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FORMULA FOR SECOND REGULARIZED TRACE OF THE STURM-LIOUVILLE EQUATION WITH SPECTRAL PARAMETER IN THE BOUNDARY CONDITIONS

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Abstract. In the paper we calculate formula for the second regularized trace of the problem generated by Sturm-Liouville operator equation and with spectral parameter dependent boundary condition.

1. Introduction

The study of regularized traces of ordinary differential operators has a long history and there are a large number of papers and books studying this issue. The regularized trace of the differential operators can be regarded as a generalization of the traces of matrices and operators .The trace formula for the scalar differential operators have been first found by Gelfand and Levitan [15] .The formula obtained there gave rise to a large and very important theory, which started from the investigation of specific operators and further embraced the analysis of regularized traces of discrete operators in general form. In a short time, a number of authors turned their attention to trace theory and obtained interesting results. For example, Dikii provided a proof of the Gelfand-Levitan formula in [11] on the basis of direct methods of perturbation theory, and in [12], he derived trace formulas of all orders for the Sturm-Liouville operator by constructing the fractional powers of the operator in closed form and by computing an analytic extension for its zeta function. Later, Levitan [17] suggested one more method for computing the traces of the Sturm-Liouville operator: by matching the expressions for the characteristic determinant via the solution of an appropriate Cauchy problem and via the corresponding infinite product, he found and compared the coefficients of the asymptotic expansions of these expressions thus obtaining trace formulas. Gasymov's paper [14] was the first paper in which a singular differential operator with discrete spectrum was considered. Afterwards these investigations were continued in many directions, such as Dirac operators, differential operators with abstract operator-valued coefficients, and the case of matrix-valued Sturm-Liouville operators (see, [21]). In [18] , the trace of the Sturm–Liouville operator with unbounded operator coefficient has been first calculated by F.G. Maksudov,M. Bayramoglu and A.A. Adigezalov. Higher order

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regularized traces investigated for example, in [1, 7, 10 , 13] . In [7] , M. Bayramoglu and N.M.Aslanova found a formula for the second regularized trace of the problem generated by a Sturm-Liouville operator equation and a spectral parameter dependent boundary condition.The trace formulas for differential operators with operator coefficient are investigated in the works [1-5,7-10, 18,19].

In the present paper we consider an operator different from operator in [7] by boundary condition.The main goal of the paper is to establish a formula for the second regularized trace of that operator. A formula for the first regularized trace is obtained in [19].

2. Problem statement

Let H be a separable Hilbert space. Denote by $(\cdot , \cdot)$ and $\| \cdot \|$ the scalar product and the norm in H, respectively. In the Hilbert space $L_2\left(H,\,(0,\,1)\right)$ we consider the following boundary value problem

$$l\left[y\right] \equiv -y''(t) + Ay(t) + q(t)y(t) = \lambda y(t) \tag{2.1}$$

$$y'(0) = 0 \tag{2.2}$$

$$ay(1) + y'(1) = -\lambda y'(1) \tag{2.3}$$

where A is a self-adjoint positive-definite operator in H $(A > E\,,\ E$ is an identity operator in $H)$ with a compact inverse, $q\left(t\right)$ is a selfadjoint operator-valued function in H for each t. Suppose that $q\left(t\right)$ is weakly measurable and the following conditions are satisfied:

1) There exist fourth order weak derivatives on $[0,\,1]$ denoted by $q^{(k)}\left(t\right)$ which is from $\sigma_1\left(H\right)$ and $\left\|q^{(k)}\left(t\right)\right\|_{\sigma_1(H)} \le const$ for each $t \in [0,\,1]\,,\ (k = \overline{0,4})$, $A(q)^{(k)}\left(t\right) \in \sigma_1\left(H\right),\ \left\|Aq^{(k)}\left(t\right)\right\|_{\sigma_1(H)} \le const$ for $k = 0,\,1,\,2$. Note that, $\sigma_1\left(H\right)$ is a trace class (see [16], p.88), a class of compact operators in separable Hilbert space H , whose singular values form a convergent series. In [16] this class is denoted by $\sigma_1\left(H\right)$.

