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ON ASYMPTOTIC BEHAVIOR OF LOCAL PROBABILITIES OF NONLINEAR BOUNDARY CROSSING BY A RANDOM WALK

FADA G. RAHIMOV, TARANA E. HASHIMOVA, AND VUGAR S. KHALILOV

Abstract. In the paper a theorem on asymptotic behavior of the density of joint distribution of the first passage time and overshoot for a nonlinear boundary in the random walk, is proved. Limit behavior of the marginal and conditional density of the overshoot and also of the law of distribution of the first passage time are studied by means of this theorem.

1. Introduction

Let $\xi_n, \ \ n \geq 1$ be the sequence of independent identically distributed random variables determined on some probability space $(\Omega, \mathcal{F}, P)$.

Let

$$S_n = \sum_{k=1}^{n} \xi_k, \ n \geq 1,$$

and consider the first passage time

$$\tau_a = \inf \{n \geq 1 : S_n \geq f_a(n)\} \tag{1.1}$$

of the random walk $S_n, \ n \geq 1$ for the nonlinear boundary $f_a(t) \ , \ \ t > 0$ dependent on some growing parameter $a > 0$. As always, we assume $\inf \{\emptyset\} = \infty$.

The family of the first passage times $\tau_a; \ \ a \geq 0$ was the object of study of a lot of papers ([1], [3-9]).

For the case $f_a(t) = f(t)$ the asymptotic behavior of the probability $P(\tau \geq n)$ and the issue of finiteness of $E\tau$ were studied in the paper [4] at different assumptions for the boundary.

In the papers [5], [6], integral and local limit theorems and also asymptotic behavior of the joint distribution τ_a and overshoot $\chi_a = S_{\tau_a} - f_a(\tau_a)$ as $a \to \infty$ were studied for a rather wide class of family of boundaries $f_a(t)$.

In the present paper we prove a theorem on asymptotic behavior of the density of joint distribution τ_a and χ_a as $a \to \infty$. By means of this theorem we study

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the limit behavior of the marginal and conditional density of the overshoot χ_a, and also of the probability $P(\tau_a = n)$ as $a \to \infty$.

Note that the proof of the mentioned theorem is based on the result of the paper [1] in which the limit behavior of the conditional probability of nonlinear boundary crossing $P(\tau_a \geq n | S_n = x)$ as $x = x(a) \to \infty$ and $a \to \infty$ was studied.

2. Conditions and formulation of the main result

We'll assume that $\mu = E\xi_1 > 0$, $\sigma^2 = D\xi_1 < \infty$ and the boundary $f_a(t)$ satisfies the following regularity conditions:

1) For any a the function $f_a(t)$ monotonically increases, is continuously differentiable for $t > 0$, moreover $f_a(1) \uparrow \infty$ as $a \to \infty$;

2) For rather large a the function $\frac{f_a(t)}{t}$ monotonically decreases to zero as $t \to \infty$;

3) For each function $n = n(a)$ from a such that $n = n(a) \to \infty$ and $\frac{1}{n} f_a(n) \to \mu$ as $a \to \infty$, it is fulfilled $f_a'(n) \to \theta \in [0, \mu)$, $a \to \infty$;

4) The function $f_a'(t)$ weakly oscillates at infinity, i.e. for any functions $n = n(a) \to \infty$ and $m = m(a) \to \infty$ as $a \to \infty$ such that $\frac{n}{m} \to 1$, it is fulfilled $\frac{f_a(n)}{f_a(m)} \to 1$ as $a \to \infty$.

Note that from conditions 1) and 2) it follows that the equation $f_a(n) = n\mu$ with respect to n has a unique solution $N_a = N_a(\mu)$. We also note that the family of functions $f_a(t) = at^\beta$, $0 \leq \beta < 1$ satisfies conditions 1)-4). The other examples are given in the papers [4] and [5].

In what follows, we assume that for some $m \geq 1$

$$\int_{-\infty}^{\infty} |u(t)|^m \, dt < \infty, \tag{2.1}$$

where $u(t) = Me^{it\xi_1}$, $t \in R = (-\infty, \infty)$, $i^2 = -1$.

It follows from (2.1) that the sum S_n, $n \geq m$ has the continuous and bounded density $P_n(x)$ ([3]).

Assume that

$$w_a(n, r) = \frac{d}{dr} P(\tau_a = n, \chi_a \leq r)$$

is the density of the joint distribution τ_a and χ_a:

$$h(r) = \frac{P\left(S'_{\tau_+} > r\right)}{ES'_{\tau_+}}$$

is the density of limit distribution of the overshoot of the random walk $S'_n = S_n - n\theta$, $n \geq 1$ for an infinitely distant barrier, where $\tau_+ = \inf\left\{n \geq 1 : S'_n > 0\right\}$ [8].

