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MULTILINEAR FRACTIONAL INTEGRAL OPERATORS WITH ROUGH KERNEL ON MODIFIED MORREY SPACES

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Abstract. In this paper the boundedness of multi-sublinear fractional maximal operator $M_{\Omega,\alpha,m}$ and multilinear fractional integral operator $I_{\Omega,\alpha,m}$ with rough kernels on product modified Morrey spaces $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)\times \ldots\times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ are studied. The authors study necessary and sufficient conditions on the parameters of the boundedness on product modified Morrey spaces for $I_{\Omega,\alpha,m}$ and $M_{\Omega,\alpha,m}$.

1. Introduction

Suppose that Ω is homogeneous of degree zero on $\mathbb{R}^n$ and $\Omega \in L^s(\mathbb{S}^{n-1})$ with $1 < s \le \infty$, where $\mathbb{S}^{n-1}$ denote the unit sphere of $\mathbb{R}^n$. Moreover, $m \ge 1$ will denote an integer, $\theta_j (j = 1, ..., m)$ will be fixed, distinct, and nonzero real numbers, and $0 < \alpha < n$. We denote $\mathbf{f} = (f_1, ..., f_m)$, then the multilinear fractional integral operator on $\mathbb{R}^n$ is given by the formula

$$I_{\Omega,\alpha,m}\mathbf{f}(x) = \int_{\mathbb{R}^n} \frac{\Omega(y)}{|y|^{n-\alpha}} \prod_{j=1}^{m} f_j(x - \theta_j y)dy,$$

and the multilinear fractional maximal operator $M_{\Omega,\alpha,m}$

$$M_{\Omega,\alpha,m}\mathbf{f}(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{B(0,r)} |\Omega(y)| \prod_{j=1}^{m} |f_j(x - \theta_j y)|dy.$$

If $\alpha = 0$, then $M_{\Omega,m} \equiv M_{\Omega,0,m}$ is the multilinear maximal operator.

When $m = 1$ and $\Omega \equiv 1$, if let $\theta_1 = 1$, $I_{\Omega,\alpha,m}$ will be the Riesz potential operator I_α [5, 6] given by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(x-y)}{|y|^{n-\alpha}} dy.$$

Proposition 1.1. [3] *Let $0 < \alpha < n$, $1 \le p < n/\alpha$, $0 \le \lambda < n - \alpha p$.*

(i) *If $p > 1$, then condition $\alpha/n \le 1/p - 1/q \le \alpha(n - \lambda)$ is necessary and sufficient for the boundedness of the operator I_α from $\widetilde{L}^{p,\lambda}$ to $\widetilde{L}^{q,\lambda}$.*

(ii) *If $p = 1$, then condition $\alpha/n \le 1 - 1/q \le \alpha/(n - \lambda)$ s necessary and sufficient for the boundedness of the operator I_α from $\widetilde{L}^{1,\lambda}$ to $W\widetilde{L}^{q,\lambda}$.*

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Theorem 1.1. *Let* $0 < \alpha < n$, $\Omega \in L^s(\mathbb{S}^{n-1})$ *with* $1 < s \le \infty$, p *be the harmonic mean of* $p_1, ..., p_m > 1$, $0 \le \lambda < n - \alpha p$, $1 \le p < n/\alpha$ *and satisfy*

$$\frac{\lambda}{p} = \sum_{j=1}^{m} \frac{\lambda_j}{p_j} \; for \; 0 \le \lambda_j < n. \tag{1.1}$$

(i) *If* $p > s'$, *then the condition* $\alpha/n \le 1/p - 1/q \le \alpha(n-\lambda)$ *is necessary and sufficient for the boundedness of the operator* $T_{\Omega,\alpha,m}$ *from* $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times ... \times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ *to* $\widetilde{L}^{q,\lambda}(\mathbb{R}^n)$.

(ii) *If* $p = s'$, *then the condition* $\alpha/n \le 1/s' - 1/q \le \alpha(n-\lambda)$ *is necessary and sufficient for the boundedness of the operator* $T_{\Omega,\alpha,m}$ *from* $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times ... \times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ *to* $W\widetilde{L}^{q,\lambda}(\mathbb{R}^n)$.

Moreover, similar estimates hold for $M_{\Omega,\alpha,m}$.

Theorem 1.2. *Let* $0 < \alpha < n$, $0 \le n-\alpha$, $\Omega \in L^s(\mathbb{S}^{n-1})$ *with* $1 < s \le \infty$, p *be the harmonic mean of* $p_1, ..., p_m > 1$ *and satisfy* (1.1). *If* $s' \le (n-\lambda)/\alpha \le p \le n/\alpha$, *then the operator* $M_{\Omega,\alpha,m}$ *is bounded from* $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times ... \times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ *to* $L^\infty(\mathbb{R}^n)$

Remark 1.1. Note that, the statements of the Theorems 1.1 and 1.2 in the case of Morrey spaces were proved in [4].

Throughout this paper, we assume the letter C always remains to denote a positive constant that may vary at each occurrence but is independent of the essential variables.

2. Boundedness of $M_{\Omega,m}$ on modified Morrey spaces

In this part, we investigate the boundedness of maximal operator $M_{\Omega,m}$ (see Section 1) on Morrey and modified Morrey spaces defined by the following definitions.