2) $q'\left(0\right) = q'\left(1\right) = q\left(1\right) = 0$;

3) $\int_0^1 \left(q\left(t\right)f,\,f\right)dt = 0$ for each $f \in H$.

In direct sum $L_2 = L_2\left(H,\,(0,\,1)\right) \bigoplus H$ associate with problem (2.1)-(2.3) for $q\left(t\right) \equiv 0$ the operator L_0 defined as

$$D(L_0) = \{Y : Y = \{y(t),\,y_1\}/ - y''(t) + Ay(t) \in L_2(H,(0,1)),$$

$$y'(0) = 0,\ y_1 = -y'(1)\}, \tag{2.4}$$

$$L_0 Y = \{-y''(t) + Ay(t),\ ay(1) + y'(1)\}. \tag{2.5}$$

The operator corresponding to the case $q(t) \not\equiv 0$ denote by $L = L_0 + Q$, where $Q\{y\left(t\right),\,-y'\left(1\right)\} = \{q\left(t\right)y\left(t\right),\,0\}$. The scalar product in L_2 defined as

$$(Y,\,Z)_{L_2} = \int_0^1 (y(t),\,z(t))dt + \frac{1}{a}(y_1,\,z_1) \tag{2.6}$$

where $Y = \{y(t),\, y_1\}$, $Z = \{z(t),\, z_1\}$, $y(t),\, z(t) \in L_2(H,\, (0,\, 1))$, $y_1,\, z_1 \in H$, $a > 0$.

It is known that [6] operators L_0 and L have a discrete spectrum. Denote their eigenvalues by $\mu_1 \le \mu_2 \le \dots$ and $\lambda_1 \le \lambda_2 \le \dots$, respectively.

3. Auxiliary facts

Denote the eigenvalues and eigen-vectors of operator A by $\gamma_1 \le \gamma_2 \le \dots$ and $\varphi_1, \varphi_2, \dots$, respectively.

Let R_λ^0 be resolvent of operator L_0^2. In view of asymptotics for μ_k, it follows that R_λ^0 is from $\sigma_1(H)$. In [20] the following theorem was proved.

Theorem 3.1. *Let $D(A_0) \subset D(B)$, where A_0 is a self-adjoint positive discrete operator in separable Hilbert space H, such that $A_0^{-1} \in \sigma_1(H)$ and let B be a perturbation operator. Assume that there exist a number $\delta \in [0;\, 1)$ such that $BA_0^{-\delta}$ is continuable to bounded operator and some number $\omega \in [0;\, 1)$, $\omega + \delta < 1$, such that $A_0^{-(1-\delta-\omega)}$ is a trace class operator. Then there exist subsequence of natural numbers $\{n_m\}_{m=1}^{\infty}$ and sequence of closed contours $\Gamma_m \in \mathbb{C}$ such that for $N \ge \frac{\delta}{\omega}$*

$$\lim_{m\to\infty}\left(\sum_{j=1}^{n_m}(\mu_j - \lambda_j) + \frac{1}{2\pi i}\int_{\Gamma_m}\sum_{k=1}^{N}\frac{(-1)^{k-1}}{k} tr\,(BR_0(\lambda))^k\, d\lambda\right) = 0$$

where $\{\mu_n\}$ and $\{\lambda_n\}$ are eigenvalues of $A_0 + B$ and A_0, respectively, arranged in ascending order of their real parts, $R_0(\lambda)$ is a resolvent of A_0.

