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A NEW CONSTRUCTIVE METHOD FOR SOLVING OF BISINGULAR INTEGRAL EQUATIONS WITH CAUCHY KERNEL

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Abstract. In the present paper, the bisingular integral operator with Cauchy kernel S is approximated by a sequence of operators of the special form, it is proved that, the approximating operators S_n strongly converges to the operator S, for a trigonometric polynomial of degree not higher than n, the operators S_n and S coincide, and is given a new method for the approximate solution of bisingular integral equations of the first kind with Cauchy kernel.

1. Introduction

The constructive methods of solution of the bisingular integral equations the theory of which very well described in the works [3, 5, 6, 13, 17-19, 24, 26] have found wide application in aerodynamics, in theory of elasticity, electrodynamics and in other applied fields (see [3, 13, 18-21]), and many works (see [4, 7-12, 16-18, 22-25, 27]) were devoted to their construction. In the work [1] R.A. Aliev worked out a new constructive method for solution of the singular integral equations with Cauchy kernel. In this work, singular integral operator is approximated with operators preserving main properties of the singular integral operator, and that enables to obtain more exact results. In the work [2] this method was worked out and is justified for singular integral equations with Hilbert kernel. In the present paper the constructive method worked out in the work [1] is applied to the bisingular integral equations with Cauchy kernel.

Let $L_2 = L_2(\Gamma^2)$ be the space of quadratically-summable functions on the set Γ^2, where $\Gamma = \{t \in C \; : \; |t| = 1\}$. Let us consider bisingular integral operator with Cauchy kernel in L_2

$$(R\varphi)(t_1,\, t_2) = a_0(t_1,\, t_2)\varphi(t_1,\, t_2) + b_1(t_1,\, t_2)(S^{(1)}\varphi)(t_1,\, t_2)+$$
$$+b_2(t_1,\, t_2)(S^{(2)}\varphi)(t_1,\, t_2) + b_0(t_1,\, t_2)(S\varphi)(t_1,\, t_2),$$

where

$$(S^{(1)}\varphi)(t_1,\, t_2) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau_1, t_2)}{\tau_1 - t_1}d\tau_1, (S^{(2)}\varphi)(t_1,\, t_2) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(t_1, \tau_2)}{\tau_2 - t_2}d\tau_2,$$

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$$(S\varphi)(t_1,\, t_2) = \frac{1}{(\pi i)^2} \int_{\Gamma^2} \frac{\varphi(\tau_1, \tau_2)}{(\tau_1 - t_1)(\tau_2 - t_2)} d\tau_1 d\tau_2,$$

$a_0(t_1,\, t_2)$, $b_i(t_1,\, t_2)$, $i = 0,\, 1,\, 2$ are the known continuous functions , moreover

$$\Delta(t_1,\, t_2) = \prod_{\nu_1, \nu_2 = \pm 1} a_{\nu_1, \nu_2}(t_1,\, t_2) \neq 0 \text{ for any } (t_1, t_2) \in \Gamma^2, \tag{1.1}$$

$$ind_{t_1} \frac{a_{1,1}(t_1, t_2)}{a_{-1,1}(t_1, t_2)} = ind_{t_1} \frac{a_{1,-1}(t_1, t_2)}{a_{-1,-1}(t_1, t_2)} =$$

$$= ind_{t_2} \frac{a_{1,1}(t_1, t_2)}{a_{1,-1}(t_1, t_2)} = ind_{t_2} \frac{a_{-1,1}(t_1, t_2)}{a_{-1,-1}(t_1, t_2)} = 0, \tag{1.2}$$

$$ind\, R = ind_{t_1} \frac{a_{1,1}(t_1, t_2)}{a_{-1,-1}(t_1, t_2)} \cdot ind_{t_2} \frac{a_{1,1}(t_1, t_2)}{a_{-1,-1}(t_1, t_2)} = 0, \tag{1.3}$$

$a_{\nu_1, \nu_2}(t_1,\, t_2) = a_0(t_1,\, t_2) + \nu_1 b_1(t_1,\, t_2) + \nu_2 b_2(t_1,\, t_2) + \nu_1 \nu_2 b_0(t_1,\, t_2), \nu_1, \nu_2 = \pm 1.$

Let us note that the conditions (1.1), (1.2) are necessary and sufficient for Noetherity, and conditions (1.1), (1.3) for Fredholm property of the operator R (see [5]).

In the present paper the operator R is approximated by a sequence of operators of the form

$$(R_n \varphi)(t_1, t_2) = \sum_{k_1, k_2 = 0}^{2n-1} \alpha_{k_1, k_2}^{(n)}(t_1, t_2)\, \varphi(\tau_{k_1}^{(t_1)}, \tau_{k_2}^{(t_2)}),$$

where $\tau_k^{(t)} = e^{k\theta i} \cdot t$, $k = \overline{0\,,\, 2n}$, $\theta = \frac{\pi}{n}$, $n \in N$, and the functions $\alpha_{k_1, k_2}^{(n)}(t_1, t_2) \in C(\Gamma^2)$, $k_1,\, k_2 = \overline{0\,,\, 2n-1}$, $n \in N$ are expressed in terms of the given functions, and it is proved that, under the conditions indicated above, the sequence of operators $\{R_n\}$ strongly converges to the operator R in L_2, the operators R_n are invertible for sufficiently large n, and the sequence of operators $\left\{R_n^{-1}\right\}$ strongly converges to the operator R^{-1} as $n \to \infty$. It should be noted that, in this method, the determination of the inverse operator is equivalent to the study of the equation

$$\sum_{k_1, k_2 = 0}^{2n-1} \alpha_{k_1, k_2}^{(n)}(t_1, t_2)\, \varphi\left(\tau_{k_1}^{(t_1)}, \tau_{k_2}^{(t_2)}\right) = f(t_1, t_2), (t_1, t_2) \in \Gamma^2,$$

at the points $\left(\tau_{m_1}^{(t_1)}, \tau_{m_2}^{(t_2)}\right)$, $m_1, m_2 = \overline{0\,,\, 2n-1}$ because solving the resulting system of linear algebraic equations

$$\sum_{k_1, k_2 = 0}^{2n-1} \alpha_{k_1, k_2}^{(n)}\left(\tau_{m_1}^{(t_1)}, \tau_{m_2}^{(t_2)}\right) \varphi\left(\tau_{k_1+m_1}^{(t_1)}, \tau_{k_2+m_2}^{(t_2)}\right) =$$

$$= f(\tau_{m_1}^{(t_1)}, \tau_{m_2}^{(t_2)}),\ m_1, m_2 = \overline{0\,,\, 2n-1}$$

with respect to $\left(\varphi\left(\tau_0^{(t_1)}, \tau_0^{(t_2)}\right), \varphi\left(\tau_0^{(t_1)}, \tau_1^{(t_2)}\right), \ldots, \varphi\left(\tau_{2n-1}^{(t_1)}, \tau_{2n-1}^{(t_2)}\right)\right)$, we obtain the function $\varphi(t_1, t_2) = \varphi\left(\tau_0^{(t_1)}, \tau_0^{(t_2)}\right)$.

2. Approximation of a bisingular integrals with Cauchy kernel

Consider the following bisingular integral operators acting in L_2:

$$(S^{(1)}\varphi)(t_1,\, t_2) = \frac{1}{\pi i}\int_{\Gamma}\frac{\varphi(\tau_1, t_2)}{\tau_1 - t_1}d\tau_1, \quad (S^{(2)}\varphi)(t_1,\, t_2) = \frac{1}{\pi i}\int_{\Gamma}\frac{\varphi(t_1, \tau_2)}{\tau_2 - t_2}d\tau_2,$$

$$(S\varphi)\,(t_1,\, t_2) = \frac{1}{(\pi i)^2}\int_{\Gamma^2}\frac{\varphi\,(\tau_1, \tau_2)}{(\tau_1 - t_1)\,(\tau_2 - t_2)}d\tau_1 d\tau_2,\ (t_1, t_2) \in \Gamma^2.$$

It is well known (see [28]) that the operators $S^{(1)}$, $S^{(2)}$ and S acts in L_2, $\left\|S^{(1)}\right\|_{L_2\to L_2} = \left\|S^{(2)}\right\|_{L_2\to L_2} = \|S\|_{L_2\to L_2} = 1$ and $\left(S^{(1)}\right)^2 = \left(S^{(2)}\right)^2 = S^2 = I$ in L_2.