Definition 2.1. [1] Let $1 \le p < \infty$, $0 \le \lambda \le n$. We denote by $L^{p,\lambda} = L^{p,\lambda}(\mathbb{R}^n)$ the Morrey space, and by $WL^{p,\lambda} = WL^{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space, as the set of locally integrable functions $f(x), x \in \mathbb{R}^n$, with the finite norms

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, t>0} \Big(\frac{1}{t^\lambda}\int_{B(x,t)} |f(y)|^p dy\Big)^{\frac{1}{p}},$$

$$\|f\|_{WL^{p,\lambda}(\mathbb{R}^n)} = \sup_{r>0} r \sup_{x\in\mathbb{R}^n,\ t>0} \Big(\frac{1}{t^\lambda}|\{y \in B(x,t) : |f(y)| > r)\}|\Big)^{\frac{1}{p}}$$

respectively.

Definition 2.2. [3] Let $1 \le p < \infty$, $0 \le \lambda \le n$, $[t]_1 = \min\{1, t\}$. We denote by $\widetilde{L}^{p,\lambda} = \widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ the modified Morrey space, and by $W\widetilde{L}^{p,\lambda} = W\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ the weak modified Morrey space, as the set of locally integrable functions $f(x), x \in \mathbb{R}^n$, with the finite norms

$$\|f\|_{\widetilde{L}^{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, t>0} \Big(\frac{1}{[t]_1^\lambda}\int_{B(x,t)} |f(y)|^p dy\Big)^{\frac{1}{p}},$$

$$\|f\|_{W\widetilde{L}^{p,\lambda}(\mathbb{R}^n)} = \sup_{r>0} r \sup_{x\in\mathbb{R}^n, t>0} \Big(\frac{1}{[t]_1^{\lambda}} |\{y \in B(x,t : |f(y)| > r)\}|\Big)^{\frac{1}{p}}$$

respectively.

It is easy to see that $L^{p,0}(\mathbb{R}^n) = \widetilde{L}^{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, $WL^{p,0} = W\widetilde{L}^{p,0}(\mathbb{R}^n) = WL^p(\mathbb{R}^n)$. And if $\lambda < 0$ or $\lambda > n$, then $\widetilde{L}^{p,\lambda} = L^{p,\lambda}(\mathbb{R}^n) = \Theta$ where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$. In addition, from [3], we know

$$\widetilde{L}^{p,\lambda}(\mathbb{R}^n) \hookrightarrow L^{p,\lambda}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n),\ \max\{\|f\|_{L^{p,\lambda}}, \|f\|_{L^p}\} \leq \|f\|_{\widetilde{L}^{p,\lambda}}.$$

Recall the definition of $M_{\Omega,m}$, as a special case when $m = 1$, $\Omega \equiv 1$ and $\theta_1 = 1$, $M_{\Omega,m}$ is the Hardy-Littlewood maximal operator $\mathcal{M}$. The boundedness of fractional integral operators on the classical Morrey spaces was studied by Adams [1], Chiarenza and Frasca [2]. In [2], by establishing a pointwise estimate of fractional integrals in terms of the Hardy-Littlewood maximal function, they showed the boundedness of fractional integral operators on the Morrey spaces. Guliyev, Hasanov, Zeren [3] studied the boundedness of the operators $\mathcal{M}$ and I_α on the modified Morrey spaces $\widetilde{L}^{p,\lambda}$.

Lemma 2.1. [3] *Let $1 \leq p < \infty$ and $0 \leq \lambda < n$. Then for $p > 1$, $\mathcal{M}$ is bounded from $\widetilde{L}^{p,\lambda}$ to $\widetilde{L}^{p,\lambda}$ and for $p = 1$, to $\mathcal{M}$ is bounded from $\widetilde{L}^{1,\lambda}$ to $W\widetilde{L}^{1,\lambda}$.*

When $m \geq 2$ and $\Omega \in L^s(\mathbb{S}^{n-1})$, we find out $M_{\Omega,m}$ also have the same properties by providing the following multi-version of the Lemma 2.1.

Theorem 2.1. *Let $\Omega \in L^s(\mathbb{S}^{n-1})$ with $1 < s \leq \infty, 0 \leq \lambda < n$, p be the harmonic mean of $p_1, ..., p_m > 1, p \geq s'$ and satisfy* (1.1).

(i) *If $p > s'$, there exists a positive constant C such that*

$$\|M_{\Omega,m}\boldsymbol{f}\|_{\widetilde{L}^{p,\lambda}} \leq C \prod_{j=1}^{m} \|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}.$$

(ii) *If $p = s'$, there exists a positive constant C such that*

$$\|M_{\Omega,m}\boldsymbol{f}\|_{W\widetilde{L}^{p,\lambda}} \leq C \prod_{j=1}^{m} \|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}.$$

Here, we give the proof of Theorem 2.1.

Proof. Since $\Omega \in L^s(\mathbb{S}^{n-1})$ with $s > 1$, Hölder's inequality yields that

$$\frac{1}{r^n} \int_{B(0,r)} |\Omega(y)| \prod_{j=1}^{m} |f_j(x - \theta_j y)| dy$$
$$\leq \frac{1}{r^n} \bigg(\int_{B(0,r)} \prod_{j=1}^{m} |f_j(x - \theta_j y)^{s'}| dy \bigg)^{1/s'} \Big(\int_{B(0,r)} |\Omega(y)|^s dy \Big)^{1/s}$$

$$= \frac{1}{r^n}\bigg(\int_{B(0,r)}\prod_{j=1}^m |f_j(x-\theta_j y)^{s'}|dy\bigg)^{1/s'}\Big(\int_0^r\int_{\mathbb{S}^{n-1}}|\Omega(\xi)|^s\tau^{n-1}d\xi d\tau\Big)^{1/s}$$

