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ON SMOOTH SOLUTIONS OF A CLASS OF BOUNDARY VALUE PROBLEMS ON A FINITE SEGMENT FOR SECOND ORDER EQUATIONS WITH OPERATOR COEFFICIENTS

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Abstract. Algebraic conditions providing the existence and uniqueness of smooth solutions of a boundary value problem for an elliptic type second order operator-differential equations in finite segment are found.

1. Introduction

In separable Hilbert space H consider a boundary value problem

$$-u''(t) + A^2u(t) + A_1u'(t) + A_2u(t) = f(t),\ t \in (0,1), \tag{1.1}$$

$$u'(0) = 0,\ u'(1) = 0, \tag{1.2}$$

where $f(t)$, $u(t)$ are the functions determined on the interval (0,1) almost everywhere, with the values in H, A is a self-adjoint positive operator, the operators A_1 and A_2 are linear in H.

Denote by $L_2((0,1);H)$ Hilbert space of functions $g(t)$ determined on the interval (0,1) almost everywhere, with the values in H and Bochner quadratically integrable with the norm

$$\|g\|_{L_2((0,1);H)} = \left(\int_0^1 \|g(t)\|^2\,dt\right)^{1/2} < \infty.$$

Then we introduce the Hilbert space [1]

$$W_2^m\left((0,1);H\right) = \left\{u : A^m u \in L_2\left((0,1);H\right), \quad u^{(m)} \in L_2\left((0,1);H\right)\right\}$$

with the norm

$$\|u\|_{W_2^m((0,1);H)} = \left(\left\|u^{(m)}\right\|^2_{L_2((0,1);H)} + \|A^m u\|^2_{L_2((0,1);H)}\right)^{1/2},$$

where $m \geq 1$ is an integer.

Here and in the sequel, the derivatives are understood in the sense of distributions [5]. Assume $L_2(X,Y)$ is a space of linear bounded operators acting from Hilbert space X to the Hilbert space Y.

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The spaces $L_2(R;H)$ and $W_2^m(R;H)$ for $R=(-\infty;\infty)$ are introduced in the same way.

Denote by H_γ a unit of Hilbert spaces generated by the operator A, i.e. $H_\gamma = D(A^\gamma)$, $(x,y)_\gamma = (A^\gamma x, A^\gamma y)$, $x,y\in D(A^\gamma)$, $\gamma\geq 0$ for $\gamma=0$ we consider $H_0 = H, (x,y)_0=(x,y)$.

2. Main results

Definition 2.1. *Let $f(t)\in W_2^1((0,1);H)$. If there exists a function $u(t)\in W_2^3((0,1),H)$ satisfying equation (1.1) in the interval (0,1) identically, then $u(t)$ is said to be a smooth solution of equation (1.1).*

Definition 2.2. *If for any $f(t)\in W_2^1((0,1),H)$ there exists a smooth solution $u(t)$ of equation (1.1) satisfying boundary conditions (1.2) in the sense*

$$\lim_{t\to+0}\left\|u'(t)\right\|_{3/2}=0,\quad \lim_{t\to 1-0}\left\|u'(t)\right\|_{3/2}=0$$

and the estimate

$$\|u\|_{W_2^3((0,1),H)}\leq const\,\|f\|_{W_2^1((0,1),H)}$$

holds, then problem (1.1)-(1.2) is said to be well-posed in the space $W_2^3((0,1);H)$.

Existence and uniqueness of smooth solutions for some boundary value problems, in an infinite domain was studied in the papers [1-4],[6],[7].

Such problems have not been studied in finite domain. In the present paper we find sufficient conditions providing well-posedness of problem (1.1)-(1.2) in the space $W_2^3((0,1);H)$.

For this purpose we introduce the following total subspace of the space $W_2^3((0,1);H)$:

$$\overset{\circ}{W}{}_2^3((0,1);H)=\left\{u: u\in W_2^3((0,1);H),\ \ u'(0)=u'(1)=0\right\}.$$

Assume that the following conditions are fulfilled:

1)A is a positive-definite operator in H;

2)

$$A_1\in L(H_1,H)\bigcap L(H_2,H_1),\ A_2\in L(H_2,H)\bigcap L(H_3,H_1).$$

Subject to condition (1.2), we can define the following operators

$$P_0u=P_0(d/dt)\,u=-u''(t)+A^2u(t)$$

and

$$P_1u=P_1(d/dt)\,u=A_1u'(t)+A_2u(t)$$

acting from the space $\overset{\circ}{W}{}_2^3((0,1),H)$ to $W_2^1((0,1);H)$. The following theorem is valid.