It is appropriate to note that in the paper [7] (see also [9]) it is proved a theorem on the existence of limit distribution of the overshoot of the perturbed random walk according to which for twice continuously differentiable functions satisfying conditions 1)-4) and the condition $\sup\limits_{t:|t-N_a|\le M\sqrt{N_a}}\left|N_a f_a''(t)\right| < \infty$ for any $M > 0$, it is fulfilled $P(\chi_a \le r) \to \int_0^r h(u)\,du$ as $a \to \infty$.

Theorem 2.1. *Let all the listed above conditions be fulfilled with regard to the boundary $f_a(t)$ and distribution of the random variable ξ_1, and let*

$$n = n(a) = N_a + \upsilon_a\sqrt{N_a}, \tag{2.2}$$

where $\upsilon_a \to \upsilon \in R$ as $a \to \infty$.

Then

$$w_a(n,r) \sim \frac{\lambda}{\sigma\sqrt{n}}\varphi\left(\frac{\upsilon\lambda}{\sigma}\right)h(r) \quad as \ a \to \infty \tag{2.3}$$

uniformly with respect to υ from the bounded set in R, where $\lambda = \mu - \theta$, and

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}, x \in R.$$

Corollary 2.1. *Let the conditions of the theorem be fulfilled. Then*

$$\int_0^\infty |h_a(r) - h(r)|\,dr \to 0$$

as $a \to \infty$, where $h_a(r)$ is the marginal density of the overshoot $\chi_a, a > 0$.

Corollary 2.2. *Let the conditions of the theorem be fulfilled. Then*

$$P(\tau_a = n) \sim \frac{\lambda}{\sigma\sqrt{n}}\varphi\left(\frac{\upsilon\lambda}{\sigma}\right) asa \to \infty.$$

Corollary 2.3. *If the assumptions of theorem are satisfied, then*

$$h_a(r/n) \to h(r)\ ,\ r > 0,$$

where $h_a(r/n)$ is the conditional density of the overshoot χ_a provided that $\tau_a = n$.

Note that the statement of corollary 2.2 is called a local limit theorem for the first exit time τ_a([4]).

3. Auxiliary facts

For proving the theorem and corollaries we need the following statements formulated as lemmas.

Lemma 3.1. *Let condition* (2.1) *be fulfilled. Then*

$$P_n(x) = \frac{1}{\sigma\sqrt{n}}\varphi\left(\frac{x - n\mu}{\sigma\sqrt{n}}\right) + o\left(1/\sqrt{n}\right)\ ,\ \ n \to \infty.$$

This statement follows from the local limit theorem for the sum S_n([3]).

Lemma 3.2. *Let conditions 1)-3) be fulfilled with respect to the boundary $f_a(t)$and $\sigma^2 = D\xi_1 < \infty,\ \ \mu = E\xi_1 > 0$. Then*

$$\lim_{a\to\infty} P\left(\frac{\tau_a - N_a}{\sqrt{N_a}} \le x\right) = \Phi\left(\frac{\lambda}{\sigma}x\right),$$

where $\lambda = \mu - \theta$*, and* $\Phi(x)$ *is a distribution function of normal law with parameters (0,1)*

The statement of this lemma follows from the paper [5].

Lemma 3.3. *Let for* $g_n(x)$, $n \geq 1$ *and* $g(x)$, $x \in R$ *non-negative measurable functions it hold the convergence* $g_n(x) \to g(x)$ *as* $n \to \infty$ *almost everywhere with respect to the Lebesgue measure. If* $\int_{-\infty}^{\infty} g_n(x)\,dx \to \int_{-\infty}^{\infty} g(x)\,dx$ *as* $n \to \infty$*, then* $\int_{-\infty}^{\infty} |g_n(x) - g(x)|\,dx \to 0$ *as* $n \to \infty$*.*

The statement of lemma 3.3 follows from the Scheffe [2] theorem, (see also [8]).

Remark 3.1. If the functions $g_n(x)$ and $g(x)$ in lemma 3.3 are the densities of the probability distribution F_n and F, respectively, lemma 3.3 affirms that from the convergence of densities $g_n(x) \to g(x)$ as $n \to \infty$ almost everywhere with respect to the Lebesgue measure follows that the sequence of distributions $F_n, n \geq 1$ strongly converges to the distribution F, i.e. $F_n(B) \to F(x)$ as $n \to \infty$ uniformly for all $B \in \beta(R)$, where $\beta(R)$ is σ algebra of Borel sets in R [2].