The conditions of this theorem are satisfied for L_0^2 and L^2. Really, if we take $A_0 = L_0^2$, $B = L_0Q + QL_0 + Q^2$, $\left(L^2 = A_0 + B\right)$ and $\delta = \frac{1}{2}$, provided $L_0QL_0^{-1}$ is bounded, BA_0^{-1} is also bounded and for $\omega \in [0;\, 1)$, $\omega < \frac{1}{2} - \frac{2+\alpha}{4\alpha}$, $A_0^{-(1-\delta-\omega)} = L_0^{-2(1-\delta-\omega)}$ is an operator of the trace class. Thus by statement of Theorem 3.1 for $N > \frac{1}{2\omega}$

$$\lim_{m\to\infty}\left(\sum_{n=1}^{n_m}\left(\lambda_n^2 - \mu_n^2\right) + \frac{1}{2\pi i}\int_{\Gamma_m}\sum_{k=1}^{N}\frac{(-1)^{k-1}}{k}\times\right.$$
$$\left.\times tr\left[\left(L_0Q + QL_0 + Q^2\right)R_0(\lambda)\right]^k d\lambda\right) = 0. \tag{3.1}$$

4. Second regularized trace of the operator L

Let's call

$$\lim_{m\to\infty}\left(\sum_{n=1}^{n_m}\left(\lambda_n^2 - \mu_n^2 - \int_0^1 trq^2(t)\,dt\right) + \right.$$
$$\left. + \frac{1}{2\pi i}\int_{\Gamma_m}\sum_{k=2}^{N}\frac{(-1)^{k-1}}{k} tr\left[\left(L_0Q + QL_0 + Q^2\right)R_0(\lambda)\right]^k d\lambda\right) \tag{4.1}$$

a second regularized trace of L and denote it by $\sum_{n=1}^{\infty}\left(\lambda_n^{(2)} - \mu_n^{(2)}\right)$. Further, we will show that it has finite value which doesn't depend on choice of $\{n_m\}$.

By virtue of [20, lemma 3] for great m the number of eigenvalues of L_0^2 and L^2 inside the contour Γ_m is the same and equals to n_m.

In view of (3.1)

$$\lim_{m\to\infty}\left(\sum_{n=1}^{n_m}\left(\lambda_n^2-\mu_n^2-\int_0^1 trq^2\left(t\right)dt\right)+\right.$$
$$\left.+\frac{1}{2\pi i}\int_{\Gamma_m}\sum_{k=2}^{N}\frac{(-1)^{k-1}}{k}tr\left[\left(L_0Q+QL_0+Q^2\right)R_0\left(\lambda\right)\right]^k d\lambda\right)=$$
$$=\lim_{m\to\infty}\left(-\frac{1}{2\pi i}\int_{\Gamma_m}tr\left[\left(L_0Q+QL_0+Q^2\right)R_0\left(\lambda\right)\right]d\lambda-\sum_{n=1}^{n_m}\int_0^1 trq^2\left(t\right)dt\right). \tag{4.2}$$

Denote the eigenvectors of L_0 by $\psi_1,\,\psi_2,\,\ldots$. By our assumption operator $L_0QL_0^{-1}$ is bounded, so $\left(L_0Q+QL_0+Q^2\right)R_\lambda^0$ is trace class operator and thus eigenvectors of L_0 form a basis in L_2. From (4.2) we get

$$-\frac{1}{2\pi i}\int_{\Gamma_m}tr\left[\left(L_0Q+QL_0+Q^2\right)R_0\left(\lambda\right)\right]\,d\lambda=$$
$$=\sum_{n=1}^{n_m}\left(\left[L_0Q+QL_0+Q^2\right]\psi_n,\psi_n\right)_{L_2}. \tag{4.3}$$

Note that in [19] the orthonormal eigenvectors of the operator L is obtained and are of the form:

$$\psi_n=\sqrt{\frac{4ax_{k,n}}{2ax_{k,n}+a\sin\,2x_{k,n}+4x_{k,n}^3sin^2\;x_{k,n}}}\left\{cos\left(x_{k,n}t\right)\varphi_k,x_{k,n}\sin x_{k,n}\;\varphi_k\right\},$$
$$\left(\begin{array}{c} n=\overline{0,\,\infty},\;k=\overline{N,\,\infty}\\ n=\overline{1,\,\infty},\;k=\overline{1,\,N-1}\end{array}\right) \tag{4.4}$$

where $x_{k,\,n}$ are the roots (see [19]) of the equation

$$a\cos\,z-z\sin\,z-\left(z^2+\gamma_k\right)z\sin z=0,\;\;z=\sqrt{\lambda-\gamma_k}. \tag{4.5}$$

The following lemma is true.