Theorem 2.1. *The operator P_0 isomorphically maps the space $\overset{\circ}{W}{}_2^3((0,1);H)$ into $W_2^1((0,1);H)$.*

Proof. As the general solution of the equation $P_0(d/dt)\,u(t)=0$ from the space $W_2^3((0,1);H)$ is of the form

$$u_0(t)=e^{-tA}x_1+e^{(t-1)A}x_2$$

where $x_1, x_2 \in H_{5/2}$, then from condition (1.2) it follows that $-Ax_1 + Ae^{-A}x_2 = 0$ and $-Ae^{-A}x_1 + Ax_2 = 0$, i.e. $-x_1 + e^{-A}x_2 = 0$ and $-e^{-A}x_1 + x_2 = 0$. Hence we get $-x_1 + e^{-2A}x_1 = 0$, i.e. $\left(E - e^{-2A}\right)x_1 = 0$. As the operator $E - e^{-2A}$ is invertible in $H_{5/2}$, then $x_1 = 0$. Then and $x_2 = 0$ as will. Consequently, $u_0(t) = 0$ i.e. $Ker P_0 = \{0\}$. Show that $\mathrm{Im} P_0 = W_2^1\left((0,1);H\right)$. From the extension theorem [1] it follows that there exists a function $f_1(t) \in W_2^1\left(R,H\right)$ such that $f_1(t) = f(t)$ for $t \in (0,1)$ and $\|f_1\|_{W_2^1(R;H)} \le const\,\|f\|_{W_2^1((0,1);H)}$. Let us consider the equation

$$P_0(d/dt)u(t) = f_1(t), \quad t \in R, \tag{2.1}$$

and denote by

$$u_1(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(\xi^2 E + A^2\right)^{-1}\hat{f}_1(\xi)e^{i\xi t}d\xi, \quad t \in R, \tag{2.2}$$

where $\hat{f}_1(\xi)$ is the Fourier transformation of the function $f_1(t)$. Show that $u_1(t) \in W_2^3\left(R,H\right)$. Indeed, it follows from (2.2) that $\hat{u}_1(\xi) = \left(\xi^2 E + A^2\right)^{-1}\hat{f}_1(\xi)$. Then by the Plancharel theorem we have

$$\begin{aligned}
\|u_1\|^2_{W_2^3(R;H)} &= \left\|u_1'''\right\|^2_{L_2(R;H)} + \left\|A^3 u_1\right\|^2_{L_2(R;H)} = \left\|\xi^3 \hat{u}_1(\xi)\right\|^2_{L_2(R;H)} \\
&+ \left\|A^3 \hat{u}_1(\xi)\right\|_{L_2(R;H)} = \left\|\xi^3(\xi^2 E + A^2)^{-1}\hat{f}_1(\xi)\right\|^2_{L_2(R;H)} \\
&+ \left\|A^3\left(\xi^2 E + A^2\right)^{-1}\hat{f}_1(\xi)\right\|^2_{L_2(R;H)} \\
&\le \sup_{\xi\in R}\left\|\xi^2(\xi^2 E + A^2)^{-1}\right\|^2 \cdot \left\|\xi\hat{f}_1(\xi)\right\|^2_{L_2(R;H)} \\
&+ \sup_{\xi\in R}\left\|A^2(\xi^2 E + A^2)^{-1}\right\|^2 \cdot \left\|A\hat{f}_1(\xi)\right\|^2_{L_2(R;H)} \\
&\le \left\|\xi\hat{f}_1(\xi)\right\|^2_{L_2(R;H)} + \left\|A\hat{f}_1(\xi)\right\|^2_{L_2(R;H)}) \le const\,\|f\|_{W_2^1((0,1);H)}.
\end{aligned}$$

Here we used the following inequalities

$$\sup_{\xi\in R}\left\|\xi^2(\xi^2 E + A^2)^{-1}\right\| \le 1, \quad \sup_{\xi\in R}\left\|A^2\left(\xi^2 E + A^2\right)^{-1}\right\| \le 1.$$