$$= C\bigg(\frac{1}{r^n}\int_{B(0,r)}\prod_{j=1}^m |f_j(x-\theta_j y)^{s'}|dy\bigg)^{1/s'}$$

$$\le C\prod_{j=1}^m\Big(\frac{1}{r^n}\int_{B(0,r)}|f_j(x-\theta_j y)|^{s'p_j/p}dy\Big)^{p/s'p_j}$$

$$\le C\prod_{j=1}^m\Big[\mathcal{M}(f_j^{s'p_j/p})(x)\Big]^{p/s'p_j},$$

which implies a pointwise estimate

$$M_{\Omega,m}\mathbf{f}(x)\le C\prod_{j=1}^m\Big[\mathcal{M}(f_j^{s'p_j/p})(x)\Big]^{p/s'p_j}. \tag{2.1}$$

(i) If $p>s'$, by (2.1) and the Hölder inequality, we get

$$\frac{1}{[t]_1^\lambda}\int_{B(x,t)}|M_{\Omega,m}\mathbf{f}(y)|^p dy\le C\frac{1}{[t]_1^\lambda}\int_{B(x,t)}\prod_{j=1}^m\Big[\mathcal{M}(f_j^{s'p_j/p})(y)\Big]^{p^2/s'p_j}dy$$

$$\le C\prod_{j=1}^m\Big(\frac{1}{[t]_1^\lambda}\int_{B(x,t)}\Big[\mathcal{M}(f_j^{s'p_j/p})(y)\Big]^{p/s'}dy\Big)^{p/p_j}$$

for all $x\in\mathbb{R}^n$ and $t>0$. Taking the p-th root of both sides and applying Lemma 2.1 with $p/s'>1$ and the fact $f_j^{s'p_j/p}\in L^{p/s',\lambda_j}$, we get

$$\|M_{\Omega,m}\mathbf{f}\|_{\widetilde{L}^{p,\lambda}}=\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{[t]_1^\lambda}\int_{B(x,t)}|M_{\Omega,m}f(y)|^p dy\Big)^{1/p}$$

$$\le C\prod_{j=1}^m\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{[t]_1^\lambda}\int_{B(x,t)}\Big[\mathcal{M}(f_j^{s'p_j/p})(y)\Big]^{p/s'}dy\Big)^{1/p_j}$$

$$= C\prod_{j=1}^m\Big\|\mathcal{M}(f_j^{s'p_j/p})\Big\|_{\widetilde{L}^{p/s',\lambda_j}}^{p/s'p_j}\le C\prod_{j=1}^m\Big\|f_j^{s'p_j/p}\Big\|_{\widetilde{L}^{p/s',\lambda_j}}^{p/s'p_j}=C\prod_{j=1}^m\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}},$$

which is the desired inequality.

(ii) If $p=s'$, for any $\beta>0$, let $\varepsilon_0=\beta$, $\varepsilon_m=1$ and $\varepsilon_1,\varepsilon_2,...,\varepsilon_{m-1}>0$ be arbitrary which will be chosen later. From the pointwise estimate (2.1), we get

$$\{y\in B(x,t):|M_{\Omega,m}\mathbf{f}(y)|>\beta\}$$

$$\subset\bigcup_{j=1}^m\Big\{y\in B(x,t):\Big[\mathcal{M}(f_j^{s'p_j/p})(y)\Big]^{p/s'p_j}>\frac{\varepsilon_{j-1}}{[t]_1^{(\lambda-\lambda_j)/p_j}\varepsilon_j}\Big\}.$$

Let us now take $\varepsilon_1,\varepsilon_2,...,\varepsilon_{m-1}>0$ such that

$$\frac{\varepsilon_j}{\varepsilon_{j=1}}=\frac{\Big[\prod_{j=1}^m\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}\Big]^{s'/p_j}}{\beta^{s'/p_j}\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}},\ j=1,2,...,m.$$

Then, applying Lemma 2.1 with $p/s' = 1$ and the fact $f_j^{p_j} \in \widetilde{L}^{1,\lambda_j}$, we get

$$\begin{aligned}
&\{y \in B(x,t) : |M_{\Omega,m}\mathbf{f}(y)| > \beta\} \\
&\le C \sum_{j=1}^{m} \Big| \Big\{ y \in B(x,t) : \mathcal{M}(f_j^{p_j})(y) > \Big(\frac{\varepsilon_{j-1}}{[t]_1^{(\lambda-\lambda_j)/p_j} \varepsilon_j} \Big)^{p_j} \Big\} \Big| \\
&\le C \sum_{j=1}^{m} [t]_1^{\lambda_j} \Big(\frac{[t]^{(\lambda-\lambda_j)/p_j} \varepsilon_j}{\varepsilon_{j-1}} \Big)^{p_j} \| f_j^{p_j} \|_{\widetilde{L}^{1,\lambda_j}} \\
&= C \sum_{j=1}^{m} [t]_1^{\lambda_j} \Big(\frac{\varepsilon_j}{\varepsilon_{j-1}} \Big)^{p_j} \| f_j \|_{\widetilde{L}^{p_j,\lambda_j}}^{p_j} = C \sum_{j=1}^{m} [t]_1^{\lambda_j} \Big[\Big(\frac{\varepsilon_j}{\varepsilon_{j-1}} \Big)^{p_j} \| f_j \|_{\widetilde{L}^{p_j,\lambda_j}} \Big]^{p_j} \\
&= C \sum_{j=1}^{m} [t]_1^{\lambda_j} \left(\frac{1}{\beta} \prod_{j=1}^{m} \| f_j \|_{\widetilde{L}^{p_j,\lambda_j}} \right)^{s'} = C [t]_1^{\lambda_j} \left(\frac{1}{\beta} \prod_{j=1}^{m} \| f_j \|_{\widetilde{L}^{p_j,\lambda_j}} \right)^{p}.
\end{aligned}$$