4. Proof of the theorem.

Denote

$$R_a(n,r) = P(\tau_a = n, \chi_a \geq r) = P(\tau_a = n,\ S_n \geq f_a(n) + r)\ ,\ \ r > 0.$$

Taking into account $\{\tau_a > n\} \subset \{S_n < f_a(n)\}$, we can write

$$R_a(n,r) = P(\tau_a \geq n, S_n \geq f_a(n) + r).$$

By the total probability formula for $n \geq m$, we have

$$R_a(n,r) = \int_{\frac{r+f_a(n)}{n}}^{\infty} P\left(\tau_a \geq n | \bar{S}_n = x\right)\ \ nP_n(nx)\,dx.$$

Hence, by using the differentiation formula with respect to r, we find

$$w_a(n,r) = P\left(\tau_a \geq n | \bar{S}_n = \frac{r + f_a(n)}{n}\right) P_n(r + f_a(n)). \tag{4.1}$$

From lemma 3.1

$$P_m(r + f_a(n)) = \frac{1}{\sigma\sqrt{n}}\varphi\left(\frac{r + f_a(n) - n\mu}{\sigma\sqrt{n}}\right) + o\left(\frac{1}{\sqrt{n}}\right). \tag{4.2}$$

We have

$$f_a(n) - n\mu = f_a(N_a) - n\mu + (f_a(n_a) - f_a(N_a)) =$$

$$= \mu(N_a - n) + f'(\gamma_a)(n - N_a) = (n - N_a)\left(f_a'(\gamma_a) - \mu\right),$$

whereγ_a is an intermediate point between n and N_a.

From (2.2) we find

$$f_a(n) - n\mu = \upsilon_a\sqrt{N_a}\left(f_a'(\gamma_a) - \mu\right).$$

Show that $f_a^{'}(\gamma_a) \to \theta$as $a \to \infty$. For definiteness we assume $n \le \gamma_a \le N_a$. By condition (2.2) for the boundary $f_a(t)$ we have

$$\frac{f_a(n)}{n} \ge \frac{f_a(\gamma_a)}{\gamma_a} \ge \frac{f_a(N_a)}{N_a} = \mu.$$

Therefore, by (2.2) and condition 4) we obtain $\frac{f_a(\gamma_a)}{\gamma_a} \to \mu$ as $a \to \infty$.

Hence, from condition (2.3) we get $f_a^{'}(\gamma_a) \to \theta < \mu$ as $a \to \infty$.

Consequently, by (4.2) we get

$$P_n(r + f_a(n)) = \frac{1}{\sigma\sqrt{n}}\varphi\left(\frac{\upsilon(\mu-\theta)}{\sigma}\right)(1 + o(1)) \text{ as } a \to \infty \tag{4.3}$$

By result of the paper [1] we have

$$P\left(\tau_a \ge n|\bar{S}_n = \frac{r + f_a(n)}{n}\right) \to (\mu - \theta)\, h(r) \text{ as } \to \infty \tag{4.4}$$

for all $r > 0$.

Then the statement of the proved theorem follows from (4.1), (4.3) and (4.4).

Now prove the corollaries of the theorem.

Proof of Corollary 2.1. For each $c > 0$ we have

$$h_a(r) = \sum_n w_a(n,r) = \sum_{n:|n-N_a| \le c\sqrt{N_a}} w_a(n,r) + \sum_{n:|n-N_a| > c\sqrt{N_a}} w_a(n,r). \tag{4.5}$$

Denote

$$h_{a,1}(r) = \sum_{n:|n-N_a| \le c\sqrt{N_a}} w_a(n,r),$$

$$h_{a,2}(r) = \sum_{n:|n-N_a| > c\sqrt{N_a}} w_a(n,r).$$

According to the theorem and lemma 3.2, for any $c > 0$

$$h_{a,1}(r) \to h(r)\left(\Phi\left(\frac{\lambda c}{\sigma}\right) - \Phi\left(\frac{-\lambda c}{\sigma}\right)\right)$$

as $a \to \infty$.

From the last relation, for $c = c(a) \to \infty$ we get

$$h_{a,1}(r) \to h(r) \text{ as } a \to \infty \tag{4.6}$$

In what follows, from lemma 3.2 for $c = c(a) \to \infty$ we have

$$\int_0^\infty h_{a,2}(r)\,dr = P\left(\left|\frac{\tau_a - N_a}{\sqrt{N_a}}\right| > c\right) \to 0 \tag{4.7}$$

as $a \to \infty$.