Thus, $u_1(t) \in W_2^3\left(R;H\right)$. Then it is obvious that $u_1(t)$ satisfies the equation $P_0(d/dt)u(t) = f_1(t)$, identically, as $f_1(t)$ is a continuous function. Denote by $\alpha(t)$ the contraction of $u_1(t)$ on the segment [0,1], i.e. $\alpha(t) = u_1(t)$, $t \in [0,1]$. Then $\alpha(t) \in W_2^3\left((0,1);H\right)$ from the traces theorem it follows that $\alpha^{(j)}(0) \in H_{3-j-1/2}$, $\alpha^{(j)}(1) \in H_{3-j-1/2}$, $j = 0,1,2$.

Obviously, for $t \in [0,1]$ $P_0(d/dt)\alpha(t) = f(t)$. Now we will look for the solution of the equation $P_0 u = f$ in the from $u(t) = \alpha(t) + e^{-tA}x_1 + e^{(t-1)A}x_2$, $t \in (0,1)$ where x_1 and $x_2 \in H_{5/2}$ are unknown vectors, $x_2 = e^{-A}x_1 - A^{-1}\alpha'(1)$. Then from condition (1.2) it follows that $-Ax_1 + Ae^{-A}x_2 = -\alpha'(0)$ and $-Ae^{-A}x_1 + Ax_2 = -\alpha'(1)$. Thus $x_1 - e^{-A}x_2 = A^{-1}\alpha'(0)$ and $e^{-A}x_1 - x_2 = A^{-1}\alpha'(1)$. Hence we obtain $x_1 - e^{-2A}x_1 = A^{-1}\alpha'(0) - e^{-A}(A^{-1}\alpha'(1)) \equiv \psi \in H_{5/2}$. Then $x_1 = \left(E - e^{-2A}\right)^{-1}\psi$, $x_2 = e^{-A}\left(E - e^{-2A}\right)^{-1}\psi + A^{-1}\alpha'(1)$. Obviously,

x_1 and $x_2 \in H_{5/2}$. Therefore $u(t) \in W_2^3((0,1);H)$, i.e. $P_0 u = f$ and $\mathrm{Im} P_0 = W_2^1((0,1);H)$. On the other hand, it is easy to see that

$$\|P_0 u\|_{W_2^1((0,1);H)} \le const\, \|u\|_{W_2^3((0,1);H)}.$$

Therefore, the theorem statement follows from the Banach theorem on an inverse operator. Now we consider the well-posedness of problem (1.1), (1.2) in $W_2^3((0,1);H)$. The following lemma holds:

Lemma 2.1. *For all $u \in W_2^3((0,1);H)$ the equality holds*

$$\|P_0 u\|^2_{W_2^1((0,1);H)} = \|u\|^2_{W_2^3((0,1);H)} + 3\left\|Au'\right\|^2_{W_2^1((0,1);H)}. \tag{2.3}$$

Proof. Let $u \in W_2^3((0,1);H)$ $(u'(0) = u'(1) = 0)$. Then

$$\|P_0 u\|^2_{W_2^1((0,1);H)} = \left\|-u'' + A^2 u\right\|^2_{W_2^1((0,1);H)} = \left\|-u''' + A^2 u'\right\|^2_{L_2((0,1);H)}$$

$$+\left\|-Au'' + A^3 u\right\|^2_{L_2((0,1);H)} = \left\|u'''\right\|^2_{L_2((0,1);H)} + \left\|A^3 u\right\|^2_{L_2((0,1);H)}$$

$$+\left\|A^2 u'\right\|^2_{L_2((0,1);H)} + \left\|Au''\right\|^2_{L_2((0,1);H)}$$

$$-2Re\left(u''', A^2 u'\right)_{L_2((0,1);H)} - 2Re\left(Au'', A^3 u\right)_{L_2((0,1);H)}. \tag{2.4}$$

