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LIE DERIVATIVES OF ALMOST CONTACT STRUCTURE AND ALMOST PARACONTACT STRUCTURE WITH RESPECT TO X^V AND X^H ON TANGENT BUNDLE $T(M)$

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Abstract. The differential geometry of tangent bundles was studied by several authors, for example: D. E. Blair [1], V. Oproiu [3], A. Salimov [5], Yano and Ishihara [8] and among others. It is well known that differant structures deffined on a manifold M can be lifted to the same type of structures on its tangent bundle. The main aim of this paper is to study Lie derivatives with respect to $X^V$and $X^H$of almost contact structure and almost paracontact structure on tangent bundle $T(M)$.

1. Introduction

Let M be an $n-$dimensional differentiable manifold of class C^∞ and let $T_p(M)$ be the tangent space of M at a point p of M. Then the set [8]

$$T(M) = \underset{p\in M}{\cup} T_p(M) \tag{1.1}$$

is called the tangent bundle over the manifold M. For any point $\tilde{p}$ of $T(M)$, the correspondence $\tilde{p} \to p$ determines the bundle projection $\pi : T(M) \to M$. The set $\pi^{-1}(p)$ is called the fibre over $p \in M$ and M the base space.

Suppose that the base space M is covered by a system of coordinate neighbourhoods $\{U; x^h\}$, where (x^h) is a system of local coordinates defined in the neighbourhood U of M. The open set $\pi^{-1}(U) \subset T(M)$ is naturally differentiably homeomorphic to the direct product $U \times R^n$, R^n being the $n-$dimensional vector space over the real field R, in such a way that a point $\tilde{p} \in T_p(M)(p \in U)$ is represented by an ordered pair (P, X) of the point $p \in U$, and a vector $X \in R^n$, whose components are given by the cartesian coordinates (y^h) of $\tilde{p}$ in the tangent space $T_p(M)$ with respect to the natural base $\{\partial_h\}$, where $\partial_h = \frac{\partial}{\partial x^h}$. Denoting by (x^h) the coordinates of $p = \pi(\tilde{p})$ in U and establishing the correspondence $(x^h, y^h) \to \tilde{p} \in \pi^{-1}(U)$, we can introduce a system of local coordinates (x^h, y^h) in the open set $\pi^{-1}(U) \subset T(M)$. Here we cal (x^h, y^h) the coordinates in $\pi^{-1}(U)$ induced from (x^h) or simply, the induced coordinates in $\pi^{-1}(U)$.

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We denote by $\Im_s^r(M)$ the set of all tensor fields of class $C^\infty$and of type (r,s) in M. We now put $\Im(M)=\sum\limits_{r,s=0}^{\infty}\Im_s^r(M)$, which is the set of all tensor fields in M. Similarly, we denote by $\Im_s^r(T(M))$ and $\Im(T(M))$ respectively the corresponding sets of tensor fields in the tangent bundle $T(M)$.

1.1. Vertical lifts. If f is a function in M, we write f^v for the function in $T(M)$ obtained by forming the composition of $\pi:T(M)\to M$ and $f:M\to R$, so that

$$f^v=fo\pi. \tag{1.2}$$

Thus, if a point $\tilde{p}\in\pi^{-1}(U)$ has induced coordinates (x^h,y^h), then

$$f^v(\tilde{p})=f^v(x,y)=fo\pi(\tilde{p})=f(p)=f(x). \tag{1.3}$$

Thus the value of $f^v(\tilde{p})$ is constant along each fibre $T_p(M)$ and equal to the value $f(p)$. We call f^v the vertical lift of the function f [8].

Let $\tilde{X}\in\Im_0^1(T(M))$ be such that $\tilde{X}f^v=0$ for all $f\in\Im_0^0(M)$. Then we say that $\tilde{X}$ is a vertical vector field. Let $\binom{\tilde{X}^h}{\tilde{X}^{\bar{h}}}$ be components of $\tilde{X}$ with respect to the induced coordinates. Then $\tilde{X}$ is vertical if and only if its components in $\pi^{-1}(U)$ satisfy

$$\begin{pmatrix}\tilde{X}^h\\ \tilde{X}^{\bar{h}}\end{pmatrix}=\begin{pmatrix}0\\ X^{\bar{h}}\end{pmatrix}. \tag{1.4}$$

Suppose that $X\in\Im_0^1(M)$, so that is a vector field in M. We define a vector field X^v in $T(M)$ by

$$X^v(\iota\,\omega)=(\omega X)^v \tag{1.5}$$

ω being an arbitrary $1-$form in M. We cal X^v the vertical lift of X [8].

Let $\tilde{\omega}\in\Im_1^0(T(M))$ be such that $\tilde{\omega}(X)^v=0$ for all $X\in\Im_0^1(M)$. Then we say that $\tilde{\omega}$ is a vertical $1-$form in $T(M)$. We define the vertical lift ω^v of the $1-$form ω by

$$\omega^v=(\omega_i)^v(dx^i)^v \tag{1.6}$$

in each open set $\pi^{-1}(U)$, where $(U;x^h)$ is coordinate neighbourhood in M and ω is given by $\omega=\omega_i dx^i$ in U. The vertical lift ω^v of ω with local expression $\omega=\omega_i dx^i$ has components of the form

$$\omega^v:(\omega^i,0) \tag{1.7}$$

with respect to the induced coordinates in $T(M)$.