Consider the sequence of operators

$$(S_n^{(1)}\varphi)(t_1,\, t_2) = \frac{1}{\pi i}\sum_{k=0}^{n-1}\frac{\varphi(\tau_{2k+1}^{(t_1)}, t_2)}{\tau_{2k+1}^{(t_1)} - t_1}\cdot\Delta\tau_{2k+1}^{(t_1)},$$

$$(S_n^{(2)}\varphi)(t_1,\, t_2) = \frac{1}{\pi i}\sum_{k=0}^{n-1}\frac{\varphi(t_1,\, \tau_{2k+1}^{(t_2)})}{\tau_{2k+1}^{(t_2)} - t_2}\cdot\Delta\tau_{2k+1}^{(t_2)},$$

$$(S_n\varphi)(t_1,\, t_2) = \frac{1}{(\pi i)^2}\sum_{k_1,k_2=0}^{n-1}\frac{\varphi(\tau_{2k_1+1}^{(t_1)}, \tau_{2k_2+1}^{(t_2)})}{(\tau_{2k_1+1}^{(t_1)} - t_1)(\tau_{2k_2+1}^{(t_2)} - t_2)}\cdot\Delta\tau_{2k_1+1}^{(t_1)}\Delta\tau_{2k_2+1}^{(t_2)},$$

where $\tau_k^{(t)} = e^{k\theta i}\cdot t$, $\Delta\tau_k^{(t)} = \left(\tau_{k+1}^{(t)} - \tau_{k-1}^{(t)}\right)\cdot\frac{\theta}{\sin\theta} = 2ie^{k\theta i}\cdot t\cdot\theta$, $k = \overline{0\,,\,2n}$, $\theta = \frac{\pi}{n}$, $n \in N$.

Let us calculate $S_n^{(1)}\left(t_1^m\cdot t_2^p\right)$, $S_n^{(2)}\left(t_1^m\cdot t_2^p\right)$, $S_n\left(t_1^m\cdot t_2^p\right)$ for any $m, p \in Z$ (Z is the set of integer real numbers):

$$S_n^{(1)}\left(t_1^m\cdot t_2^p\right) = \frac{1}{\pi i}\sum_{k=0}^{n-1}\frac{\left(\tau_{2k+1}^{(t_1)}\right)^m t_2^p}{\tau_{2k+1}^{(t_1)} - t_1}\cdot\Delta\tau_{2k+1}^{(t_1)} =$$

$$= \frac{1}{\pi i}\sum_{k=0}^{n-1}\frac{e^{m(2k+1)\theta i}\cdot t_1^m}{\tau_{2k+1}^{(t_1)} - t_1}\cdot\Delta\tau_{2k+1}^{(t_1)}\cdot t_2^p = \lambda_m^{(n)}\cdot t_1^m\cdot t_2^p, \tag{2.1}$$

$$S_n^{(2)}\left(t_1^m\cdot t_2^p\right) = \frac{1}{\pi i}\sum_{k=0}^{n-1}\frac{t_1^m\left(\tau_{2k+1}^{(t_2)}\right)^p}{\tau_{2k+1}^{(t_2)} - t_2}\cdot\Delta\tau_{2k+1}^{(t_2)} =$$

$$= \frac{1}{\pi i}\sum_{k=0}^{n-1}\frac{e^{p(2k+1)\theta i}\cdot t_2^p}{\tau_{2k+1}^{(t_2)} - t_2}\cdot\Delta\tau_{2k+1}^{(t_2)}\cdot t_1^m = \lambda_p^{(n)}\cdot t_1^m\cdot t_2^p, \tag{2.2}$$

$$S_n\left(t_1^m\cdot t_2^p\right) = \frac{1}{(\pi i)^2}\sum_{k_1,k_2=0}^{n-1}\frac{\left(\tau_{2k_1+1}^{(t_1)}\right)^m\left(\tau_{2k_2+1}^{(t_2)}\right)^p}{\left(\tau_{2k_1+1}^{(t_1)} - t_1\right)\left(\tau_{2k_2+1}^{(t_2)} - t_2\right)}\cdot\Delta\tau_{2k_1+1}^{(t_1)}\Delta\tau_{2k_2+1}^{(t_2)} =$$

$$= \frac{1}{\pi i}\sum_{k_1=0}^{n-1}\frac{e^{m(2k+1)\theta i}\cdot t_1^m}{\left(\tau_{2k_1+1}^{(t_1)} - t_1\right)}\cdot\Delta\tau_{2k_1+1}^{(t_1)}\cdot\frac{1}{\pi i}\sum_{k_2=0}^{n-1}\frac{e^{p(2k+1)\theta i}\cdot t_2^p}{\left(\tau_{2k_2+1}^{(t_2)} - t_2\right)}\cdot\Delta\tau_{2k_2+1}^{(t_2)} = \lambda_m^{(n)}\cdot t_1^m\cdot\lambda_p^{(n)}\cdot t_2^p, \tag{2.3}$$

where

$$\lambda_m^{(n)} = \frac{1}{\pi i}\sum_{k=0}^{n-1}\frac{e^{m(2k+1)\theta i}}{\left(\tau_{2k+1}^{(t)} - t\right)}\cdot\Delta\tau_{2k+1}^{(t)} = \frac{1}{\pi i}\sum_{k=0}^{n-1}\frac{e^{m(2k+1)\theta i}}{\left(\tau_{2k+1}^{(1)} - 1\right)}\cdot\Delta\tau_{2k+1}^{(1)}.$$

Calculating $\lambda_m^{(n)}$, we find that $\lambda_m^{(n)} = 1$ for $m = \overline{0,\, n-1}$, $\lambda_m^{(n)} = -1$ for $m = \overline{n,\, 2n-1}$, and $\lambda_{m\pm 2n}^{(n)} = \lambda_m^{(n)}$ for all $m \in Z$.

Suppose that

$$\varphi(t_1, t_2) = \sum_{m,p=-\infty}^{+\infty} c_{k,p} t_1^m t_2^p.$$

Then, taking (2.1)-(2.3) into account, we obtain

$$\left(S_n^{(1)}\varphi\right)(t_1, t_2) = \sum_{m,p=-\infty}^{+\infty} c_{k,p}\lambda_m^{(n)} t_1^m t_2^p,$$

$$\left(S_n^{(2)}\varphi\right)(t_1, t_2) = \sum_{m,p=-\infty}^{+\infty} c_{k,p}\lambda_p^{(n)} t_1^m t_2^p,$$

$$\left(S_n\varphi\right)(t_1, t_2) = \sum_{m,p=-\infty}^{+\infty} c_{k,p}\lambda_m^{(n)}\lambda_p^{(n)} t_1^m t_2^p.$$

This implies the following properties of the operators $S_n^{(1)}$, $S_n^{(2)}$ and S_n.

Properties 2.1. The following relations hold:

$$\left(S_n^{(1)}\right)^2 = \left(S_n^{(2)}\right)^2 = (S_n)^2 = I \text{ in } L_2,$$

$$\left\|S_n^{(1)}\right\|_{L_2\to L_2} = \left\|S_n^{(2)}\right\|_{L_2\to L_2} = \|S_n\|_{L_2\to L_2} = 1,$$

and for any algebraic polynomials $P_{n-1}(t_1, t_2) = \sum_{k_1,k_2=-n+1}^{n-1}\alpha_{k_1,k_2} t_1^{k_1} t_2^{k_2}$ of degree not higher than $n-1$ the equalities

$$(S_n^{(1)}P)(t_1,\, t_2) = (S^{(1)}P)(t_1,\, t_2),\;\; (S_n^{(2)}P)(t_1,\, t_2) = (S^{(2)}P)(t_1,\, t_2),$$
$$(S_nP)(t_1,\, t_2) = (SP)(t_1,\, t_2)$$

holds.

Suppose that

$$E_n^{(2)}(\varphi) = \inf_{P_n\in T_n}\|\varphi - P_n\|_{L_2}$$

is the best approximation of the function $\varphi \in L_2$ by polynomials from T_n, where the T_n is the set of polynomials of the form $\sum_{k_1,k_2=-n}^{n}\alpha_{k_1,k_2} t_1^{k_1} t_2^{k_2}$, $\alpha_{k_1,k_2} \in C$.