Hence, we obtain the following inequality

$$\begin{aligned}
\| M_{\Omega,\alpha,m}\mathbf{f} \|_{W\widetilde{L}^{q,\lambda}} &= \sup_{\beta>0} \beta = \sup_{x\in\mathbb{R}^n, t>0} \Big(\frac{1}{[t]_1^{\lambda}} |\{ y \in B(x,t) : |M_{\Omega,\alpha,m}\mathbf{f}(y)| > \beta \}| \Big)^{\frac{1}{p}} \\
&\le C \prod_{j=1}^{m} \| f_j \|_{\widetilde{L}^{p_j,\lambda_j}}.
\end{aligned}$$

This is the conclusion (ii) of Theorem 2.1. □

3. Boundedness of $I_{\Omega,\alpha,m}$ and $M_{\Omega,\alpha,m}$ on modified Morrey spaces

In this part, we get the boundedness on modified Morrey spaces and show the proof of the Theorems 1.1 and 1.2. With the same arguments on Morrey spaces, we also begin with a requisite Hedberg's type estimates on modified Morrey spaces.

Lemma 3.1. *Let $0 < \alpha < n$, $\Omega \in L^s(\mathbb{S}^{n-1})$ with $1 < s \le \infty$, p be the harmonic mean of $p_1, ..., p_m > 1$, $0 \le \lambda < n - \alpha p$, $s' \le p < n/\alpha$ and satisfy* (1.1)*, then there exists a positive constant C such that*

$$|I_{\Omega,\alpha,m}\boldsymbol{f}(x)| \le C (M_{\Omega,m}\boldsymbol{f}(x))^{p/q} \prod_{j=1}^{m} \| f_j \|_{\widetilde{L}^{p_j,\lambda_j}}^{1-p/q}.$$

Proof. For any $\delta > 0$, we split the integral into two parts:

$$\begin{aligned}
T_{\Omega,\alpha,m}\mathbf{f}(x) &= \Big(\int_{B(0,\delta)} + \int_{\mathbb{R}^n \setminus B(0,\delta)} \Big) \frac{\Omega(y)}{|y|^{n-\alpha}} \prod_{j=1}^{m} f_j(x - \theta_j y) dy \\
&=: A(x,\delta) + B(x,\delta).
\end{aligned}$$

For $A(x,\delta)$, we have

$$\begin{aligned}
|A(x,\delta)| &\le \int_{B(0,\delta)} \frac{\Omega(y)}{|y|^{n-\alpha}} \prod_{j=1}^{m} |f_j(x-\theta_j y)|dy \\
&\le \sum_{i=0}^{\infty} \int_{B(0,2^{-i}\delta)\setminus B(0,2^{-i-1}\delta)} \frac{\Omega(y)}{|y|^{n-\alpha}} \prod_{j=1}^{m} |f_j(x-\theta_j y)|dy \\
&\le \sum_{i=0}^{\infty} \left(2^{-i-1}\delta\right)^{\alpha-n} \int_{B(0,2^{-i}\delta)} |\Omega(y)| \prod_{j=1}^{m} |f_j(x-\theta_j y)|dy \\
&\le \sum_{i=0}^{\infty} \left(2^{-i-1}\delta\right)^{\alpha-n} \left(2^{-i}\delta\right)^{n} M_{\Omega,m}\mathbf{f}(x) \\
&\le 2^{n-\alpha}\delta^{\alpha} M_{\Omega,m}\mathbf{f}(x) \sum_{i=0}^{\infty} 2^{-i\alpha} \le C\delta^{\alpha} M_{\Omega,m}\mathbf{f}(x).
\end{aligned}$$

Recalling the conditions of Lemma 3.1, we can see $s' \le p < (n-\lambda)/\alpha$, which implies $\alpha < (n-\lambda)/p \le (n-\lambda)/s'$, then we get