Vertical lifts to a unique algebraic isomorphism of the tensor algebra $\Im(M)$ into the tensor algebra $\Im(T(M))$ with respect to constant coefficients by the conditions

$$(P\otimes Q)^V=P^V\otimes Q^V,\ (P+R)^V=P^V+R^V \tag{1.8}$$

P,Q and R being arbitrary elements of $\Im(M)$. The vertical lifts F^V of an element $F\in\Im_1^1(M)$ with lokal components F_i^h has components of the form [8]

$$F^V:\begin{pmatrix}0&0\\ F_i^h&0\end{pmatrix}.$$

Vertical lift has the following formulas ([4],[8]):

$$\begin{aligned} (fX)^v &= f^v X^v,\ I^v X^v = 0,\ \eta^v (X^v) = 0, \\ (f\eta)^v &= f^v \eta^v,\ [X^v, Y^v] = 0,\ \varphi^v X^v = 0, \\ X^v f^v &= 0,\ X^v f^v = 0 \end{aligned} \tag{1.9}$$

hold good, where $f \in \Im_0^0(M_n)$, $X, Y \in \Im_0^1(M_n)$, $\eta \in \Im_1^0(M_n)$, $\varphi \in \Im_1^1(M_n)$, $I = id_{M_n}$.

1.2. Complete lifts. If f is a function in M, we write f^c for the function in $T(M)$ defined by

$$f^c = \iota(df) \tag{1.10}$$

and call f^c the comple lift of the function f. The complete lift f^c of a function f has the lokal expression

$$f^c = y^i \partial_i f = \partial f \tag{1.11}$$

with respect to the induced coordinates in $T(M)$, where ∂f denotes $y^i \partial_i f$.

Suppose that $X \in \Im_0^1(M)$. We define a vector field X^c in $T(M)$ by

$$X^c f^c = (Xf)^c, \tag{1.12}$$

f being an arbitrary function in M and call X^c the complete lift of X in $T(M)$ ([2],[8]). The complete lift X^c of X with components x^h in M has components

$$X^c = \begin{pmatrix} X^h \\ \partial X^h \end{pmatrix} \tag{1.13}$$

with respect to the induced coordinates in $T(M)$.

Suppose that $\omega \in \Im_1^0(M)$, then a $1-$form ω^c in $T(M)$ defined by

$$\omega^c(X^c) = (\omega X)^c \tag{1.14}$$

X being an arbitrary vector field in M. We call ω^c the complete lift of ω. The complete lift ω^c of ω with components ω_i in M has components of the form

$$\omega^c : (\partial\omega_{i,}\omega_i) \tag{1.15}$$

with respect to the induced coordinates in $T(M)$ [2].

The complete lifts to a unique algebra isomorphism of the tensor algebra $\Im(M)$ into the tensor algebra $\Im(T(M))$ with respect to constant coefficients, is given by the conditions

$$(P \otimes Q)^C = P^C \otimes Q^V + P^V \otimes Q^C,\ (P + R)^C = P^C + R^C, \tag{1.16}$$

where P, Q and R being arbitrary elements of $\Im(M)$. The complete lifts F^C of an element $F \in \Im_1^1(M)$ with lokal components F_i^h has components of the form

$$F^C : \begin{pmatrix} F_i^h & 0 \\ \partial F_i^h & F_i^h \end{pmatrix}.$$

In addition, we know that the complete lifts are defined by ([4],[8]):

$$\begin{aligned}(fX)^c &= f^cX^v + f^vX^c = (Xf)^c, \\ X^cf^v &= (Xf)^v,\ \eta^v(x^c) = (\eta(x))^v, \\ X^vf^c &= (Xf)^v,\ \varphi^vX^c = (\varphi X)^v, \\ \varphi^cX^v &= (\varphi X)^v,\ (\varphi X)^c = \varphi^cX^c, \\ \eta^v(X^c) &= (\eta(X))^c,\ \eta^c(X^v) = (\eta(X))^v, \\ [X^v, Y^c] &= [X,Y]^v,\ I^c = I,\ I^vX^c = X^v,\ [X^c,Y^c] = [X,Y]^c.\end{aligned} \tag{1.17}$$

1.3. Horizontal lifts. The horizontal lift f^H of $f \in \Im_0^0(M)$ to the tangent bundle $T(M)$ is given by

$$f^H = f^C - \nabla_\gamma f, \tag{1.18}$$

where

$$\nabla_\gamma f = \gamma \nabla f. \tag{1.19}$$

Let $X \in \Im_0^1(M)$. Then the horizontal lift X^H of X defined by

$$X^H = X^C - \nabla_\gamma X \tag{1.20}$$

in $T(M)$, where

$$\nabla_\gamma X = \gamma \nabla X. \tag{1.21}$$

İf we compare horizontal and complete lift, we obtain

$$X^H = (\hat{\nabla}_X)^C$$

for any $X \in \Im_0^1(M_n)$, where $\hat{\nabla}$ is an affine connection in M_n defined by

$$\hat{\nabla}_X Y = \nabla_Y X + [X,Y]$$

or

$$(\nabla_Y X)^v = (\hat{\nabla}_X Y)^v + [Y,X]^v \tag{1.22}$$

and $(\hat{\nabla}_X)^C$ is the complete lift of the derivation $\hat{\nabla}_X$. The horizontal lift X^H of X has the components