As for $u \in \overset{\circ}{W}{}_2^3((0,1);H)$ we have

$$\left(Au'', A^3 u\right)_{L_2((0,1);H)} = \left(A^{3/2} u'(t), A^{5/2} u(t)\right)\Big|_0^1$$

$$-\left\|A^2 u'\right\|^2_{L_2((0,1);H)} = -\left\|A^2 u'\right\|^2_{L_2((0,1);H)} \tag{2.5}$$

and

$$\left(u''', A^2 u'\right)_{L_2((0,1);H)} = \left(A^{1/2} u''(t), A^{3/2} u'(t)\right)\Big|_0^1 - \left\|Au''\right\|^2_{L_2((0,1);H)}. \tag{2.6}$$

Then taking into account equalities (2.5) and (2.6) in (2.4), we have

$$\|P_0 u\|^2_{W_2^1((0,1);H)} = \|u\|^2_{W_2^3((0,1);H)} + 3\left(\left\|Au''\right\|^2_{L_2((0,1);H)} + \left\|A^2 u'\right\|^2_{L_2((0,1);H)}\right).$$

Since

$$\left\|Au'\right\|^2_{W_2^1((0,1);H)} = \left\|Au''\right\|^2_{L_2((0,1);H)} + \left\|A^2 u'\right\|^2_{L_2((0,1);H)},$$

then

$$\|P_0 u\|^2_{W_2^1((0,1);H)} = \|u\|^2_{W_2^3((0,1);H)} + 3\left\|Au'\right\|^2_{W_2^1((0,1);H)}.$$

Lemma 2.2. *For all $u \in \overset{\circ}{W}{}_2^3((0,1);H)$ it holds the inequalities*

$$\left\|A^2 u\right\|_{W_2^1((0,1);H)} \le c_0 \|P_0 u\|_{W_2^1((0,1);H)} \tag{2.7}$$

$$\left\|Au'\right\|_{W_2^1((0,1);H)} \le c_1 \|P_0 u\|_{W_2^1((0,1);H)}. \tag{2.8}$$

where $c_0 = 1$, $c_1 = \frac{1}{2}$.

Proof. Let $u \in W_2^3((0,1);H)$. Then from lemma 2.1 we get

$$\left\|A^2u\right\|^2_{W_2^1((0,1);H)} = \left\|A^2u'\right\|^2_{L_2((0,1);H)} + \left\|A^3u\right\|^2_{L_2((0,1);H)}$$

$$\leq \left\|Au'\right\|^2_{W_2^1((0,1);H)} + \|u\|^2_{W_2^3((0,1);H)} \leq \|P_0u\|^2_{W_2^1((0,1);H)}.$$

i.e. inequality (2.7) is valid. Then taking into account $u'(0) = u'(1) = 0$ we obtain

$$\left\|Au'\right\|^2_{W_2^1((0,1);H)} = \left\|Au''\right\|^2_{L_2((0,1);H)} + \left\|A^2u'\right\|^2_{L_2((0,1);H)}$$

$$= (Au'', Au'')_{L_2((0,1);H)} + (A^2u', A^2u')_{L_2((0,1);H)}$$

$$= \left(A^{3/2}u'(t), A^{1/2}u''(t)\right)\Big|_0^1 - (A^2u', u''')_{L_2((0,1);H)}$$

$$+ \left(A^{5/2}u(t), A^{3/2}u'(t)\right)\Big|_0^1 - \left(A^3u, Au''\right)_{L_2((0,1);H)}$$

$$\leq \frac{1}{2}\left(\left\|A^2u'\right\|^2_{L_2((0,1);H)} + \left\|u'''\right\|^2_{L_2((0,1);H)}\right)$$

$$+\frac{1}{2}\left(\left\|A^3u\right\|^2_{L_2((0,1);H)} + \left\|Au''\right\|^2_{L_2((0,1);H)}\right)$$

$$= \frac{1}{2}\left(\|u\|^2_{W_2^3((0,1);H)} + \left\|Au'\right\|^2_{W_2^1((0,1);H)}\right).$$

Taking into account equality (2.3) in the last inequality, we get

$$\left\|Au'\right\|^2_{W_2^1((0,1);H)} \leq \frac{1}{2}\left(\|P_0u\|^2_{W_2^1((0,1);H)} - 2\left\|Au'\right\|^2_{W_2^1((0,1);H)}\right).$$

Hence we have

$$\left\|Au'\right\|^2_{W_2^1((0,1);H)} \leq \frac{1}{4}\|P_0u\|^2_{W_2^1((0,1);H)},$$

and consequently

$$\left\|Au'\right\|_{W_2^1((0,1);H)} \leq \frac{1}{2}\|P_0u\|_{W_2^1((0,1);H)}.$$

The lemma is proved.