Theorem 2.1. *The sequences of operators* $\left\{S_n^{(1)}\right\}$, $\left\{S_n^{(2)}\right\}$, $\{S_n\}$ *strongly converges to the operators* $S^{(1)}$, $S^{(2)}$ *and* S *respectively, and, for any* $\varphi \in L_2$, *the following estimates holds:*

$$\left\|S^{(1)}\varphi - S_n^{(1)}\varphi\right\|_{L_2} \le 2E_{n-1}^{(2)}(\varphi),\;\; \left\|S^{(2)}\varphi - S_n^{(2)}\varphi\right\|_{L_2} \le 2E_{n-1}^{(2)}(\varphi),$$

$$\|S\varphi - S_n\varphi\|_{L_2} \le 2E_{n-1}^{(2)}(\varphi). \tag{2.4}$$

Proof. Suppose that $P_{n-1}(t_1,t_2)$ is the best approximation polynomial for the function $\varphi\in L_2$ from T_{n-1}. In properties 2.1 we have

$$\left(S^{(1)}\varphi-S_n^{(1)}\varphi\right)(t_1,t_2)=S^{(1)}(\varphi-P_{n-1})(t_1,t_2)-S_n^{(1)}(\varphi-P_{n-1})(t_1,t_2).$$

Then

$$\left\|S^{(1)}\varphi-S_n^{(1)}\varphi\right\|_{L_2}\le\left(\left\|S^{(1)}\right\|_{L_2\to L_2}+\left\|S_n^{(1)}\right\|_{L_2\to L_2}\right)\cdot\|\varphi-P_{n-1}\|_{L_2}=2E_{n-1}^{(2)}(\varphi).$$

Thus, we have shown that the sequence of operators $\left\{S_n^{(1)}\right\}$ strongly converges to the operator $S^{(1)}$ in L_2. The remaining inequalities are proved similarly. This completes the proof of the theorem 2.1. □

Consider the regular integral

$$(\mathrm{K}\varphi)(t_1,\,t_2)=\int_{\Gamma^2}K(t_1,t_2,\tau_1,\tau_2)\,\varphi(\tau_1,\tau_2)\,d\tau_1d\tau_2,\,(t_1,t_2)\in\Gamma^2,$$

where $K(t_1,t_2,\tau_1,\tau_2)$ is a continuous function, and the sequence of operators

$$(\mathrm{K}_n\varphi)(t_1,\,t_2)=\sum_{k_1,k_2=0}^{2n-1}K\left(t_1,t_2,\tau_{k_1}^{(t_1)},\tau_{k_2}^{(t_2)}\right)\times$$

$$\times\varphi\left(\tau_{k_1}^{(t_1)},\tau_{k_2}^{(t_2)}\right)\left(\frac{1}{2}\Delta\tau_{k_1}^{(t_1)}\right)\left(\frac{1}{2}\Delta\tau_{k_2}^{(t_2)}\right),\ (t_1,t_2)\in\Gamma^2,\ n\in N.$$

Suppose that

$$\|K\|_\infty=\max\left\{|K(t_1,t_2,\tau_1,\tau_2)|\ :\ t_1,t_2,\tau_1,\tau_2\in\Gamma\right\},$$

$$E_n(K)=\inf\left\{\left\|K(t_1,t_2,\tau_1,\tau_2)-\sum_{p_1,p_2=-n}^{n}h_{p_1,p_2}(t_1,t_2)\,\tau_1^{p_1}\tau_2^{p_2}\right\|_\infty :\right.$$

$$\left.h_{p_1,p_2}\in T_n,\ p_1,p_2=\overline{-n,\,n}\ \right\}.$$

Theorem 2.2. *The sequence of the operators* $\{\mathrm{K}_n\}$ *strongly converges to the operator* K *in* L_2 *and, for any* $\varphi\in L_2$*, the following estimate holds:*

$$\|\mathrm{K}\varphi-\mathrm{K}_n\varphi\|_{L_2}\le 8\pi^2\,\|K\|_\infty\cdot E_{n-1}^{(2)}(\varphi)+8\pi^2\cdot E_{n-1}(K)\left\{E_{n-1}^{(2)}(\varphi)+\|\varphi\|_{L_2}\right\}. \tag{2.5}$$

Proof. Let us calculate $\sum_{k=0}^{2n-1}\left(\tau_k^{(t)}\right)^m\left(\frac{1}{2}\Delta\tau_k^{(t)}\right)$ for any $m\in Z$:

$$\sum_{k=0}^{2n-1}\left(\tau_k^{(t)}\right)^m\left(\frac{1}{2}\Delta\tau_k^{(t)}\right)=i\theta\cdot t^{m+1}\sum_{k=0}^{2n-1}e^{(m+1)k\theta i}=$$

$$=\begin{cases}2\pi i\cdot t^{m+1}, & m=-1\ (mod\,2n),\\ 0, & m\ne-1\ (mod\,2n).\end{cases} \tag{2.6}$$

Since

$$\int_\Gamma\tau^m d\tau=\begin{cases}2\pi i, & m=-1,\\ 0, & m\ne-1,\end{cases}$$

it follows from (2.6) that, for any polynomial $q_{2n-2}(t) = \sum_{k=-2n+2}^{2n-2} \alpha_k t^k$ the following relation holds:

$$\int_{\Gamma} q_{2n-2}(\tau) d\tau = \sum_{k=0}^{2n-1} q_{2n-2}(\tau_k^{(t)}) \left(\frac{1}{2} \Delta \tau_k^{(t)} \right). \tag{2.7}$$

Suppose that

$$E_{n-1}^{(2)}(\varphi) = \| \varphi - P_{n-1} \|_{L_2},$$

$$E_n(K) = \left\| K(t_1, t_2, \tau_1, \tau_2) - \sum_{p_1, p_2 = -n}^{n} h_{p_1, p_2}(t_1, t_2) \tau_1^{p_1} \tau_2^{p_2} \right\|_{\infty}.$$

Then from relation (2.7), we obtain

$$(\mathrm{K}\varphi)(t_1, t_2) - (\mathrm{K}_n \varphi)(t_1, t_2) = (\mathrm{K} - \mathrm{K}_n)(\varphi - P_{n-1})(t_1, t_2) +$$

$$+ \int_{\Gamma^2} \left[K(t_1, t_2, \tau_1, \tau_2) - \sum_{p_1, p_2 = -n}^{n} h_{p_1, p_2}(t_1, t_2) \tau_1^{p_1} \tau_2^{p_2} \right] \cdot P_{n-1}(\tau_1, \tau_2) d\tau_1 d\tau_2 +$$

$$+ \sum_{k_1, k_2 = 0}^{2n-1} \left[K(t_1, t_2, \tau_{k_1}^{(t_1)}, \tau_{k_2}^{(t_2)}) - \sum_{p_1, p_2 = -n}^{n} h_{p_1, p_2}(t_1, t_2) \left(\tau_{k_1}^{(t_1)} \right)^{p_1} \left(\tau_{k_2}^{(t_2)} \right)^{p_2} \right] \times$$

$$\times P_{n-1} \left(\tau_{k_1}^{(t_1)}, \tau_{k_2}^{(t_2)} \right) \left(\frac{1}{2} \Delta \tau_{k_1}^{(t_1)} \right) \left(\frac{1}{2} \Delta \tau_{k_2}^{(t_2)} \right).$$

Hence

$$\| \mathrm{K}\varphi - \mathrm{K}_n \varphi \|_{L_2} \le \left\{ \| \mathrm{K} \|_{L_2 \to L_2} + \| \mathrm{K}_n \|_{L_2 \to L_2} \right\} \| \varphi - P_{n-1} \|_{L_2} +$$

$$+ \left\| \int_{\Gamma^2} \left[K(t_1, t_2, \tau_1, \tau_2) - \sum_{p_1, p_2 = -n}^{n} h_{p_1, p_2}(t_1, t_2) \tau_1^{p_1} \tau_2^{p_2} \right] \cdot P_{n-1}(\tau_1, \tau_2) d\tau_1 d\tau_2 \right\|_{L_2} +$$

$$+ \left\| \sum_{k_1, k_2 = 0}^{2n-1} \left[K(t_1, t_2, \tau_{k_1}^{(t_1)}, \tau_{k_2}^{(t_2)}) - \sum_{p_1, p_2 = -n}^{n} h_{p_1, p_2}(t_1, t_2) \left(\tau_{k_1}^{(t_1)} \right)^{p_1} \left(\tau_{k_2}^{(t_2)} \right)^{p_2} \right] \times \right.$$

$$\left. \times P_{n-1} \left(\tau_{k_1}^{(t_1)}, \tau_{k_2}^{(t_2)} \right) \left(\frac{1}{2} \Delta \tau_{k_1}^{(t_1)} \right) \left(\frac{1}{2} \Delta \tau_{k_2}^{(t_2)} \right) \right\|_{L_2}.$$

Taking into account the inequalities $\| \mathrm{K} \|_{L_2 \to L_2} \le 4\pi^2 \| K \|_{\infty}$, $\| \mathrm{K}_n \|_{L_2 \to L_2} \le 4\pi^2 \| K \|_{\infty}$, it follows estimation (2.5). This completes the proof of the theorem 2.2. □

3. Construction and justification of a new constructive method of solving bisingular integral equations

Suppose that A is a linearly continuous operator in a Banach space X and $\{A_n\}_{n=1}^{\infty}$ is the sequence of linear continuous operators in X.