$$X^H : \begin{pmatrix} X^h \\ -\Gamma_i^h X^i \end{pmatrix} \tag{1.23}$$

with respect to the induced coordinates in $T(M)$, where

$$\Gamma_i^h = y^i \Gamma_{ji}^h\ . \tag{1.24}$$

Let $\omega \in \Im_1^0(M)$ with affine connection ∇. Then the horizontal lift ω^H of ω is defined by

$$\omega^H = \omega^C - \nabla_\gamma \omega \tag{1.25}$$

in $T(M)$, where $\nabla_\gamma \omega = \gamma \nabla \omega$. The horizontal lift ω^H of ω has component of the form

$$\omega^H : (\Gamma_i^h \omega_h, \omega_i) \tag{1.26}$$

with respect to the induced coordinates in $T(M)$.

Suppose there is given a tensor field

$$S = S_{k...j}^{i...h} \frac{\partial}{\partial x^i} \otimes ... \otimes \frac{\partial}{\partial x^h} \otimes dx^k \otimes ... \otimes dx^j \tag{1.27}$$

in M with affine connection ∇ and in $T(M)$ a tensor field $\nabla_\gamma S$ is defined by

$$\nabla_\gamma S = y^l \nabla_l S_{k...j}^{i...h} \frac{\partial}{\partial y^i} \otimes ... \otimes \frac{\partial}{\partial y^h} \otimes dx^k \otimes ... \otimes dx^j \tag{1.28}$$

with respect to the induced coordinates (x^h, y^h) in $\pi^{-1}(U)$. In addition, we define a tensor field $\gamma_X S$ in $\pi^{-1}(U)$ by

$$\gamma_X S = (X^l S_{lk...j}^{i...h}) \frac{\partial}{\partial y^i} \otimes ... \otimes \frac{\partial}{\partial y^h} \otimes dx^k \otimes ... \otimes dx^j$$

and a tensor field γS in $\pi^{-1}(U)$ by

$$\gamma S = (y^l S_{lk...j}^{i...h}) \frac{\partial}{\partial y^i} \otimes ... \otimes \frac{\partial}{\partial y^h} \otimes dx^k \otimes ... \otimes dx^j$$

with respect to the induced coordinates (x^h, y^h), U being an arbitrary coordinate neighborhood in M. Then we have

$$\gamma_X S = (S_X)^V$$

for any $X \in \Im_0^1(M)$ and $S \in \Im_s^0(M)$ or $\Im_s^1(M)$, where $S_X \in \Im_{s-1}^0(M)$ or $\Im_{s-1}^1(M)$ [8].

The horizontal lift S^H of a tensor field S of arbitrary type in M to $T(M)$ is defined by

$$S^H = S^C - \nabla_\gamma S. \tag{1.29}$$

For any $P, Q \in T(M)$, we have

$$\begin{aligned} \nabla_\gamma (P \otimes Q) &= (\nabla_\gamma P) \otimes Q^V + P^V \otimes (\nabla_\gamma Q), \\ (P \otimes Q)^H &= P^H \otimes Q^V + P^V \otimes Q^H. \end{aligned} \tag{1.30}$$

We also know that the horizontal lifts are defined by ([8],[4])

$$\begin{aligned} I^H &= I,\ I^H X^v = X^V,\ I^v X^H = X^v,\ I^H X^H = X^H, \\ X^H f^v &= (Xf)^v,\ (fX)^H = f^v X^H,\ \omega^H(X^H) = 0, \\ \omega^v(X^H) &= (\omega(X))^v,\ \omega^H(X^v) = (\omega(X))^v, \\ F^H X^v &= (FX)^v,\ F^H X^H = (FX)^H. \end{aligned} \tag{1.31}$$

Let M be an $n-$dimensional differentiable manifold of class C^∞. Differantial transformation, defined by $D = L_X$, is called as Lie derivation with respect to vector field X if

$$\begin{aligned} L_X f &= Xf,\ \forall f \in \Im_0^0(M), \\ L_X Y &= [X, Y],\ \forall X, Y \in \Im_0^1(M). \end{aligned} \tag{1.32}$$

$[X, Y]$ is called by Lie bracked. The Lie derivative $L_X F$ of a tensor field F of type $(1, 1)$ with respect to a vector field X is defined by [8]

$$(L_X F)Y = [X, FY] - F[X, Y]. \tag{1.33}$$

Differantial transformation of algebra $\Im(M)$, defined by

$$D = \nabla_X : \Im(M) \to \Im(M),\ X \in \Im_0^1(M),$$

is called as covariant derivation with respect to vector field X if

$$\begin{aligned} \nabla_{fX+gY} t &= f \nabla_X t + g \nabla_Y t, \\ \nabla_X f &= Xf, \end{aligned} \tag{1.34}$$

where $\forall f, g \in \Im_0^0(M), \forall X, Y \in \Im_0^1(M), \forall t \in \Im(M)$.

On the other hand, a transformation, defined by

$$\nabla : \Im_0^1(M) \times \Im_0^1(M) \to \Im_0^1(M),$$

is called as affin connection ([5],[8]).