$$n - \alpha s' > n - (n-\lambda)s'/p \ge n - (n-\lambda) = \lambda.$$

In order to estimate $B(x,\delta)$, we choose a real number σ such that

$$n - \alpha s' > \sigma > n - (n-\lambda)s'/p \ge \lambda.$$

One can then see from the chose of σ that

$$n - (n - \alpha - \sigma/s')s < 0 \tag{3.1}$$

and

$$(n-\sigma)/s' - n/p \le (n-\sigma)/s' - (n-\lambda)/p < 0. \tag{3.2}$$

If $p > s'$, by the Hölder inequality and the fact (3.2), we obtain

$$\begin{aligned}
F_\sigma(x,\delta) &\le \sum_{i=0}^{\infty} (2^i\delta)^{-\sigma/s'} \left(\int_{|y|<2^{i+1}\delta} \prod_{j=1}^{m} |f_j(x-\theta_j y)|^{s'} dy \right)^{1/s'} \\
&\le C \sum_{i=0}^{\infty} (2^i\delta)^{(n-\sigma)/s'-n/p} \left(\int_{|y|<2^{i+1}\delta} \prod_{j=1}^{m} |f_j(x-\theta_j y)|^{p} dy \right)^{1/p} \\
&\le C \sum_{i=0}^{\infty} (2^i\delta)^{(n-\sigma)/s'-n/p} [2^{i+1}\delta]_1^{\lambda/p} \left(\frac{1}{[2^{i+1}\delta]_1^{\lambda}} \int_{|y|<2^{i+1}\delta} \prod_{j=1}^{m} |f_j(x-\theta_j y)|^{p} dy \right)^{1/p} \\
&\le C \sum_{i=0}^{\infty} (2^i\delta)^{(n-\sigma)/s'-n/p} [2^{i+1}\delta]_1^{\lambda/p} \prod_{j=1}^{m} \left(\frac{1}{[2^{i+1}\delta]_1^{\lambda}} \int_{|y|<2^{i+1}\delta} |f_j(x-\theta_j y)|^{p_j} dy \right)^{1/p_j} \\
&\le C \sum_{i=0}^{\infty} (2^i\delta)^{(n-\sigma)/s'-n/p} [2^{i+1}\delta]_1^{\lambda/p} \prod_{j=1}^{m} \|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}
\end{aligned}$$

$$
\begin{aligned}
&= C\prod_{j=1}^{m}\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}\sum_{i=0}^{\infty}(2^i\delta)^{(n-\sigma)/s'-n/p}[2^{i+1}\delta]_1^{\lambda/p}\\
&\le C\prod_{j=1}^{m}\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}\begin{cases}\sum\limits_{i=0}^{\infty}(2^i\delta)^{(n-\sigma)/s'-n/p}, & if\quad \delta\ge 1/2\\ \sum\limits_{i=0}^{[\log_2\frac{1}{2\delta}]}(2^i\delta)^{(n-\sigma)/s'-(n-\lambda)/p} \\ \qquad +\sum\limits_{i=[\log_2\frac{1}{2\delta}]+1}^{\infty}(2^i\delta)^{(n-\sigma)/s'-n/p}, & if\quad 0<\delta<1/2\end{cases}\\
&\le C\prod_{j=1}^{m}\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}\begin{cases}\delta^{(n-\sigma)/s'-n/p}, & if\quad \delta\ge 1/2\\ \delta^{(n-\sigma)/s'-(n-\lambda)/p}+\delta^{(n-\sigma)/s'-n/p}, & if\quad 0<\delta<1/2\end{cases}\\
&\le C\prod_{j=1}^{m}\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}\begin{cases}\delta^{(n-\sigma)/s'-n/p}, & if\quad \delta\ge 1/2\\ \delta^{(n-\sigma)/s'-(n-\lambda)/p}, & if\quad 0<\delta<1/2\end{cases}\\
&\le C\delta^{(n-\sigma)/s'-n/p}[2\delta]_1^{\lambda/p}\prod_{j=1}^{m}\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}.
\end{aligned}
$$

If $p=s'$, using the Hölder inequality and the fact $0\le\lambda<\sigma$, we get

$$
\begin{aligned}
F_\sigma(x,\delta) &\le \sum_{i=0}^{\infty}(2^i\delta)^{-\sigma/s'}\left(\int_{|y|<2^{i+1}\delta}\prod_{j=1}^{m}|f_j(x-\theta_j y)|^{s'}dy\right)^{1/s'}\\
&\le C\sum_{i=0}^{\infty}(2^i\delta)^{-\sigma/s'}[2^{i+1}\delta]_1^{\lambda/s'}\left(\frac{1}{[2^{i+1}\delta]_1^{\lambda}}\int_{|y|<2^{i+1}\delta}\prod_{j=1}^{m}|f_j(x-\theta_j y)|^{s'}dy\right)^{1/s'}\\
&\le C\sum_{i=0}^{\infty}(2^i\delta)^{-\sigma/s'}[2^{i+1}\delta]_1^{\lambda/s'}\prod_{j=1}^{m}\left(\frac{1}{[2^{i+1}\delta]_1^{\lambda}}\int_{|y|<2^{i+1}\delta}|f_j(x-\theta_j y)|^{p_j}dy\right)^{1/p_j}\\
&\le C\prod_{j=1}^{m}\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}\sum_{i=0}^{\infty}(2^i\delta)^{-\sigma/s'}[2^{i+1}\delta]_1^{\lambda/s'}\\
&\le C\prod_{j=1}^{m}\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}\begin{cases}\sum\limits_{i=0}^{\infty}(2^i\delta)^{-\sigma/s'}, & if\quad \delta\ge 1/2\\ \sum\limits_{i=0}^{[\log_2\frac{1}{2\delta}]}(2^i\delta)^{(n-\sigma)/s'p} \\ \qquad +\sum\limits_{i=[\log_2\frac{1}{2\delta}]+1}^{\infty}(2^i\delta)^{(n-\sigma)/s'}, & if\quad 0<\delta<1/2\end{cases}\\
&\le C\prod_{j=1}^{m}\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}\begin{cases}\delta^{-\sigma/s'}, \ if\ \delta\ge 1/2\\ \delta^{(\lambda-\sigma)/s'}+\delta^{-\sigma/s'}, \ if\ 0<\delta<1/2\end{cases}
\end{aligned}
$$

$$\leq C\prod_{j=1}^{m}\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}\begin{cases}\delta^{-\sigma/s'}, \; if\ \delta\geq 1/2\\ \delta^{(\lambda-\sigma)/s'}, \; if\ 0<\delta<1/2\end{cases}$$

$$\leq C\delta^{-\sigma/s'}[2\delta]_1^{\lambda/p}\prod_{j=1}^{m}\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}\leq C\delta^{(n-\sigma)/s'-n/p}[2\delta]_1^{\lambda/p}\prod_{j=1}^{m}\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}.$$