Now prove the main theorem of this paper.

Theorem 2.2. *Let conditions 1), 2) fulfilled and the following inequality hold:*

$$q = \frac{1}{2}\max\left(\|A_1\|_{H_1\to H}, \|A_1\|_{H_2\to H_1}\right) + \max\left(\|A_2\|_{H_2\to H}, \|A_2\|_{H_3\to H_1}\right) < 1$$

Then (1.1)-(1.2) is well defined in $W_2^3((0,1);H)$.

Proof. Write problem (1.1)-(1.2) in to form of the equation $P_0u + P_1u = f$, where $f \in W_2^1((0,1);H)$, $u \in \overset{\circ}{W}{}_2^3((0,1);H)$. After substitution $\omega = P_0u$ and using Theorem 1, we get the equation $\omega + P_1P_0^{-1}\omega = f$ in $W_2^1((0,1);H)$. As for any $\omega \in W_2^1((0,1);H)$, we have

$$\left\|P_1P_0^{-1}\omega\right\|_{W_2^1((0,1);H)} = \|P_1u\|_{W_2^1((0,1);H)} \leq \left\|A_1u'\right\|_{W_2^1((0,1);H)}$$

$$+\|A_2u\|_{W_2^1((0,1);H)} = \left(\left\|A_1u''\right\|^2_{L_2((0,1);H)} + \left\|AA_1u'\right\|^2_{L_2((0,1);H)}\right)^{1/2}$$

$$+\left(\left\|A_2u'\right\|^2_{L_2((0,1);H)}+\left\|AA_2u\right\|^2_{L_2((0,1);H)}\right)^{1/2}$$

$$\leq\left(\left\|A_1\right\|^2_{H_2\to H_1}\cdot\left\|Au''\right\|^2_{L_2((0,1);H)}+\left\|A_1\right\|^2_{H_1\to H}\left\|A^2u'\right\|^2_{L_2((0,1);H)}\right)^{1/2}$$

$$+\left(\left\|A_2\right\|^2_{H_2\to H}\cdot\left\|A^2u'\right\|^2_{L_2((0,1);H)}+\left\|A_2\right\|^2_{H_3\to H_1}\cdot\left\|A^3u\right\|^2_{L_2((0,1);H)}\right)^{1/2}$$

$$\leq\max\left(\left\|A_1\right\|_{H_2\to H_1},\left\|A_1\right\|_{H_1\to H}\right)\left\|Au'\right\|_{W_2^1((0,1);H)}$$

$$+\max\left(\left\|A_2\right\|_{H_3\to H_1},\left\|A_2\right\|_{H_2\to H}\right)\cdot\left\|A^2u\right\|_{W_2^1((0,1);H)}.$$

Taking into account (2.7) and (2.8), from Lemma 2.2 we get

$$\left\|P_1P_0^{-1}\omega\right\|_{W_2^1((0,1);H)}\leq\frac{1}{2}\max\left(\left\|A_1\right\|_{H_2\to H_1},\left\|A_1\right\|_{H_1\to H}\right)$$

$$\times\left\|P_0u\right\|_{W_2^1((0,1);H)}+\max\left(\left\|A_2\right\|_{H_3\to H_1},\left\|A_2\right\|_{H_2\to H}\right)$$

$$\times\left\|P_0u\right\|_{W_2^1((0,1);H)}=q\cdot\left\|P_0u\right\|_{W_2^1((0,1);H)}=q\cdot\left\|\omega\right\|_{W_2^1((0,1);H)}.$$

As $q<1$, then $E+P_1P_0^{-1}$ is invertible in $W_2^1\left((0,1);H\right)$. Then

$\omega=\left(E+P_1P_0^{-1}\right)^{-1}f$ and $u=P_0^{-1}\left(E+P_1P_0^{-1}\right)^{-1}f$. Hence we have

$$\left\|u\right\|_{W_2^3((0,1);H)}\leq const\left\|f\right\|_{W_2^1((0,1);H)}.$$

The theorem is proved.

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