Definition 3.1. We say that the approximation method involving the system of operators $\{A_n\}_{n=1}^{\infty}$ can be applied to the invertible operator A if there exists an $n_0 \in N$ such that the operators A_n are invertible for $n \geq n_0$ and, for any $y \in X$, the solutions $x_n \in X$ of the equation $A_n x_n = y\,, \ \ n \geq n_0$, converge in the norm of the space X to the solution of the equation $Ax = y$.

Let us present the following assertions from the theory of projective methods [14].

Proposition 3.1. Suppose that the sequence of operators $\{A_n\}_{n=1}^{\infty}$ strongly converges to an invertible operator A. The approximation method involving the system of operators $\{A_n\}_{n=1}^{\infty}$ can be applied to the operator A if and only if there exists an $n_0 \in N$ such that the sequence $\{A_n\}_{n \geq n_0}$ is uniformly invertible, moreover, if x^* and x_n^* are the solutions of the equations $Ax = y$ and $A_n x = y$, respectively, then

$$\|x^* - x_n^*\|_X \leq const \ \ \|(A_n - A)x\|_X \,. \tag{3.1}$$

Proposition 3.2. Let the approximation method involving the system of operators $\{A_n\}_{n=1}^{\infty}$ be applicable to the invertible operator A. Then, for any system $\{B_n\}_{n=1}^{\infty}$ of linear continuous operators in the space X satisfying the condition $\lim\limits_{n\to\infty} \|B_n\| = 0$, the approximation method involving the system of operators $\{A_n + B_n\}_{n=1}^{\infty}$ can be applied to the operator A.

Consider the following sequences of linear operators acting in L_2:

$$\left(\mathrm{K}_n^{(1)}\varphi\right)(t_1,t_2) = \sum_{k_1,k_2=0}^{2n-1} K(t_1,t_2,\tau_{k_1}^{(1)},\tau_{k_2}^{(1)})\tau_{k_1}^{(1)}\tau_{k_2}^{(1)} \int\limits_{\tau_{k_1}^{(1)}\tau_{k_1+1}^{(1)}} \int\limits_{\tau_{k_2}^{(1)}\tau_{k_2+1}^{(1)}} \frac{\varphi(\tau_1,\tau_2)}{\tau_1\cdot\tau_2}d\tau_1 d\tau_2,$$

$$\left(\mathrm{K}_n^{(2)}\varphi\right)(t_1,t_2) =$$

$$= \sum_{k_1,k_2=0}^{2n-1} \int_0^{\theta}\int_0^{\theta} K\left(e^{i\sigma_1}\tau_{m_1}^{(1)},\, e^{i\sigma_2}\tau_{m_2}^{(1)}, \tau_{k_1+m_1}^{(1)}, \tau_{k_2+m_2}^{(1)}\right) d\sigma_1 d\sigma_2\, \varphi(\tau_{k_1}^{(t_1)}, \tau_{k_2}^{(t_2)}),$$

where $\tau_k^{(1)} = e^{k\theta i}, \ \theta = \frac{\pi}{n}$, and m_1, m_2 are numbers such that $t_i \in \tau_{m_i}^{(1)}\tau_{m_i+1}^{(1)}\,, \ i = \overline{1\,,2}$.

The uniform convergence of the operators $\left\{\mathrm{K}_n^{(1)}\right\}$ to the operator K follows from the inequality

$$\left\|\mathrm{K} - \mathrm{K}_n^{(1)}\right\|_{L_2\to L_2} \leq 4\pi^2 \max_{\substack{t_1\in\Gamma \\ t_2\in\Gamma}} \max_{\substack{|\tau_1-\tau_1'|\leq\theta \\ |\tau_2-\tau_2'|\leq\theta}} \left|K(t_1,t_2,\tau_1,\tau_2)\tau_1\tau_2 - K(t_1,t_2,\tau_1',\tau_2')\tau_1'\tau_2'\right|,$$

and the strong convergence of the operators $\left\{\mathrm{K}_n^{(2)}\right\}$ to the operator K follows from the theorem 2.2 and from the inequality

$$\left\|\mathrm{K}_n - \mathrm{K}_n^{(2)}\right\|_{L_2\to L_2} \le 4\pi^2 \max_{\substack{|t_1-t_1'|\le\theta \\ |t_2-t_2'|\le\theta}} \max_{\substack{|\tau_1-\tau_1'|\le\theta \\ |\tau_2-\tau_2'|\le\theta}} \left|K(t_1,t_2,\tau_1,\tau_2)\tau_1\tau_2 - K(t_1',t_2',\tau_1',\tau_2')\tau_1'\tau_2'\right|. \tag{3.2}$$

Lemma 3.1. *If the inverse operator $(I+\mathrm{K})^{-1}$ exists, then, the for large values of n, the operators $(I+\mathrm{K}_n)$ are also invertible and the sequence of the operators $\left\{(I+\mathrm{K}_n)^{-1}\right\}$ strongly converges to the operator $(I+\mathrm{K})^{-1}$ in L_2.*

Proof. Suppose that the inverse operator $(I+\mathrm{K})^{-1}$ exist. Since the sequence of operators $\left\{\mathrm{K}_n^{(1)}\right\}$ uniformly convergences to the operator K, it follows that, for large values of n $(\ge n_0)$, the operators $I+\mathrm{K}_n^{(1)}$ are uniformly invertible.

For any $f\in L_2$ consider the equation

$$\left(I+\mathrm{K}_n^{(2)}\right)\varphi_n(t_1,t_2) = f(t_1,t_2), (t_1,t_2)\in\Gamma^2. \tag{3.3}$$

Considering equation (3.3) at the points $\left(\tau_{-m_1+k_1}^{(t_1)}, \tau_{-m_2+k_2}^{(t_2)}\right)\in\Gamma^2$, $k_1, k_2 = \overline{0, 2n-1}$, where m_1 and m_2 are the indexes satisfying $t_i\in \tau_{m_i}^{(1)}\tau_{m_i+1}^{(1)}$, $i=\overline{1,2}$, we can write this equation in the following equivalent form:

$$\Phi_n G_n = F_n, \tag{3.4}$$

where $\Phi_n = (\varphi_n(\tau_{-m_1}^{(t_1)}, \tau_{-m_2}^{(t_2)}), \ldots, \varphi_n(\tau_{-m_1}^{(t_1)}, \tau_{-m_2+2n-1}^{(t_2)}), \varphi_n(\tau_{-m_1+1}^{(t_1)}, \tau_{-m_2}^{(t_2)}), \ldots,$ $\varphi_n(\tau_{-m_1+2n-1}^{(t_1)}, \tau_{-m_2+2n-1}^{(t_2)}))$,

$$G_n = \begin{pmatrix} 1+g_{0,0} & g_{0,1} & \cdots & g_{0,4n^2-1} \\ g_{1,0} & 1+g_{1,1} & \cdots & g_{1,4n^2-1} \\ \cdots & \cdots & \cdots & \cdots \\ g_{4n^2-1,0} & g_{4n^2-1,1} & \cdots & g_{4n^2-1,4n^2-1} \end{pmatrix},$$

$$g_{(2n)i_1+i_2,(2n)j_1+j_2} =$$

$$= -\int_0^\theta\int_0^\theta K(e^{i\sigma_1}\tau_{j_1}^{(1)}, e^{i\sigma_2}\tau_{j_2}^{(1)}, \tau_{i_1}^{(1)}, \tau_{i_2}^{(1)})\,\tau_{i_1}^{(1)}\tau_{i_2}^{(1)} d\sigma_1 d\sigma_2, \quad i_1,i_2,j_1,j_2 = \overline{0, 2n-1},$$

$$F_n = \left(f\left(\tau_{-m_1}^{(t_1)}, \tau_{-m_2}^{(t_2)}\right), \ldots, f\left(\tau_{-m_1}^{(t_1)}, \tau_{-m_2+2n-1}^{(t_2)}\right),\right.$$
$$\left. f\left(\tau_{-m_1+1}^{(t_1)}, \tau_{-m_2}^{(t_2)}\right), \ldots, f\left(\tau_{-m_1+2n-1}^{(t_1)}, \tau_{-m_2+2n-1}^{(t_2)}\right)\right).$$

Let us prove that $def\, G_n \ne 0$ for any $n\ge n_0$. To do this, we must show that the equation

$$(x_0, x_1, \ldots, x_{4n^2-1})\cdot G_n = (d_0, d_1, \ldots, d_{4n^2-1}) \tag{3.5}$$

is solvable for any right-hand side. For any vector $(d_0, d_1, \ldots, d_{4n^2-1})$, we take the function

$$f^{(0)}(t_1, t_2) = \frac{1}{\theta^2} d_{2ni_1+i_2} \text{ for } t_1\in \tau_{i_1}^{(1)}\tau_{i_1+1}^{(1)},\ t_2\in\tau_{i_2}^{(1)}\tau_{i_2+1}^{(1)},\ \ i_1, i_2 = \overline{0, 2n-1},$$