Proposition 1.1. *For any* $X, Y \in \Im_0^1(M_n)$ [8]

$$\begin{aligned} i)\ [X^V, Y^H] &= [X, Y]^V - (\nabla_X Y)^V = -(\hat{\nabla}_Y X)^V, \qquad (1.35)\\ ii)\ [X^C, Y^H] &= [X, Y]^H - \gamma(L_X Y),\\ iii)\ [X^H, Y^V] &= [X, Y]^V + (\nabla_Y X)^V,\\ iv)\ [X^H, Y^H] &= [X, Y]^H - \gamma\hat{R}(X, Y), \end{aligned}$$

where $\hat{R}$ denotes the curvature tensor of the affine connection $\hat{\nabla}$.

2. Main Results

Let an $n-$dimensional diferentiable manifold M_n be endowed with a tensor field φ of type $(1,1)$, a vector field ξ, a $1-$form η, I the identity and let them satisfy

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \varphi(\xi) = 0, \qquad \eta o \varphi = 0, \qquad \eta(\xi) = 1. \qquad (2.1)$$

Then (φ, ξ, η) define almost contact structure on M_n ([8]). From (2.1), we get on taking horizontal and vertical lifts

$$\begin{aligned} (\varphi^H)^2 &= -I + \eta^v \otimes \xi^H + \eta^H \otimes \xi^v, \qquad (2.2)\\ \varphi^H \xi^v &= 0,\ \varphi^H \xi^H = 0,\ \eta^v o \xi^H = 0,\\ \eta^H o \varphi^H &= 0,\ \eta^v(\xi^v) = 0,\ \eta^v(\xi^H) = 1,\\ \eta^H(\xi^v) &= 1,\ \eta^H(\xi^H) = 0. \end{aligned}$$

We now define a $(1,1)$ tensor field J on $\Im(M_n)$ by

$$J = \varphi^H - \xi^v \otimes \eta^v + \xi^H \otimes \eta^H. \qquad (2.3)$$

Then it is easy to show that $J^2 X^v = -X^v$ and $J^2 X^H = -X^H$, which give that J is an almost contact structure on $\Im(M_n)$. We get from (2.3)

$$\begin{aligned} JX^v &= (\varphi X)^v + (\eta(X)\xi)^H,\\ JX^H &= (\varphi X)^H - (\eta(X)\xi)^v \end{aligned}$$

for any $X \in \Im_0^1(M_n)$ [4].

Theorem 2.1. *For L_X the operator Lie derivation with respect to X , $J \in \Im_1^1(\Im(M_n))$ defined by (2.3) and $\eta(Y) = 0$, we have*

$$\begin{aligned} i)\ (L_{X^H} J)Y^v &= ((\hat{\nabla}_X \varphi)Y)^v + ((\hat{\nabla}_X \eta)Y)^v \xi^H,\\ ii)\ (L_{X^H} J)Y^H &= ((L_X \varphi)Y)^H - \gamma\hat{R}(X, \varphi Y) + \varphi^H \gamma\hat{R}(X, Y) - ((L_X \eta)Y)^v \xi^v\\ &\quad -(\eta^v \gamma\hat{R}(X, Y))\xi^v + (\eta^H \gamma\hat{R}(X, Y))\xi^H,\\ iii)\ (L_{X^v} J)Y^v &= 0,\\ iv)\ (L_{X^v} J)Y^H &= ((L_X \varphi)Y)^v - ((\nabla_X \varphi)Y)^v + ((L_X \eta)Y)^v \xi^H - ((\nabla_X \eta)Y)^v \xi^H, \end{aligned}$$

where $X, Y \in \Im_0^1(M_n)$, a tensor field $\varphi \in \Im_1^1(M_n)$, a vector field ξ and a $1-$form $\eta \in \Im_1^0(M_n)$.

Proof. For $J = \varphi^H - \xi^v \otimes \eta^v + \xi^H \otimes \eta^H$ and $\eta(Y) = 0$, we get