Lemma 4.1. *If properties 1,2 hold, and* $\gamma_k\sim gk^\alpha,\;\;0<g<\infty,\;\;2<\alpha<\infty,$ *then the following series is absolutely convergent*

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left|\left(x_{k,n}^2+\gamma_k\right)\frac{2ax_{k,n}\int_0^1\cos\,\left(2x_{k,n}t\right)f_k\left(t\right)dt}{2ax_{k,n}+a\sin 2x_{k,n}+4x_{k,n}^3sin^2\;\,x_{k,n}}\right|+$$
$$+\sum_{k=N}^{\infty}\left|\left(x_{k,0}^2+\gamma_k\right)\frac{2ax_{k,0}\int_0^1\cos\,\left(2x_{k,0}t\right)f_k\left(t\right)dt}{2ax_{k,0}+a\sin\,2x_{k,0}+4x_{k,0}^3sin^2\,x_{k,0}}\right|+$$
$$+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left|\frac{4ax_{k,n}\int_0^1cos^2\left(x_{k,n}t\right)g_k\left(t\right)dt}{2ax_{k,n}+a\sin 2x_{k,n}\;+4x_{k,n}^3sin^2\;x_{k,n}}-\int_0^1 g_k\left(t\right)dt\right|+$$

$$+\sum_{k=N}^{\infty}\left|\frac{4ax_{k,0}\int_0^1\cos^2\left(x_{k,0}t\right)g_k\left(t\right)dt}{2ax_{k,0}+a\sin\,2x_{k,0}+4x_{k,0}^3\sin^2\,x_{k,0}}-\int_0^1 g_k\left(t\right)dt\right|<\infty. \tag{4.6}$$

where $f_k\left(t\right)=\left(q\left(t\right)\varphi_k,\,\varphi_k\right),\;g_k\left(t\right)=\left(q^2\left(t\right)\varphi_k,\,\varphi_k\right).$

Proof. Let's denote the sums on the left of (4.6) by d_1, d_2, d_3, d_4 according to their order. By virtue of property 2, integrating by parts at first twice, then four times, we have

$$\int_0^1\cos\,2x_{k,n}tf_k\left(t\right)dt\;=-\frac{1}{\left(2x_{k,n}\right)^2}\int_0^1\cos\,2x_{k,n}tf_k''\left(t\right)dt. \tag{4.7}$$

$$\int_0^1\cos\,2x_{k,n}tq_k\left(t\right)dt=-\frac{1}{\left(2x_{k,n}\right)^3}f_k''\left(1\right)\sin\,2x_{k,n}-$$

$$-\frac{1}{\left(2x_{k,n}\right)^4}\cos\,2x_{k,n}tf_k'''\left(t\right)|_0^1+\frac{1}{\left(2x_{k,n}\right)^4}\int_0^1\cos\,2x_{k,n}tf_k^{(IV)}\left(t\right)dt. \tag{4.8}$$

In virtue of estimate

$$\frac{2ax_{k,0}}{2ax_{k,0}+a\sin\,2x_{k,0}+4x_{k,0}^3\sin^2\,x_{k,0}}=1+O\left(\frac{1}{x_{k,0}}\right). \tag{4.9}$$