Since $f^{(0)} \in L_2$, it follows that the equation

$$\left(I+\mathrm{K}_n^{(1)}\right)\varphi_n^{(0)}(t_1,t_2)=f^{(0)}(t_1,t_2)\,,(t_1,t_2)\in\Gamma^2 \tag{3.6}$$

is uniquely solvable with respect to $\varphi_n^{(0)}$ in L_2 for $n \geq n_0$. Writing $t_1 = e^{is_1}$, $t_2 = e^{is_2}$ and integrating (3.6) on the quadrate $[p_1\theta,\ (p_1+1)\,\theta] \times [p_2\theta,\ (p_2+1)\,\theta]$, $p_1,\, p_2 = \overline{0,\, 2n-1}$ we have

$$\left(\int_0^\theta\int_0^\theta \varphi_n^{(0)}(e^{i\sigma_1}\tau_0^{(1)},\, e^{i\sigma_2}\tau_0^{(1)})\,d\sigma_1 d\sigma_2,\ldots,\right.$$
$$\left.\int_0^\theta\int_0^\theta \varphi_n^{(0)}(e^{i\sigma_1}\tau_{2n-1}^{(1)},\, e^{i\sigma_2}\tau_{2n-1}^{(1)})\,d\sigma_1 d\sigma_2\right)\cdot G_n=$$
$$=\left(\int_0^\theta\int_0^\theta f^{(0)}(e^{i\sigma_1},\, e^{i\sigma_2})\,d\sigma_1 d\sigma_2,\ldots,\right.$$
$$\left.\int_{(2n-1)\theta}^{2n\theta}\int_{(2n-1)\theta}^{2n\theta} f^{(0)}(e^{i\sigma_1},\, e^{i\sigma_2})\,d\sigma_1 d\sigma_2\right)=(d_0,\ldots,d_{4n^2-1})\,, \tag{3.7}$$

that is the equation (3.5) is solvable for any right-hand side; therefore, $def\, G_n \neq 0$. Then equation (3.4) and, therefore, equation (3.3) is solvable for any $f \in L_2$ for almost all $(t_1, t_2) \in \Gamma^2$. Next we prove the uniform invertibility of the operators $\left(I+\mathrm{K}_n^{(2)}\right)$. Suppose that $\varphi_n\,(t_1,\, t_2)$ is a solution of equation (3.3). Then, from (3.4), we obtain

$$\Phi_n=F_n\cdot G_n^{-1}=\left(f(\tau_{-m_1}^{(t_1)},\,\tau_{-m_2}^{(t_2)}),\ldots,f(\tau_{-m_1+2n-1}^{(t_1)},\,\tau_{-m_2+2n-1}^{(t_2)})\right)\cdot G_n^{-1}.$$

Let $G_n^{-1}=\left((G_n^{-1})^{(0)},(G_n^{-1})^{(1)},\,\ldots,(G_n^{-1})^{(4n^2-1)}\right)$, where $\left(G_n^{-1}\right)^{(k)}$ is the $(k+1)$-th column of the matrix G_n^{-1}, $k=\overline{0,\,4n^2-1}$. Then for almost all $(t_1,\, t_2)\in\Gamma^2$, we can write

$$\varphi_n(t_1,\,t_2)=\left(f(\tau_{-k_1}^{(t_1)},\,\tau_{-k_2}^{(t_2)}),\ldots,f(\tau_{-k_1+2n-1}^{(t_1)},\,\tau_{-k_2+2n-1}^{(t_2)}\right)\left(G_n^{-1}\right)^{(k_2+2nk_1)}$$

for $(t_1,\, t_2)\in\left(\tau_{k_1}^{(1)},\,\tau_{k_1+1}^{(1)}\right)\times\left(\tau_{k_2}^{(1)},\,\tau_{k_2+1}^{(1)}\right)$, $k_1,\, k_2=\overline{0,\, 2n-1}$. Then

$$\|\varphi_n\|_{L_2}^2=\frac{1}{4\pi^2}\sum_{k_1,k_2=0}^{2n-1}\int_{\tau_{k_1}^{(1)}\tau_{k_1+1}^{(1)}}\int_{\tau_{k_2}^{(1)}\tau_{k_2+1}^{(1)}}\left|\left(G_n^{-1}\right)^{(k_2+2nk_1)}\right|^2\times$$
$$\times\left|\left(f(\tau_{-k_1}^{(t_1)},\,\tau_{-k_2}^{(t_2)}),\ldots,f(\tau_{-k_1+2n-1}^{(t_1)},\,\tau_{-k_2+2n-1}^{(t_2)})\right)\right|^2|dt_1|\,|dt_2|=$$
$$=\frac{1}{4\pi^2}\int_{\tau_0^{(1)}\tau_1^{(1)}}\int_{\tau_0^{(1)}\tau_1^{(1)}}\sum_{k_1,k_2=0}^{2n-1}\left|\left(G_n^{-1}\right)^{(k_2+2nk_1)}\right|^2\times$$
$$\times\left|\left(f(\tau_0^{(t_1)},\,\tau_0^{(t_2)}),\ldots,f(\tau_{2n-1}^{(t_1)},\,\tau_{2n-1}^{(t_2)})\right)\right|^2|dt_1|\,|dt_2|\,. \tag{3.8}$$

From (3.7) we find that, for any vector $(d_0,\, d_1,\,\ldots,d_{4n^2-1})$,

$$\int_0^\theta\int_0^\theta \varphi_n^{(0)}(e^{i\sigma_1}\tau_{k_1}^{(1)},e^{i\sigma_2}\tau_{k_2}^{(1)})d\sigma_1 d\sigma_2=$$
$$=(d_0,\,d_1,\,\ldots,d_{4n^2-1})\,\left(G_n^{-1}\right)^{k_2+2nk_1},\ k_1,k_2=\overline{0,\,2n-1}, \tag{3.9}$$

where $\varphi_n^{(0)}(t_1, t_2) = (I + \mathrm{K}_n^{(1)})^{-1} f^{(0)}(t_1, t_2)$.

Since the family of operators $\left\{(I + \mathrm{K}_n^{(1)})^{-1}\right\}$ is uniformly bounded, i.e., there is an $M_0 < +\infty$ such that, for any $n \geq n_0$ the inequality $\left\|(I + \mathrm{K}_n^{(1)})^{-1}\right\|_{L_2 \to L_2} \leq M_0$ holds, it follows that

$$\left\|\varphi_n^{(0)}\right\|_{L_2} = \left\|(I + \mathrm{K}_n^{(1)})^{-1} f^{(0)}\right\|_{L_2} \leq M_0 \cdot \left\|f^{(0)}\right\|_{L_2} = M_0 \cdot \left(\frac{1}{4\pi^2\theta^2} \sum_{k=0}^{4n^2-1} |d_k|^2\right)^{\frac{1}{2}}.$$

On the other hand, from (3.9) we obtain

$$\sum_{k_1,k_2=0}^{2n-1} \left|(d_0,\, d_1,\, \ldots, d_{4n^2-1})\ \left(G_n^{-1}\right)^{(k_2+2nk_1)}\right|^2 =$$

$$= \sum_{k_1,k_2=0}^{2n-1} \left|\int_0^\theta \int_0^\theta \varphi_n^{(0)}(e^{i\sigma_1}\tau_{k_1}^{(1)}, e^{i\sigma_2}\tau_{k_2}^{(1)}) d\sigma_1 d\sigma_2\right|^2 \leq$$

$$\leq 4\pi^2\theta^2 \left\|\varphi_n^{(0)}\right\|^2_{L^2(\Gamma^2)} \leq M_0^2 \sum_{k=0}^{4n^2-1} |d_k|^2 . \tag{3.10}$$

In view of (3.10), from (3.8) we find that, for any $n \geq n_0$ the inequality

$$\left\|(I + \mathrm{K}_n^{(2)})^{-1}\right\|_{L_2 \to L_2} \leq M_0$$

holds. We have found that the operators $I + \mathrm{K}_n^{(2)}$ are uniformly invertible; then, by proposition 3.1, we have that, the approximation method involving the system of operators $\left\{I + \mathrm{K}_n^{(2)}\right\}$ can be applied to the operator $I + \mathrm{K}$. By proposition 3.2 from the inequality (3.2), we find that if the inverse operator $(I + \mathrm{K})^{-1}$ exists, then the approximation method involving the system of operators $I + \mathrm{K}_n$ can be applied to the operator $I + \mathrm{K}$, because $I + \mathrm{K}_n = I + \mathrm{K}_n^{(2)} + \left(\mathrm{K}_n - \mathrm{K}_n^{(2)}\right)$. This completes the proof of the lemma 3.1. □