$$
\begin{aligned}
i)\ (L_{X^H}J)Y^v &= L_{X^H}JY^v - JL_{X^H}Y^v \\
&= [X^H, (\varphi Y)^v + (\eta(Y)\xi)^H] - (\varphi^H - \xi^v \otimes \eta^v + \xi^H \otimes \eta^H)[X^H, Y^v] \\
&= [X^H, (\varphi Y)^v] + [X^H, (\eta(Y)\xi)^H] - \varphi^H[X^H, Y^v] + \eta^v([X^H, Y^v])\xi^v \\
&\quad -\eta^H([X^H, Y^v])\xi^H \\
&= [X, \varphi Y]^v + (\nabla_{\varphi Y}X)^v - \varphi^H([X, Y]^v + (\nabla_Y X)^v) + \eta^v([X, Y]^v \\
&\quad +(\nabla_Y X)^v)\xi^v - \eta^H([X, Y]^v + (\nabla_Y X)^v)\xi^H \\
&= ((L_X\varphi)Y)^v + (\varphi(L_XY))^v + (\hat{\nabla}_X\varphi Y)^v + [\varphi Y, X]^v - (\varphi L_XY)^v \\
&\quad -(\varphi\nabla_Y X)^v + \eta^v([X, Y]^v)\xi^v + (\eta^v(\nabla_Y X)^v)\xi^v \\
&\quad -(\eta[X, Y])^v\xi^H - \eta^H(\nabla_Y X)^v\xi^H \\
&= ((L_X\varphi)Y)^v + (\varphi(L_XY))^v + ((\hat{\nabla}_X\varphi)Y)^v + (\varphi\hat{\nabla}_XY)^v \\
&\quad -((L_X\varphi)Y)^v - (\varphi(L_XY))^v - (\varphi(L_XY))^v - (\varphi\nabla_Y X)^v \\
&\quad -(\eta[X, Y])^v\xi^H - (\eta^H(\nabla_Y X)^v)\xi^H \\
&= ((\hat{\nabla}_X\varphi)Y)^v + (\varphi\hat{\nabla}_XY)^v - (\varphi(L_XY))^v - \varphi^H(\nabla_Y X)^v \\
&\quad +((L_X\eta)Y)^v\xi^H - (\eta^H((\hat{\nabla}_XY)^v + [Y, X]^v))\xi^H \\
&= ((\hat{\nabla}_X\varphi)Y)^v + (\varphi\hat{\nabla}_XY)^v - (\varphi(L_XY))^v - \varphi^H((\hat{\nabla}_XY)^v + [Y, X]^v) \\
&\quad +((L_X\eta)Y)^v\xi^H - (\eta(\hat{\nabla}_XY))^v\xi^H - (\eta(L_YX))^v\xi^H \\
&= ((\hat{\nabla}_X\varphi)Y)^v + (\varphi\hat{\nabla}_XY)^v - (\varphi(L_XY))^v - (\varphi(\hat{\nabla}_XY))^v + (\varphi(L_XY))^v \\
&\quad +((L_X\eta)Y)^v\xi^H + ((\hat{\nabla}_X\eta)Y)^v)\xi^H + (\eta(L_XY))^v\xi^H \\
&= ((\hat{\nabla}_X\varphi)Y)^v + (\varphi\hat{\nabla}_XY)^v - (\varphi(\hat{\nabla}_XY))^v + ((L_X\eta)Y)^v\xi^H \\
&\quad +(((\hat{\nabla}_X\eta)Y))^v\xi^H - (((L_X\eta)Y))^v\xi^H \\
&= ((\hat{\nabla}_X\varphi)Y)^v + ((\hat{\nabla}_X\eta)Y)^v)\xi^H,
\end{aligned}
$$

$$
\begin{aligned}
ii)\ (L_{X^H}J)Y^H &= L_{X^H}JY^H - JL_{X^H}Y^H \\
&= [X^H, (\varphi Y)^H - (\eta(Y)\xi)^v] - (\varphi^H - \xi^v \otimes \eta^v + \xi^H \otimes \eta^H)[X^H, Y^H] \\
&= [X^H, (\varphi Y)^H] - [X^H, (\eta(Y)\xi)^v] - \varphi^H[X^H, Y^H] + \eta^v([X^H, Y^H])\xi^v \\
&\quad -\eta^H([X^H, Y^H])\xi^H \\
&= [X, \varphi Y]^H - \gamma\hat{R}(X, \varphi Y) - \varphi^H([X, Y]^H - \gamma\hat{R}(X, Y)) \\
&\quad +\eta^v([X, Y]^H - \gamma\hat{R}(X, Y))\xi^v - \eta^H([X, Y]^H - \gamma\hat{R}(X, Y))\xi^H \\
&= ((L_X\varphi)Y)^H + (\varphi(L_XY))^H - \gamma\hat{R}(X, \varphi Y) - (\varphi(L_XY))^H \\
&\quad +\varphi^H\gamma\hat{R}(X, Y) + (\eta L_XY)^v\xi^v - (\eta^v\gamma\hat{R}(X, Y))\xi^v \\
&\quad -\eta^H([X, Y]^H)\xi^H + (\eta^H\gamma\hat{R}(X, Y))\xi^H
\end{aligned}
$$

$$\begin{aligned} &= ((L_X\varphi)Y)^H - \gamma\hat{R}(X,\varphi Y) - ((L_X\eta)Y)^v\xi^v + \varphi^H\gamma\hat{R}(X,Y) \\ &\quad -(\eta^v\gamma\hat{R}(X,Y))\xi^v + (\eta^H\gamma\hat{R}(X,Y))\xi^H \\ &= ((L_X\varphi)Y)^H - \gamma\hat{R}(X,\varphi Y) - ((L_X\eta)Y)^v\xi^v + J(\gamma\hat{R}(X,Y)), \end{aligned}$$

$$\begin{aligned} iii)\ (L_{X^v}J)Y^v &= L_{X^v}JY^v - JL_{X^v}Y^v \\ &= [X^v,(\varphi Y)^v + (\eta(Y)\xi)^H] - (\varphi^H - \xi^v\otimes\eta^v + \xi^H\otimes\eta^H)[X^v,Y^v] \\ &= [X^v,(\varphi Y)^v] + [X^v,(\eta(Y)\xi)^H] \\ &= 0, \end{aligned}$$