Then, combining with the estimates $E_\sigma(\delta)\leq C\delta^{\alpha-(n-\sigma)/s'}$, we have

$$|S(x,\delta)|\leq C\delta^{-\sigma/s'}[2\delta]_1^{\lambda/p}\prod_{j=1}^{m}\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}},\ \ for\ every\ p\geq s'.$$

Thus

$$|I_{\Omega,\alpha,m}\mathbf{f}(x)|\leq C\left(\delta^{\alpha}M_{\Omega,m}\mathbf{f}(x)+\delta^{\alpha-n/p}[2\delta]_1^{\lambda/p}\prod_{j=1}^{m}\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}\right)$$

$$\leq C\min\left\{\delta^{\alpha}M_{\Omega,m}\mathbf{f}(x)+\delta^{\alpha-n/p}\prod_{j=1}^{m}\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}},\right.$$

$$\left.\delta^{\alpha}M_{\Omega,m}\mathbf{f}(x)+\delta^{\alpha-(n-\lambda)/p}\prod_{j=1}^{m}\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}\right\}.$$

Minimizing with respect to δ, at

$$\delta=\left[(M_{\Omega,m}\mathbf{f}(x))^{-1}\prod_{j=1}^{m}\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}\right]^{p/n}$$

and

$$\delta=\left[(M_{\Omega,m}\mathbf{f}(x))^{-1}\prod_{j=1}^{m}\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}\right]^{p/(n-\lambda)}$$

we have

$$|I_{\Omega,\alpha,m}\mathbf{f}(x)|\leq C\min\left\{\left(\frac{M_{\Omega,m}\mathbf{f}(x)}{\Pi_{j=1}^{m}\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}}\right)^{1-\frac{p\alpha}{n}},\left(\frac{M_{\Omega,m}\mathbf{f}(x)}{\Pi_{j=1}^{m}\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}}\right)^{1-\frac{p\alpha}{n-\lambda}}\right\}$$

$$\times\prod_{j=1}^{m}\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}\leq C(M_{\Omega,m}\mathbf{f}(x))^{p/q}\prod_{j=1}^{m}\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}^{1-p/q}.$$

This is the conclusion of Lemma 3.1. □

Now we give the proof of Theorem 1.1.

Proof. of Theorem 1.1. Firstly, we will devote to the proof of (i).
Sufficiency. By Lemma 3.1 and by the boundedness of $M_{\Omega,m}$ in Theorem 2.1, we

have

$$\left(\frac{1}{t^\lambda}\int_{B(x,t)}|I_{\Omega,\alpha,m}\mathbf{f}(y)|^q dy\right)^{1/q} \leq C\prod_{j=1}^m \|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}^{1-p/q}\left(\frac{1}{t^\lambda}\int_{B(x,t)}(M_\Omega\mathbf{f}(y))^p dy\right)^{1/q}$$
$$\leq C\prod_{j=1}^m \|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}^{1-p/q}\prod_{j=1}^m \|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}^{p/q} \leq C\prod_{j=1}^m \|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}.$$

Taking the supremum for $x \in \mathbb{R}^n$ and $t > 0$, we will get the desired conclusion.

Necessity. Define $\mathbf{f}_\varepsilon(x) = (f_1(\varepsilon x), ..., f_m(\varepsilon x))$ for $\varepsilon > 0$. Then it is easy to show that

$$T_{\Omega,\alpha,m}\mathbf{f}_\varepsilon(y) = \varepsilon^{-\alpha}T_{\Omega,\alpha,m}\mathbf{f}(\varepsilon y). \tag{3.3}$$

Let $[\epsilon]_{1,+} = \max\{1, \epsilon\}$, by the fact (3.3) for $f_\epsilon(x)$ with $\epsilon > 0$, (x) with $\epsilon > 0$, we get

$$\begin{aligned}
\|I_{\Omega,\alpha,m}\mathbf{f}_\epsilon\|_{\widetilde{L}^{q,\lambda}} &= \epsilon^{-\alpha}\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{[t]_1^\lambda}\int_{B(x,t)}|I_{\Omega,\alpha,m}\mathbf{f}(\epsilon y)|^q dy\Big)^{1/q}\\
&= \epsilon^{-\alpha-n/q}\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{[t]_1^\lambda}\int_{B(\epsilon x,\epsilon t)}|I_{\Omega,\alpha,m}\mathbf{f}(y)|^q dy\Big)^{1/q}\\
&= \epsilon^{-\alpha-n/q}\sup_{t>0}\Big(\frac{[\epsilon t]_1}{[t]_1}\Big)^{\lambda/q}\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{[\epsilon t]_1^\lambda}\int_{B(\epsilon x,\epsilon t)}|I_{\Omega,\alpha,m}\mathbf{f}(y)|^q dy\Big)^{1/q}\\
&= \epsilon^{-\alpha-n/q}[\epsilon]_{1,+}^{\frac{\lambda}{q}}\|I_{\Omega,\alpha,m}\mathbf{f}_\epsilon\|_{\widetilde{L}^{q,\lambda}}
\end{aligned}$$