By Π and Π_1 we denote the sets of sequences of bounded linear operators $\{B_n\}$, $\{B'_n\}$ in L_2 of the forms

$$\left(B_n\varphi\right)(t_1, t_2) = \sum_{k_1,k_2=0}^{2n-1} \alpha_{k_1,k_2}^{(n)}(t_1, t_2)\varphi(\tau_{k_1}^{(t_1)}, \tau_{k_2}^{(t_2)}) ,$$

$$\left(B'_n\varphi\right)(t_1, t_2) = \sum_{k_1,k_2=0}^{n-1} \beta_{k_1,k_2}^{(n)}(t_1, t_2)\varphi(\tau_{2k_1}^{(t_1)}, \tau_{2k_2}^{(t_2)}) ,$$

respectively, where the $\alpha_{k_1,k_2}^{(n)}(t_1, t_2)$, $k_1, k_2 = \overline{0,\, 2n-1}$, $\beta_{k_1,k_2}^{(n)}(t_1, t_2)$, $k_1, k_2 = \overline{0,\, n-1}$ are continuous functions such that the sequence of operators $\{B_n\}$, $\{B'_n\}$ strongly converges in L_2 to some linear bounded operator, while by Π^*, Π_1^* the sets of the sequences $\{M_n\} \in \Pi$ satisfying the following condition:

(*) if the inverse operator $(I + BM)^{-1}$ exists, then, for any sequence $\{B_n\} \in \Pi$, $B_n \xrightarrow{s} B$ (respectively, for any $\{B'_n\} \in \Pi_1$, $B'_n \xrightarrow{s} B$) the approximation method involving the system of operators $\{I+B_n\,M_n\}$ (respectively $\{I+B'_n\,M_n\}$), where $M_n \xrightarrow{s} M$, can be applied to the operator $I+BM$.

Lemma 3.2. *The sequence of operators* $\{\mathrm{K}_n\}$ *belongs to* Π^*.

Proof. Let the sequence of operators $\{B_n\}$ belongs to Π and $B_n \xrightarrow{s} B$. It is well known (see, for example, [15]) that the strong convergence of a sequence of linear bounded operators implies its uniform convergence on any compact set. Then the operators $B_n\mathrm{K}$ uniformly converge to the operator $B\mathrm{K}$. As the sequence of operators $\mathrm{K}_\mathrm{n}^{(1)}$ uniformly converge to K and the sequence of operators $\left\{\|B_n\|_{L_2\to L_2}\right\}$ is bounded, then we obtain that the sequence of operators $B_n\mathrm{K}_\mathrm{n}^{(1)}$ also uniformly converge to the operator $B\mathrm{K}$. Next, proceeding just as in Lemma 3.1, we have uniform invertibility of the operators $I+B_n\mathrm{K}_\mathrm{n}^{(2)}$. Since

$$\left\|B_n\left(\mathrm{K}_n^{(2)}-\mathrm{K}_n\right)\right\|_{L_2\to L_2}\to 0 \text{ as } n\to\infty,$$

In view of proposition 3.1 and 3.2, we find that the approximate method involving the system of operators $\{I+B_n\mathrm{K}_n\}$ can be applied to the operator $(I+B\mathrm{K})$. This completes the proof of the lemma 3.2. □

Similar to the proof of the Lemma 3.2 it is proved that the sequences of operators $\left\{\mathrm{K}_n^{(3)}\right\}$, $\left\{\mathrm{K}_n^{(4)}\right\}$, $\left\{\mathrm{K}_n^{(5)}\right\}$, $\left\{\mathrm{K}_n^{(6)}\right\}$ strongly converge to the operator K and the inclusions $\left\{\mathrm{K}_n^{(3)}\right\}\in\Pi_1^*$, $\left\{\mathrm{K}_n^{(4)}\right\}\in\Pi_1^*$, $\left\{\mathrm{K}_n^{(5)}\right\}\in\Pi_1^*$, $\left\{\mathrm{K}_n^{(6)}\right\}\in\Pi_1^*$ exist , where

$$\left(\mathrm{K}_n^{(3)}\varphi\right)(t_1,t_2)=\sum_{k_1,k_2=0}^{n-1}K(t_1,t_2,\tau_{2k_1}^{(t_1)},\tau_{2k_2}^{(t_2)})\,\varphi(\tau_{2k_1}^{(t_1)},\tau_{2k_2}^{(t_2)})\cdot\Delta\tau_{2k_1}^{(t_1)}\Delta\tau_{2k_2}^{(t_2)},$$

$$\left(\mathrm{K}_n^{(4)}\varphi\right)(t_1,t_2)=\sum_{k_1,k_2=0}^{n-1}K(t_1,t_2,\tau_{2k_1}^{(t_1)},\tau_{2k_2+1}^{(t_2)})\,\varphi(\tau_{2k_1}^{(t_1)},\tau_{2k_2+1}^{(t_2)})\cdot\Delta\tau_{2k_1}^{(t_1)}\Delta\tau_{2k_2+1}^{(t_2)},$$

$$\left(\mathrm{K}_n^{(5)}\varphi\right)(t_1,t_2)=\sum_{k_1,k_2=0}^{n-1}K(t_1,t_2,\tau_{2k_1+1}^{(t_1)},\tau_{2k_2}^{(t_2)})\,\varphi(\tau_{2k_1+1}^{(t_1)},\tau_{2k_2}^{(t_2)})\cdot\Delta\tau_{2k_1+1}^{(t_1)}\Delta\tau_{2k_2}^{(t_2)},$$

$$\left(\mathrm{K}_n^{(6)}\varphi\right)(t_1,t_2)=\sum_{k_1,k_2=0}^{n-1}K(t_1,t_2,\tau_{2k_1+1}^{(t_1)},\tau_{2k_2+1}^{(t_2)})\,\varphi(\tau_{2k_1+1}^{(t_1)},\tau_{2k_2+1}^{(t_2)})\cdot\Delta\tau_{2k_1+1}^{(t_1)}\Delta\tau_{2k_2+1}^{(t_2)}.$$

By Π_2 we denote the sets of sequences $\{B_n\}\in\Pi$,

$$\left(B_n\varphi\right)(t_1,t_2)=\sum_{k_1,k_2=0}^{2n-1}\alpha_{k_1,k_2}^{(n)}(t_1,t_2)\varphi(\tau_{k_1}^{(t_1)},\tau_{k_2}^{(t_2)}),$$

where $\alpha_{k_1,k_2}^{(n)}(t_1,t_2)$, $k_1,k_2=\overline{0,\,2n-1}$ are continuous functions such that the sequence of operators

$$\left(B_n^{(1)}\varphi\right)(t_1,t_2)=\sum_{k_1,k_2=0}^{n-1}\alpha_{2k_1,2k_2}^{(n)}(t_1,t_2)\varphi(\tau_{2k_1}^{(t_1)},\tau_{2k_2}^{(t_2)}),$$

$$\left(B_n^{(2)}\varphi\right)(t_1,t_2)=\sum_{k_1,k_2=0}^{n-1}\alpha_{2k_1,2k_2+1}^{(n)}(t_1,t_2)\varphi(\tau_{2k_1}^{(t_1)},\tau_{2k_2+1}^{(t_2)}),$$

$$\left(B_n^{(3)}\varphi\right)(t_1,t_2)=\sum_{k_1,k_2=0}^{n-1}\alpha_{2k_1+1,2k_2}^{(n)}(t_1,t_2)\varphi(\tau_{2k_1+1}^{(t_1)},\tau_{2k_2}^{(t_2)})$$

strongly converges in L_2 to any linear bounded operator; and by Π_2^* a set of sequences $\{M_n\}\in\Pi$, satisfying the following condition:

(**) if the inverse operator $(I+BM)^{-1}$ exist, then, for any sequence $\{B_n\}\in\Pi_2$, $B_n\xrightarrow{s}B$ the approximation method involving the system of operators $\{I+B_n M_n\}$, where $M_n\xrightarrow{s}M$, can be applied to the operator $I+BM$.

Lemma 3.3. *The sequence of operators* $\left\{\mathrm{K}_n^{(3)}\right\}$ *belongs to* Π_2^*.