From (4.5), (4.6) and (4.7) it follows that

$$\int_0^\infty h_{a,1}(r)\,dr \to 1 = \int_0^\infty h(r)\,dr.$$

Therefore, from lemma 3.3

$$\int_0^\infty |h_{a,1}(r) - h(r)|\, dr \to 0 \text{ as } a \to \infty.$$

Then, taking into account the estimation

$$\int_0^\infty |h_a(r) - h(r)|\, dr \le \int_0^\infty |h_{a,1}(r) - h(r)|\, dr + \int_0^\infty h_{a,2}(r)\, dr,$$

we get the affirmation of corollary 2.1.

Proof of Corollary 2.2. For $c > 0$ we have

$$\sqrt{n} P(\tau_a = n) = \sqrt{n} \int_0^\infty w_a(n,r)\, dr =$$

$$= \sqrt{n} \int_0^c w_a(n,r)\, dr + \sqrt{n} \int_c^\infty w_a(n,r)\, dr. \tag{4.8}$$

By the theorem on majorized convergence and asymptotics (2.3) we find that for each $c > 0$

$$\sqrt{n} \int_0^c w_a(n,r)\, dr \to H(c) \frac{\lambda}{\mu} \varphi\left(\frac{\lambda v}{\sigma}\right) \tag{4.9}$$

as $a \to \infty$.

Prove that the second term in (4.8) converges to zero as $a \to \infty$ and $c \to \infty$. Indeed, we have

$$\sqrt{n} \int_0^\infty w_a(n,r)\, dr = \sqrt{n} P(\tau_a = n, \chi_a > c) = \sqrt{n} P(\tau_a \ge n, \chi_a > c).$$

Hence, taking into account $\{\tau_a \ge n\} \subset \{S_{n-1} \le f_a(n-1)\}$, we get

$$\sqrt{n} \int_c^\infty w_a(n,r)\, dr \le \sqrt{n} P(S_{n-1} \le f_a(n-1), S_{n-1} + \xi_n > c + f_a(n)) \le$$

$$\le \sqrt{n} P(c + f_a(n) - \xi_n < S_{n-1} \le f_a(n-1), \xi_n > c) =$$

$$= \sqrt{n} \int_c^\infty P(c + f_a(n) - x < S_{n-1} \le f_a(n-1))\, dF(x), \tag{4.10}$$

where $F(x) = P(\xi_1 \le x)$. In the last equality it is taken into account that the random value ξ_n is independent of S_{n-1}.

On the other hand, for rather large a from the local limit theorem (lemma 3.1) and from $f_a(n) - f_a(n-1) \to \theta$ as $a \to \infty$ we have

$$\sqrt{n} P(c + f_a(n) - x < S_{n-1} \le f_a(n-1)) \le$$

$$\le K(x - c - \theta) \le K(x - c),$$

where R is some constant.

Then from (4.9) it follows that

$$\sqrt{n}\int_c^{\infty} w_a(n,r)\,dr \leq \sqrt{n}P\left(S_{n-1} \leq f_a(n-1), S_{n-1}+\xi_n > c+f_a(n)\right) \leq .$$

The last integral tends to zero as $c \to \infty$ since $M|\xi_1| < \infty$.

Therefore,

$$\sqrt{n}\int_c^{\infty} w_a(n,r)\,dr \to 0 \text{ as } a \to \infty \text{ and } c \to \infty.$$

Then by (4.8) and (4.9) as $c \to \infty$ the statement of corollary 2.2 follows, since $H(c) \to 1$ as $c \to \infty$.

The statement of corollary 3.3 by virtue of asymptotics (2.3) and corollary 2.2 follows from the equality

$$h_a(r,n) = \frac{w_a(n,r)}{P(\tau_a = n)}.$$

Remark 4.1. Note that corollary 2.3 shows that the conditional distribution of the overshoot χ_a, given $\tau_a = n$ strongly converges to the unconditional limit distribution $H(r)$. This means that the well known property of asymptotic independence of the overshoot χ_a and the first passage time τ_a as $a \to \infty$ ([7], [8], [9]) holds in the sense of strong convergence of probability distributions ([2]).

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Fada G. Rahimov
Baku State University, 23 Z.Khalilov str., Az 1148, Baku, Azerbaijan.
E-mail address: ragimovf@rambler.ru

Tarana E. Hashimova
Baku State University, 23 Z.Khalilov str., Az 1148, Baku, Azerbaijan.

Vugar S. Khalilov

Institute of Mathematics and Mechanics of NAS of Azerbaijan, 9 B. Vahabzadeh str., Az 1141, Baku, Azerbaijan.

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