$$\begin{aligned} iv)\ (L_{X^v}J)Y^H &= L_{X^v}JY^H - JL_{X^v}Y^H \\ &= [X^v,(\varphi Y)^H - (\eta(Y)\xi)^v] - (\varphi^H - \xi^v\otimes\eta^v + \xi^H\otimes\eta^H)[X^v,Y^H] \\ &= [X^v,(\varphi Y)^H] - [X^v,(\eta(Y)\xi)^v] - \varphi^H[X^v,Y^H] \\ &\quad +\eta^v([X^v,Y^H])\xi^v - \eta^H([X^v,Y^H])\xi^H \\ &= [X,\varphi Y]^v - (\nabla_X\varphi Y)^v - \varphi^H([X,Y]^v - (\nabla_XY)^v) \\ &\quad +\eta^v([X,Y]^v - (\nabla_XY)^v)\xi^v - \eta^H([X,Y]^v - (\nabla_XY)^v)\xi^H \\ &= ((L_X\varphi)Y)^v + (\varphi(L_XY))^v - ((\nabla_X\varphi)Y)^v - (\varphi\nabla_XY)^v \\ &\quad -(\varphi(L_XY))^v + (\varphi\nabla_XY)^v + (\eta^v([X,Y]^v))\xi^v \\ &\quad -(\eta^v(\nabla_XY)^v)\xi^v - (\eta L_XY)^v\xi^H + (\eta\nabla_XY)^v\xi^H \\ &= ((L_X\varphi)Y)^v - ((\nabla_X\varphi)Y)^v + ((L_X\eta)Y)^v\xi^H - ((\nabla_X\eta)Y)^v\xi^H, \end{aligned}$$

where $\eta L_XY = L_X\eta(Y) - (L_X\eta)Y$ and $\eta\nabla_XY = \nabla_X\eta(Y) - (\nabla_X\eta)Y$. □

Corollary 2.1. *If we put $Y=\xi$, i.e. $\eta(\xi)=1$ and ξ has the conditions of (2.1), then we get different results*

$$\begin{aligned} i)\ (L_{X^H}J)\xi^v &= (L_X\xi)^H - \gamma\hat{R}(X,\xi) + ((\hat{\nabla}_X\varphi)\xi)^v + ((\hat{\nabla}_X\eta)\xi)^v\xi^H, \\ ii)\ (L_{X^H}J)\xi^H &= -(\hat{\nabla}_X\xi)^v + ((L_X\varphi)\xi)^H - ((L_X\eta)\xi)^v\xi^v + \varphi^H\gamma\hat{R}(X,\xi) \\ &\quad -(\eta^v\gamma\hat{R}(X,\xi))\xi^v + (\eta^H\gamma\hat{R}(X,\xi))\xi^H, \\ iii)\ (L_{X^v}J)\xi^v &= -(\hat{\nabla}_\xi X)^v, \\ iv)\ (L_{X^v}J)\xi^H &= ((L_X\varphi)\xi)^v - ((\nabla_X\varphi)\xi)^v + ((L_X\eta)\xi)^v\xi^H - ((\nabla_X\eta)\xi)^v\xi^H. \end{aligned}$$

Let an $n-$dimensional diferentiable manifold M_n be endowed with a tensor field φ of type $(1,1)$, a vector field ξ, a $1-$form η, I the identity and let them satisfy

$$\varphi^2 = I - \eta\otimes\xi, \qquad \varphi(\xi)=0, \qquad \eta o\varphi = 0, \qquad \eta(\xi)=1. \tag{2.4}$$

Then (φ,ξ,η) define almost paracontact structure on M_n [7]. From (2.4), we get on taking horizontal and vertical lifts [4]

$$\begin{aligned} (\varphi^H)^2 &= I - \eta^v\otimes\xi^H - \eta^H\otimes\xi^v, \\ \varphi^H\xi^v &= 0,\ \varphi^H\xi^H = 0,\ \eta^vo\xi^H = 0, \\ \eta^Ho\varphi^H &= 0,\ \eta^v(\xi^v) = 0,\ \eta^v(\xi^H) = 1, \\ \eta^H(\xi^v) &= 1,\ \eta^H(\xi^H) = 0. \end{aligned} \tag{2.5}$$

We now define a $(1,1)$ tensor field $\widetilde{J}$ on $\Im(M_n)$ by

$$\widetilde{J} = \varphi^H - \xi^v \otimes \eta^v - \xi^H \otimes \eta^H. \tag{2.6}$$

Then it is easy to show that $\widetilde{J}^2 X^v = X^v$ and $\widetilde{J}^2 X^H = X^H$, which give that $\widetilde{J}$ is an almost product structure on $\Im(M_n)$. We get from (2.6)

$$\begin{aligned} \widetilde{J}X^v &= (\varphi X)^v - (\eta(X)\xi)^H, \\ \widetilde{J}X^H &= (\varphi X)^H - (\eta(X)\xi)^v \end{aligned}$$

for any $X \in \Im_0^1(M_n)$.

Theorem 2.2. *For L_X the operator Lie derivation with respect to X, $\widetilde{J} \in \Im_1^1(\Im(M_n))$ defined by (2.6) and $\eta(Y) = 0$, we have*

$$\begin{aligned} i)\ (L_{X^H}\widetilde{J})Y^v &= ((\hat{\nabla}_X\varphi)Y)^v - ((\hat{\nabla}_X\eta)Y)^v\xi^H, \\ ii)\ (L_{X^H}\widetilde{J})Y^H &= ((L_X\varphi)Y)^H - \gamma\hat{R}(X,\varphi Y) - ((L_X\eta)Y)^v\xi^v + \varphi^H\gamma\hat{R}(X,Y) \\ &\quad -(\eta^v\gamma\hat{R}(X,Y))\xi^v - (\eta^H\gamma\hat{R}(X,Y))\xi^H, \\ iii)\ (L_{X^v}\widetilde{J})Y^v &= 0, \\ iv)\ (L_{X^v}\widetilde{J})Y^H &= ((L_X\varphi)Y)^v - ((\nabla_X\varphi)Y)^v - ((L_X\eta)Y)^v\xi^H + ((\nabla_X\eta)Y)^v\xi^H, \end{aligned}$$

where $X, Y \in \Im_0^1(M_n)$, a tensor field $\varphi \in \Im_1^1(M_n)$, a vector field ξ and a $1-$form $\eta \in \Im_1^0(M_n)$.