Proof. Let the sequence of operators $(B_n\varphi)(t_1,t_2)=\sum\limits_{k_1,k_2=0}^{2n-1}\alpha_{k_1,k_2}^{(n)}(t_1,t_2)\varphi(\tau_{k_1}^{(t_1)},\tau_{k_2}^{(t_2)})$ belongs to Π_0, and let the sequence of operators

$$\left(B_n^{(1)}\varphi\right)(t_1,t_2)=\sum_{k_1,k_2=0}^{n-1}\alpha_{2k_1,2k_2}^{(n)}(t_1,t_2)\varphi(\tau_{2k_1}^{(t_1)},\tau_{2k_2}^{(t_2)}),$$

$$\left(B_n^{(2)}\varphi\right)(t_1,t_2)=\sum_{k_1,k_2=0}^{n-1}\alpha_{2k_1,2k_2+1}^{(n)}(t_1,t_2)\varphi(\tau_{2k_1}^{(t_1)},\tau_{2k_2+1}^{(t_2)}),$$

$$\left(B_n^{(3)}\varphi\right)(t_1,t_2)=\sum_{k_1,k_2=0}^{n-1}\alpha_{2k_1+1,2k_2}^{(n)}(t_1,t_2)\varphi(\tau_{2k_1+1}^{(t_1)},\tau_{2k_2}^{(t_2)}),$$

$$\left(B_n^{(4)}\varphi\right)(t_1,t_2)=\sum_{k_1,k_2=0}^{n-1}\alpha_{2k_1+1,2k_2+1}^{(n)}(t_1,t_2)\varphi(\tau_{2k_1+1}^{(t_1)},\tau_{2k_2+1}^{(t_2)})$$

strongly converge to the operators $B^{(1)}$, $B^{(2)}$, $B^{(3)}$ and $B^{(4)}$ in L_2, respectively. Then we have

$$\left(B_n\mathrm{K}_n^{(3)}\varphi\right)(t_1,t_2)=\left(B_n^{(1)}\mathrm{K}_n^{(3)}\varphi\right)(t_1,t_2)+\left(B_n^{(2)}\mathrm{K}_n^{(3)}\varphi\right)(t_1,t_2)+$$
$$+\left(B_n^{(3)}\mathrm{K}_n^{(3)}\varphi\right)(t_1,t_2)+\left(B_n^{(4)}\mathrm{K}_n^{(3)}\varphi\right)(t_1,t_2)=$$
$$=\left(B_n^{(1)}\mathrm{K}_n^{(3)}\varphi\right)(t_1,t_2)+\left(\tilde{B}_n^{(2)}\mathrm{K}_n^{(4)}\varphi\right)(t_1,t_2)+\left(\tilde{B}_n^{(3)}\mathrm{K}_n^{(5)}\varphi\right)(t_1,t_2)+\left(\tilde{B}_n^{(4)}\mathrm{K}_n^{(6)}\varphi\right)(t_1,t_2),$$

where

$$\left(\tilde{B}_n^{(2)}\varphi\right)(t_1,t_2)=\left(B_n^{(2)}\varphi\right)\left(t_1,\tau_{-1}^{(t_2)}\right)=\sum_{k_1,k_2=0}^{n-1}\alpha_{2k_1,2k_2+1}^{(n)}(t_1,t_2)\varphi(\tau_{2k_1}^{(t_1)},\tau_{2k_2}^{(t_2)}),$$

$$\left(\tilde{B}_n^{(3)}\varphi\right)(t_1,t_2)=\left(B_n^{(3)}\varphi\right)\left(\tau_{-1}^{(t_1)},t_2\right)=\sum_{k_1,k_2=0}^{n-1}\alpha_{2k_1+1,2k_2}^{(n)}(t_1,t_2)\varphi(\tau_{2k_1}^{(t_1)},\tau_{2k_2}^{(t_2)}),$$

$$\left(\tilde{B}_n^{(4)}\varphi\right)(t_1,t_2)=\left(B_n^{(4)}\varphi\right)\left(\tau_{-1}^{(t_1)},\tau_{-1}^{(t_2)}\right)=\sum_{k_1,k_2=0}^{n-1}\alpha_{2k_1+1,2k_2+1}^{(n)}(t_1,t_2)\varphi(\tau_{2k_1}^{(t_1)},\tau_{2k_2}^{(t_2)}).$$

Similar to the proof of Lemma 3.2, we can prove that if the inverse operator $(I+B\mathrm{K})^{-1}$ exists, then, the for large values of n, the operators $I+B_n\mathrm{K}_n^{(3)}$ are also invertible and the sequence of the operators $\left(I+B_n\mathrm{K}_n^{(3)}\right)^{-1}$ strongly converges to the operator $(I+B\mathrm{K})^{-1}$ in L_2, that is the approximation method involving the system of operators $\left\{I+B_n\mathrm{K}_n^{(3)}\right\}$, can be applied to the operator $I+B\mathrm{K}$. This completes the proof of the lemma 3.3. □

Lemma 3.4. *If, for any $m\in N$, the sequence of operators $\left\{M_n^{(m)}\right\}$ belongs to Π^*, $\{M_n\}\in\Pi$, and*

$$\lim_{m\to\infty}\sup_{n\in N}\left\|M_n^{(m)}-M_n\right\|_{L_2\to L_2}=0, \tag{3.11}$$

then $\{M_n\}\in\Pi^*$.

Proof. Suppose that conditions (3.11) are satisfied. Since any sequence of operators $\{B_n\}\in\Pi$ is uniformly bounded, it follows that, in view of relation (3.11), there exists a number $m_0\in N$ such that

$$\sup_{n\in N}\left\|B_nM_n^{(m_0)}-B_nM_n\right\|_{L_2(\Gamma^2)\to L_2(\Gamma^2)}<1.$$

Hence the family of operators $\left\{I+B_nM_n-B_nM_n^{(m_0)}\right\}$ is uniformly invertible. Using the relation

$$I+B_nM_n=I+B_nM_n-B_nM_n^{(m_0)}+B_nM_n^{(m_0)}=$$

$$=(I+B_nM_n-B_nM_n^{(m_0)})\left[I+\left(I+B_nM_n-B_nM_n^{(m_0)}\right)^{-1}B_nM_n^{(m_0)}\right],$$

we obtain that the family of operators $\{I+B_nM_n\}$ is also uniformly invertible (beginning with some $n\geq n_2$), because $\left[\left(I+B_nM_n-B_nM_n^{(m_0)}\right)^{-1}B_n\right]\in\Pi$ and $\left\{M_n^{(m)}\right\}\in\Pi^*$. This completes the proof of the lemma 3.4. □

We denote by L a set of linearly bounded operators in L_2, of the form

$$W=\sum_{i=1}^{p_1}\left(h_iI+\mathrm{D}^{(i)}+\mathrm{B}^{(i)}\mathrm{Q}^{(i)}\right)\left[\left(c_iS^{(1)}-S^{(1)}c_i\right)+\left(\mathrm{H}^{(i)}S^{(1)}-S^{(1)}\mathrm{H}^{(i)}\right)\right]+$$

$$+\sum_{i=p_1+1}^{p_2}\left(h_iI+\mathrm{Q}^{(i)}+\mathrm{B}^{(i)}\mathrm{D}^{(i)}\right)\left[\left(d_iS^{(2)}-S^{2}d_i\right)+\left(\mathrm{N}^{(i)}S^{(2)}-S^{(2)}\mathrm{N}^{(i)}\right)\right]+$$

$$+\sum_{i=p_2+1}^{p_3}\mathrm{B}^{(i)}\left[\left(c_iS^{(1)}-S^{(1)}c_i\right)+\left(\mathrm{H}^{(i)}S^{(1)}-S^{(1)}\mathrm{H}^{(i)}\right)\right]\left[\left(d_iS^{(2)}-S^{2}d_i\right)+\right.$$

$$\left.+\left(\mathrm{N}^{(i)}S^{(2)}-S^{(2)}\mathrm{N}^{(i)}\right)\right]+\sum_{i=p_3+1}^{p_4}\mathrm{B}^{(i)}\mathrm{U}^{(i)},$$

where $h_i(t_1,t_2)$, $i=\overline{1\,,\,p_2}$, $c_i(t_1,t_2)$, $i=\overline{1\,,\,p_1}$ and $i=\overline{p_2+1\,,\,p_3}$, $d_i(t_1,t_2)$, $i=\overline{p_1+1\,,\,p_3}$ are continuous functions, $B^{(i)}\in L_2$, $i=\overline{1\,,\,p_4}$ are linearly bounded operators in L_2,

$$(\mathrm{D}^{(i)}\varphi)(t_1,\,t_2)=\int_\Gamma D^{(i)}(t_1,t_2,\tau_1)\,\varphi(\tau_1,t_2)\,d\tau_1 \text{ and } D^{(i)}\in L_2\left(\Gamma^3\right),\ i=\overline{1\,,\,p_2},$$

$$(\mathrm{H}^{(i)}\varphi)(t_1,\,t_2)=\int_\Gamma H^{(i)}(t_1,t_2,\tau_1)\,\varphi(\tau_1,t_2)\,d\tau_1 \text{ and } H^{(i)}\in L_2\left(\Gamma^3\right),$$

$i=\overline{1\,,\,p_1}$ and $i=\overline{p_2+1\,,\,p_3}$,

$$(\mathrm{Q}^{(i)}\varphi)(t_1,\,t_2)=\int_\Gamma Q^{(i)}(t_1,t_2,\tau_2)\,\varphi(t_1,\tau_2)\,d\tau_2 \text{ and } Q^{(i)}\in L_2\left(\Gamma^3\right),\ i=\overline{1\,,\,p_2},$$

$$(\mathrm{N}^{(i)}\varphi)(t_1,\,t_2)=\int_\Gamma N^{(i)}(t_1,t_2,\tau_2)\,\varphi(t_1,\tau_2)\,d\tau_2 \text{ and } N^{(i)}\in L_2\left(\Gamma^3\right),\ i=\overline{p_1+1\,,\,p_3},$$

$$(\mathrm{U}^{(i)}\varphi)(t_1,\,t_2)=\int_{\Gamma^2} U^{(i)}(t_1,t_2,\tau_1,\tau_2)\,\varphi(\tau_1,\tau_2)\,d\tau_1 d\tau_2 \text{ and } U^{(i)}\in L_2\left(\Gamma^4\right),$$

$i=\overline{p_3+1\,,\,p_4}$.

