlicz spaces,* Studia Math., 174 (2006), no. 2, 147-168.
[11] M. Khabazi, *The mean convergence of trigonometric Fourier series in weighted Orlicz classes*, Proc. A. Razmadze Math. Inst., 129 (2002), 65-75.
[12] M. A. Krasnosel'skiĭ, Y. B. Rutickiĭ, *Convex Functions and Orlicz Spaces*, Noordholf, 1961.
[13] V. G. Ponomarenko, *Approximation of periodic functions in Orlicz spaces*, Translated from Sibirskii Matematicheskii Zhurnal, Vo. 7, No. 6, (1966), 1337-1346.
[14] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional integrals and derivatives, Theory and applications,* Gordon and Breach Science Publishers, 1993.
[15] R. Taberski, *Differences, moduli and derivatives of fractional orders*, Comment. Math. Prace Mat., 19 (1976/77), no. 2, 389-400.
[16] I. I. Tsyganok, A generalization of a theorem of Jackson, (Russian) Mat. Sb. (N.S.) 71 (113) (1966), 257260.
[17] Y. E. Yildirir, *Approximation of periodic functions in weighted Orlicz spaces*, Glasnik Matematički, 47(67)(2012), 401 – 413.

[18] Y. E. Yildirir, R. Cetintas, *Approximation theorems in weighted Orlicz spaces*, J. Math. Sci. Adv. Appl., V. 14, No. 1, (2012), 35-49.

[19] Y. E.Yildirir, D. M. Israfilov, *The properties of convolution type transforms in weighted Orlicz spaces*, Glasnik Matematicki, 45 (2010), 461-474.

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