Proof. For $\widetilde{J} = \varphi^H - \xi^v \otimes \eta^v - \xi^H \otimes \eta^H$ and $\eta(Y) = 0$, we get

$$\begin{aligned} i)\ (L_{X^H}\widetilde{J})Y^v &= L_{X^H}\widetilde{J}Y^v - \widetilde{J}L_{X^H}Y^v \\ &= [X^H, (\varphi Y)^v - (\eta(Y)\xi)^H] - (\varphi^H - \xi^v \otimes \eta^v - \xi^H \otimes \eta^H)[X^H, Y^v] \\ &= [X^H, (\varphi Y)^v] - [X^H, (\eta(Y)\xi)^H] - \varphi^H[X^H, Y^v] + \eta^v([X^H, Y^v])\xi^v \\ &\quad +\eta^H([X^H, Y^v])\xi^H \\ &= [X, \varphi Y]^v + (\nabla_{\varphi Y}X)^v - \varphi^H([X,Y]^v + (\nabla_Y X)^v) \\ &\quad +\eta^v([X,Y]^v + (\nabla_Y X)^v)\xi^v + \eta^H([X,Y]^v + (\nabla_Y X)^v)\xi^H \end{aligned}$$

$$
\begin{aligned}
&= ((L_X\varphi)Y)^v + (\varphi(L_XY))^v + (\hat{\nabla}_X\varphi Y)^v + [\varphi Y, X]^v - (\varphi L_X Y)^v \\
&\quad -(\varphi\nabla_Y X)^v + \eta^v([X,Y]^v)\xi^v + (\eta^v(\nabla_Y X)^v)\xi^v \\
&\quad +(\eta[X,Y])^v\xi^H + \eta^H(\nabla_Y X)^v\xi^H \\
&= ((L_X\varphi)Y)^v + (\varphi(L_XY))^v + ((\hat{\nabla}_X\varphi)Y)^v + (\varphi\hat{\nabla}_X Y)^v \\
&\quad -((L_X\varphi)Y)^v - (\varphi(L_XY))^v - (\varphi(L_XY))^v - (\varphi\nabla_Y X)^v \\
&\quad +(\eta[X,Y])^v\xi^H + (\eta^H(\nabla_Y X)^v)\xi^H \\
&= ((\hat{\nabla}_X\varphi)Y)^v + (\varphi\hat{\nabla}_X Y)^v - (\varphi(L_XY))^v - \varphi^H(\nabla_Y X)^v \\
&\quad -((L_X\eta)Y)^v\xi^H + (\eta^H((\hat{\nabla}_X Y)^v + [Y,X]^v))\xi^H \\
&= ((\hat{\nabla}_X\varphi)Y)^v + (\varphi\hat{\nabla}_X Y)^v - (\varphi(L_XY))^v - \varphi^H((\hat{\nabla}_X Y)^v + [Y,X]^v) \\
&\quad -((L_X\eta)Y)^v\xi^H + (\eta(\hat{\nabla}_X Y)^v)\xi^H + (\eta(L_Y X))^v\xi^H \\
&= ((\hat{\nabla}_X\varphi)Y)^v + (\varphi\hat{\nabla}_X Y)^v - (\varphi(L_XY))^v - (\varphi(\hat{\nabla}_X Y))^v + (\varphi(L_XY))^v \\
&\quad -((L_X\eta)Y)^v\xi^H - ((\hat{\nabla}_X\eta)Y)^v)\xi^H - (\eta(L_XY))^v\xi^H \\
&= ((\hat{\nabla}_X\varphi)Y)^v + (\varphi\hat{\nabla}_X Y)^v - (\varphi(\hat{\nabla}_X Y))^v - ((L_X\eta)Y)^v\xi^H \\
&\quad -(((\hat{\nabla}_X\eta)Y)^v)\xi^H + (((L_X\eta)Y))^v\xi^H \\
&= ((\hat{\nabla}_X\varphi)Y)^v - ((\hat{\nabla}_X\eta)Y)^v)\xi^H,
\end{aligned}
$$