Similar to the proof of Lemma 3.2 and Lemma 3.3, and using Lemma 3.4 the following lemma is proved.

Lemma 3.5. *If $W\in \mathrm{L}$ and the inverse operator $(I+W)^{-1}$ exists, then, for large values of n, the operators $I+W_n$ are also invertible and $(I+W_n)^{-1}\xrightarrow{s}(I+W)^{-1}$, where*

$$W_n=\sum_{i=1}^{p_1}\left(h_iI+\mathrm{D}_n^{(i)}+\mathrm{B}_n^{(i)}\mathrm{Q}_n^{(i)}\right)\left[\left(c_iS_n^{(1)}-S_n^{(1)}c_i\right)+\left(\mathrm{H}_n^{(i)}S_n^{(1)}-S_n^{(1)}\mathrm{H}_n^{(i)}\right)\right]+$$

$$+\sum_{i=p_1+1}^{p_2}\left(h_iI+\mathrm{Q}_n^{(i)}+\mathrm{B}_n^{(i)}\mathrm{D}_n^{(i)}\right)\left[\left(d_iS_n^{(2)}-S_n^{(2)}d_i\right)+\left(\mathrm{N}_n^{(i)}S_n^{(2)}-S_n^{(2)}\mathrm{N}_n^{(i)}\right)\right]+$$

$$+\sum_{i=p_2+1}^{p_3}\mathrm{B}_n^{(i)}\left[\left(c_iS_n^{(1)}-S_n^{(1)}c_i\right)+\left(\mathrm{H}_n^{(i)}S_n^{(1)}-S_n^{(1)}\mathrm{H}_n^{(i)}\right)\right]\times$$

$$\times\left[\left(d_iS_n^{(2)}-S_n^{(2)}d_i\right)+\left(\mathrm{N}_n^{(i)}S_n^{(2)}-S_n^{(2)}\mathrm{N}_n^{(i)}\right)\right]+\sum_{i=p_3+1}^{p_4}\mathrm{B}_n^{(i)}\mathrm{U}_n^{(i)},$$

$$\left\{B_n^{(i)}\right\}\in\Pi_2,\,B_n^{(i)}\xrightarrow{s}B,\,i=\overline{1\,,\,p_4},$$

$$\left(\mathrm{D}_n^{(i)}\varphi\right)(t_1,\,t_2)=\sum_{k_1=0}^{2n-1}D^{(i)}\left(t_1,t_2,\tau_{k_1}^{(t_1)}\right)\varphi\left(\tau_{k_1}^{(t_1)},t_2\right)\left(\frac{1}{2}\Delta\tau_{k_1}^{(t_1)}\right),i=\overline{1\,,\,p_2},$$

$\left(\mathrm{H}_n^{(i)}\varphi\right)(t_1,\,t_2)=\sum_{k_1=0}^{2n-1}H^{(i)}\left(t_1,t_2,\tau_{k_1}^{(t_1)}\right)\varphi\left(\tau_{k_1}^{(t_1)},t_2\right)\left(\frac{1}{2}\Delta\tau_{k_1}^{(t_1)}\right)$, $i=\overline{1\,,\,p_1}$ *and* $i=\overline{p_2+1\,,\,p_3}$,

$$(\mathrm{Q}_n^{(i)}\varphi)(t_1,\,t_2)=\sum_{k_2=0}^{2n-1}Q^{(i)}\left(t_1,t_2,\tau_{k_2}^{(t_2)}\right)\varphi\left(t_1,\tau_{k_2}^{(t_2)}\right)\left(\frac{1}{2}\Delta\tau_{k_2}^{(t_2)}\right),i=\overline{1\,,\,p_2},$$

$$(\mathrm{N}_n^{(i)}\varphi)(t_1,\, t_2) = \sum_{k_2=0}^{2n-1} N^{(i)}\left(t_1, t_2, \tau_{k_2}^{(t_2)}\right)\varphi\left(t_1, \tau_{k_2}^{(t_2)}\right)\left(\frac{1}{2}\Delta\tau_{k_2}^{(t_2)}\right), i = \overline{p_1+1\,,\, p_3},$$

$$\left(\mathrm{U}_n^{(i)}\varphi\right)(t_1,\, t_2) = \sum_{k_1,k_2=0}^{2n-1} U^{(i)}\left(t_1, t_2, \tau_{k_1}^{(t_1)}, \tau_{k_2}^{(t_2)}\right)\varphi\left(\tau_{k_1}^{(t_1)}, \tau_{k_2}^{(t_2)}\right)\left(\frac{1}{2}\Delta\tau_{k_1}^{(t_1)}\right)\left(\frac{1}{2}\Delta\tau_{k_2}^{(t_2)}\right),$$

$i = \overline{p_3+1\,,\, p_4}$.

Next, let us justify the applicability of the approximation method for complete bisingular integral equations with continuous coefficients.

Theorem 3.1. *If the operators Rand $R'R''R'''$are invertible in the space L_2, where*

$$R' = a_0 I + b_1 S^{(1)} - b_2 S^{(2)} - b_0 S,\; R'' = a_0 I - b_1 S^{(1)} + b_2 S^{(2)} - b_0 S,$$
$$R''' = a_0 I - b_1 S^{(1)} - b_2 S^{(2)} + b_0 S,$$

then the approximate method involving the system of operators

$$(R_n\varphi)(t_1,\, t_2) = a_0(t_1,\, t_2)\varphi(t_1,\, t_2) + b_1(t_1,\, t_2)(S_n^{(1)}\varphi)(t_1,\, t_2)+$$
$$+b_2(t_1,\, t_2)(S_n^{(2)}\varphi)(t_1,\, t_2) + b_0(t_1,\, t_2)(S_n\varphi)(t_1,\, t_2),$$

can be applied to the bisingular integral operator R, that is the operators R_n also invertible for large values of n, and $R_n^{-1} \xrightarrow{s} R^{-1}$. Moreover the following estimate holds:

$$\left\|R_n^{-1}f - R^{-1}f\right\|_{L_2} \le const \cdot E_{n-1}^{(2)}(\varphi), \tag{3.12}$$

where $\varphi = R^{-1}f$.

Proof. The operator $\frac{1}{\Delta}R'R''R'''R$ is in the form of $\frac{1}{\Delta}R'R''R'''R = I + W$, where $W \in \mathrm{L}$. By lemma 3.5, the approximate method involving the system of operators $\frac{1}{\Delta}R'_n\, R''_n\, R'''_n\, R_n = I + W_n$ can be applied to the operator $I + W$, where $R'_n = a_0 I + b_1 S_n^{(1)} - b_2 S_n^{(2)} - b_0 S_n$, $R''_n = a_0 I - b_1 S_n^{(1)} + b_2 S_n^{(2)} - b_0 S_n$, $R'''_n = a_0 I - b_1 S_n^{(1)} - b_2 S_n^{(2)} + b_0 S_n$, that is the operators $I + W_n$ also invertible for large values of n, and $(I + W_n)^{-1} \xrightarrow{s} (I + W)^{-1}$. Then it follows that the operators R_n also invertible for large values n, and the following equality holds:

$$R_n^{-1} = (I + W_n)^{-1}\,\frac{1}{\Delta}R'_n\, R''_n\, R'''_n\,.$$

Therefore the sequence of operators $\left\{R_n^{-1}\right\}$ strongly converges to R^{-1} in L_2. Estimation (3.12) follows from Remark 2.1 Chap. 2 [14] and from theorems 2.1 and 2.2. This completes the proof of the theorem 3.1. □

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