and

$$\begin{aligned}
\|I_{\Omega,\alpha,m}\mathbf{f}_\epsilon\|_{W\widetilde{L}^{q,\lambda}} &= \sup_{r>0} r\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{[t]_1^\lambda}\int_{\{y\in B(x,t):|I_{\Omega,\alpha,m}\mathbf{f}_\epsilon(y)|>r\}}dy\Big)^{\frac{1}{q}}\\
&= \sup_{r>0} r\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{[t]_1^\lambda}\int_{\{y\in B(x,t):|I_{\Omega,\alpha,m}\mathbf{f}(\epsilon y)|>r\epsilon^\alpha\}}dy\Big)^{\frac{1}{q}}\\
&= \epsilon^{-n/q}\sup_{r>0} r\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{[t]_1^\lambda}\int_{\{y\in B(\epsilon x,\epsilon t):|I_{\Omega,\alpha,m}\mathbf{f}(y)|>r\epsilon^\alpha\}}dy\Big)^{\frac{1}{q}}\\
&= \epsilon^{-\alpha-n/q}\sup_{t>0}\Big(\frac{[\epsilon t]_1}{[t]_1}\Big)^{\lambda/q}\\
&\times \sup_{r>0} r\epsilon^\alpha\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{[\epsilon t]_1^\lambda}|y\in B(\epsilon x,\epsilon t): |I_{\Omega,\alpha,m}\mathbf{f}(y)| > r\epsilon^\alpha|\Big)^{\frac{1}{q}}\\
&= \epsilon^{-\alpha-n/q}[\epsilon]_{1,+}^{\frac{\lambda}{q}}\|I_{\Omega,\alpha,m}\mathbf{f}_\epsilon\|_{W\widetilde{L}^{q,\lambda}}.
\end{aligned}$$

(i)Assume that $I_{\Omega,\alpha,m}$ is bounded from $\widetilde{L}^{p_1,\lambda_1} \times ... \times \widetilde{L}^{p_m,\lambda_m}$ to $\widetilde{L}^{q,\lambda}$ we get

$$\begin{aligned}
\|I_{\Omega,\alpha,m}\mathbf{f}\|_{\widetilde{L}^{q,\lambda}} &= \epsilon^{\alpha+n/q}[\epsilon]_{1,+}^{-\frac{\lambda}{q}}\|I_{\Omega,\alpha,m}\mathbf{f}_\epsilon\|_{\widetilde{L}^{q,\lambda}} \\
&\leq C\epsilon^{\alpha+n/q}[\epsilon]_{1,+}^{-\frac{\lambda}{q}}\prod_{j=1}^{m}\|f\|_j(\epsilon.)\|_{\widetilde{L}^{p_j,\lambda_j}} \\
&= C\epsilon^{\alpha+n/q}[\epsilon]_{1,+}^{-\frac{\lambda}{q}}\prod_{j=1}^{m}\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{[t]_1^{\lambda_j}}\int_{B(x,t)}|f_j(\epsilon y)|^{p_j}dy\Big)^{1/p_j} \\
&= C\epsilon^{\alpha+n/q}[\epsilon]_{1,+}^{-\frac{\lambda}{q}}\prod_{j=1}^{m}\epsilon^{-n/p_j}\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{[t]_1^{\lambda_j}}\int_{B(\epsilon x,\epsilon t)}|f_j(y)|^{p_j}dy\Big)^{1/p_j} \\
&\leq C\epsilon^{\alpha+n/q}[\epsilon]_{1,+}^{-\frac{\lambda}{q}}\prod_{j=1}^{m}\epsilon^{-n/p_j}\sup_{t>0}\Big(\frac{[\epsilon t]_1}{[t]_1}\Big)^{\lambda_j/p_j} \\
&\times\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{[\epsilon t]_1^{\lambda_j}}\int_{B(\epsilon x,\epsilon t)}|f_j(y)|^{p_j}dy\Big)^{1/p_j} \\
&\leq C\epsilon^{\alpha+n/q-n/p}[\epsilon]_{1,+}^{-\frac{\lambda}{q}}[\epsilon]_1^{\frac{\lambda}{p}}\prod_{j=1}^{m}\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}} \\
&\leq C\epsilon^{\alpha+n/q-n/p}[\epsilon]_{1,+}^{\frac{\lambda}{p}-\frac{\lambda}{q}}\prod_{j=1}^{m}\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}},
\end{aligned}$$

where C is independent of ϵ. When $1/p < 1/q + \alpha/n$, then for all $\mathbf{f} \in \widetilde{L}^{p_1,\lambda_1} \times ... \times \widetilde{L}^{p_m,\lambda_m}$ we have $\|I_{\Omega,\alpha,m}\mathbf{f}\|_{\widetilde{L}^{q,\lambda}} = 0$ as $\epsilon \to 0$.

When $1/p > 1/q + \alpha/(n-\lambda)$, then for all $\mathbf{f} \in \widetilde{L}^{p_1,\lambda_1} \times ... \times \widetilde{L}^{p_m,\lambda_m}$ we have $\|I_{\Omega,\alpha,m}\mathbf{f}\|_{\widetilde{L}^{q,\lambda}} = 0$ as $\epsilon \to \infty$.

Therefore we get $\alpha/n \leq 1/p - 1/q \leq \alpha/(n-\lambda)$.