$$
\begin{aligned}
ii)\ (L_{X^H}\widetilde{J})Y^H &= L_{X^H}\widetilde{J}Y^H - \widetilde{J}L_{X^H}Y^H \\
&= [X^H, (\varphi Y)^H - (\eta(Y)\xi)^v] - (\varphi^H - \xi^v\otimes\eta^v - \xi^H\otimes\eta^H)[X^H, Y^H] \\
&= [X^H, (\varphi Y)^H] - [X^H, (\eta(Y)\xi)^v] - \varphi^H[X^H, Y^H] + \eta^v([X^H, Y^H])\xi^v \\
&\quad +\eta^H([X^H, Y^H])\xi^H \\
&= [X, \varphi Y]^H - \gamma\hat{R}(X,\varphi Y) - \varphi^H([X,Y]^H - \gamma\hat{R}(X,Y)) \\
&\quad +\eta^v([X,Y]^H - \gamma\hat{R}(X,Y))\xi^v + \eta^H([X,Y]^H - \gamma\hat{R}(X,Y))\xi^H \\
&= ((L_X\varphi)Y)^H + (\varphi(L_XY))^H - \gamma\hat{R}(X,\varphi Y) - (\varphi(L_XY))^H \\
&\quad +\varphi^H\gamma\hat{R}(X,Y) + (\eta L_X Y)^v\xi^v - (\eta^v\gamma\hat{R}(X,Y))\xi^v \\
&\quad -\eta^H([X,Y]^H)\xi^H - (\eta^H\gamma\hat{R}(X,Y))\xi^H \\
&= ((L_X\varphi)Y)^H - \gamma\hat{R}(X,\varphi Y) - ((L_X\eta)Y)^v\xi^v + \varphi^H\gamma\hat{R}(X,Y) \\
&\quad -(\eta^v\gamma\hat{R}(X,Y))\xi^v - (\eta^H\gamma\hat{R}(X,Y))\xi^H \\
&= ((L_X\varphi)Y)^H - \gamma\hat{R}(X,\varphi Y) - ((L_X\eta)Y)^v\xi^v + \widetilde{J}(\gamma\hat{R}(X,Y)),
\end{aligned}
$$

$$
\begin{aligned}
iii)\ (L_{X^v}\widetilde{J})Y^v &= L_{X^v}\widetilde{J}Y^v - \widetilde{J}L_{X^v}Y^v \\
&= [X^v, (\varphi Y)^v - (\eta(Y)\xi)^H] - (\varphi^H - \xi^v\otimes\eta^v - \xi^H\otimes\eta^H)[X^v, Y^v] \\
&= [X^v, (\varphi Y)^v] - [X^v, (\eta(Y)\xi)^H] \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
iv)\ (L_{X^v}\widetilde{J})Y^H &= L_{X^v}\widetilde{J}Y^H - \widetilde{J}L_{X^v}Y^H \\
&= [X^v,(\varphi Y)^H - (\eta(Y)\xi)^v] - (\varphi^H - \xi^v\otimes\eta^v - \xi^H\otimes\eta^H)[X^v,Y^H] \\
&= [X^v,(\varphi Y)^H] - [X^v,(\eta(Y)\xi)^v] - \varphi^H[X^v,Y^H] + \eta^v([X^v,Y^H])\xi^v \\
&\quad +\eta^H([X^v,Y^H])\xi^H \\
&= [X,\varphi Y]^v - (\nabla_X\varphi Y)^v - \varphi^H([X,Y]^v - (\nabla_X Y)^v) + \eta^v([X,Y]^v \\
&\quad -(\nabla_X Y)^v)\xi^v + \eta^H([X,Y]^v - (\nabla_X Y)^v)\xi^H \\
&= ((L_X\varphi)Y)^v + (\varphi(L_XY))^v - ((\nabla_X\varphi)Y)^v - (\varphi\nabla_XY)^v \\
&\quad -(\varphi(L_XY))^v + (\varphi\nabla_XY)^v + \eta^v([X,Y]^v)\xi^v - \eta^v(\nabla_XY)^v\xi^v \\
&\quad +(\eta L_XY)^v\xi^H - (\eta\nabla_XY)^v\xi^H \\
&= ((L_X\varphi)Y)^v - ((\nabla_X\varphi)Y)^v - ((L_X\eta)Y)^v\xi^H + ((\nabla_X\eta)Y)^v\xi^H,
\end{aligned}
$$

where $\eta L_XY = L_X\eta(Y) - (L_X\eta)Y$ and $\eta\nabla_XY = \nabla_X\eta(Y) - (\nabla_X\eta)Y$. □

Corollary 2.2. *If we put $Y=\xi$, i.e. $\eta(\xi)=1$ and ξ has the conditions of (2.4), then we have*

$$
\begin{aligned}
i)\ (L_{X^H}\widetilde{J})\xi^v &= -(L_X\xi)^H + \gamma\hat{R}(X,\xi) + ((\hat{\nabla}_X\varphi)\xi)^v - ((\hat{\nabla}_X\eta)\xi)^v\xi^H, \\
ii)\ (L_{X^H}\widetilde{J})\xi^H &= -(\hat{\nabla}_X\xi)^v + ((L_X\varphi)\xi)^H - ((L_X\eta)\xi)^v\xi^v + \varphi^H\gamma\hat{R}(X,\xi) \\
&\quad -(\eta^v\gamma\hat{R}(X,\xi))\xi^v - (\eta^H\gamma\hat{R}(X,\xi))\xi^H, \\
iii)\ (L_{X^v}\widetilde{J})\xi^v &= (\hat{\nabla}_\xi X)^v, \\
iv)\ (L_{X^v}\widetilde{J})\xi^H &= ((L_X\varphi)\xi)^v - ((\nabla_X\varphi)\xi)^v - ((L_X\eta)\xi)^v\xi^H + ((\nabla_X\eta)\xi)^v\xi^H.
\end{aligned}
$$

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