(ii) Assume that $I_{\Omega,\alpha,m}$ is bounded from $\widetilde{L}^{p_1,\lambda_1} \times ... \times \widetilde{L}^{p_m,\lambda_m}$ to $W\widetilde{L}^{q,\lambda}$ we have

$$\begin{aligned}
&\|I_{\Omega,\alpha,m}\mathbf{f}\|_{W\widetilde{L}^{q,\lambda}} = \epsilon^{\alpha+n/q}[\epsilon]_{1,+}^{-\frac{\lambda}{q}}\|I_{\Omega,\alpha,m}\mathbf{f}_\epsilon\|_{W\widetilde{L}^{q,\lambda}} \\
&\leq C\epsilon^{\alpha+n/q}[\epsilon]_{1,+}^{-\frac{\lambda}{q}}\prod_{j=1}^{m}\|f_j(\epsilon.)\|_{\widetilde{L}^{p_j,\lambda_j}} \leq C\epsilon^{\alpha+n/q-n/p}[\epsilon]_{1,+}^{\frac{\lambda}{p}-\frac{\lambda}{q}}\prod_{j=1}^{m}\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}},
\end{aligned}$$

where C is independent of ϵ. When $1/p < 1/q+\alpha/n$, then for all $\mathbf{f} \in \widetilde{L}^{p_1,\lambda} \times ... \times \widetilde{L}^{p_m,\lambda}$ we have $\|I_{\Omega,\alpha,m}\mathbf{f}\|_{W\widetilde{L}^{q,\lambda}} = 0$ as $\epsilon \to 0$. When $1/p > 1/q + \alpha/(n-\lambda)$, then for all $\mathbf{f} \in \widetilde{L}^{p_1,\lambda} \times ... \times \widetilde{L}^{p_m,\lambda}$ we have $\|I_{\Omega,\alpha,m}\mathbf{f}\|_{W\widetilde{L}^{q,\lambda}} = 0$ as $\epsilon \to \infty$. Consequently, we get $\alpha/n \leq 1/p-1/q \leq \alpha/(n-\lambda)$. Next, we prove conclusions (i) and (ii) hold for $M_{\Omega,\alpha,m}$. By the same arguments as above we get the necessity part and the sufficiency part follows from the conclusion of $I_{\Omega,\alpha,m}$. This completes the proof of Theorem 1.1. □

Finally we show the proof of Theorem 1.2.

Proof. of Theorem 1.2. By the Hölder inequality, we have

$$
\begin{aligned}
M_{\Omega,\alpha,m}\mathbf{f}(x) &= \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{B(0,r)} |\Omega(y)| \prod_{j=1}^{m} |f_j(x-\theta_j y)| dy \\
&\le C \sup_{r>0} \frac{1}{r^{n-\alpha}} \Big(\int_{B(0,r)} |\Omega(y)|^{p'} dy \Big)^{1/p'} \Big(\int_{B(0,r)} | \prod_{j=1}^{m} |f_j(x-\theta_j y)|^{p} dy \Big)^{1/p} \\
&\le C \sup_{r>0} \frac{1}{r^{n-\alpha}} \Big(\int_{B(0,r)} |\Omega(y)|^{p'} dy \Big)^{1/p'} \prod_{j=1}^{m} \Big(\int_{B(0,r)} |f_j(x-\theta_j y)|^{p_j} dy \Big)^{1/p} \\
&\le C \sup_{r>0} r^{\alpha-n/p} \Big(\frac{1}{r^n} \int_{B(0,r)} |\Omega(y)|^{s} dy \Big)^{1/s} \prod_{j=1}^{m} \Big(\int_{B(0,r)} |f_j(x-\theta_j y)|^{p_j} dy \Big)^{1/p_j} \\
&\le C \sup_{r>0} r^{\alpha-n/p} \prod_{j=1}^{m} \Big(\int_{B(0,r)} |f_j(x-\theta_j y)|^{p_j} dy \Big)^{1/p_j}.
\end{aligned}
$$

(i) If $p = (n-\lambda)/\alpha \ge s'$, by the fact (1.1), we obtain

$$
\begin{aligned}
M_{\Omega,\alpha,m}\mathbf{f}(x) &\le \sup_{r>0} r^{\alpha-(n-\lambda)/p} \prod_{j=1}^{m} \Big(\frac{1}{r^{\lambda_j}} \int_{B(0,r)} |f_j(x-\theta_j y)|^{p_j} dy \Big)^{1/p_j} \\
&= C \prod_{j=1}^{m} \|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}.
\end{aligned}
$$

(ii) If $p = (n-\lambda)/\alpha \ge p \ge n/\alpha$, using the fact (1.1), we get

$$
\begin{aligned}
M_{\Omega,\alpha,m}\mathbf{f}(x) &\le C \sup_{r>0} r^{\alpha-n/p} [r]_1^{\lambda/p} \prod_{j=1}^{m} \Big(\frac{1}{[r]_1^{\lambda_j}} \int_{B(0,r)} |f_j(x-\theta_j y)|^{p_j} dy \Big)^{1/p_j} \\
&\le C \sup_{r>0} r^{\alpha-n/p} [r]_1^{\lambda/p} \prod_{j=1}^{m} \|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}. \\
&\le \prod_{j=1}^{m} \|f_j\|_{\widetilde{L}^{p_j,\lambda_j}} \max\Big\{ \sup_{0<r<1} r^{\alpha-\frac{n-\lambda}{p}}, \sup_{r\ge 1} r^{\alpha-n/p} \Big\} = \prod_{j=1}^{m} \|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}.
\end{aligned}
$$

Therefore, we complete the proof of Theorem 1.2. □

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