Taking into property 1 and (4.7) we have

$$\sum_{k=N}^{\infty}\left|\frac{2ax_{k,0}\gamma_k\int_0^1\cos\,2x_{k,0}tf_k\left(t\right)dt}{2ax_{k,0}+a\sin\,2x_{k,0}+4x_{k,0}^3\sin^2\,x_{k,0}}\right|\leq$$

$$\leq\sum_{k=N}^{\infty}\gamma_k\left(1+O\left(\frac{1}{x_{k,0}}\right)\right)\int_0^1\left|f_k\left(t\right)\right|dt<\infty,$$

$$\sum_{k=N}^{\infty}\left|\frac{2ax_{k,0}^3\int_0^1\cos\,2x_{k,0}tf_k\left(t\right)dt}{2ax_{k,0}+a\sin\,2x_{k,0}+4x_{k,0}^3\sin^2\,x_{k,0}}\right|\leq$$

$$\leq\sum_{k=N}^{\infty}\left(\frac{1}{2}+O\left(\frac{1}{x_{k,0}}\right)\right)\int_0^1\left|f_k''\left(t\right)\right|dt.$$

So, we get that series denoted by d_2 is absolutely convergent.

Then by using relation (4.8) and asymptotics $x_{k,n}\sim\pi n$ (see [6]), property $\left\|Aq''\left(t\right)\right\|_{\sigma_1(H)}\leq const$ and (4.7) the following estimate holds

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left|\frac{2ax_{k,n}\gamma_k\int_0^1\cos\,2x_{k,n}tf_k\left(t\right)dt}{2ax_{k,n}+a\sin\,2x_{k,n}+4x_{k,n}^3\sin^2\,x_{k,n}}\right|=$$

$$=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\gamma_k\left(1+O\left(\frac{1}{n}\right)\right)O\left(\frac{1}{n^2}\right)\int_0^1\left|f_k''\left(t\right)\right|dt=$$

$$=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}O\left(\frac{1}{n^2}\right)\int_0^1\left|\left(Aq''\left(t\right)\varphi_k,\,\varphi_k\right)\right|dt<const \tag{4.10}$$

Since $\left\|q^{(k)}(t)\right\|_{\sigma_1(H)} \leq const\,(k=2,3,4)$, in virtue of asymptotics $x_{k,n}$ and (4.8), we get

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left|\frac{2ax_{k,n}^3\int_0^1 \cos\,2x_{k,n}tf_k(t)\,dt}{2ax_{k,n}+a\sin\,2x_{k,n}+4x_{k,n}^3\sin^2\,x_{k,n}}\right|=$$

$$=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left(1+O\left(\frac{1}{n}\right)\right)\left[\frac{1}{(2x_{k,n})^2}\left(\left|f_k'''(1)\right|+\right.\right.$$

$$\left.\left.+\left|f_k'''(0)\right|\right)+\frac{1}{(2x_{k,n})^2}\int_0^1\left|f_k^{(IV)}(t)\right|dt\right]<\infty \tag{4.11}$$

Obviously that $\sin\,2x_{k,n}=0$. From (4.10) and (4.11) it follows that series denoted by d_1 is also convergent.
Then

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left|\frac{4ax_{k,n}\int_0^1 \cos^2(x_{k,n}t)\,g_k(t)\,dt}{2ax_{k,n}+a\sin\,2x_{k,n}+4x_{k,n}^3\sin^2\,x_{k,n}}-\int_0^1 g_k(t)\,dt\right|=$$

$$=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left|\frac{2ax_{k,n}\int_0^1 (1+\cos\,2x_{k,n}t)\,g_k(t)\,dt}{2ax_{k,n}+a\sin\,2x_{k,n}+4x_{k,n}^3\sin^2\,x_{k,n}}-\int_0^1 g_k(t)\,dt\right|=$$

$$=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left|\left(1+O\left(\frac{1}{n}\right)\right)\int_0^1(1+\cos\,2x_{k,n}t)\,g_k(t)\,dt-\int_0^1 g_k(t)\,dt\right|\leq$$

$$\leq\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left(1+O\left(\frac{1}{n}\right)\right)\int_0^1\left|\cos\,2x_{k,n}tg_k(t)\right|dt+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}O\left(\frac{1}{n}\right)\int_0^1\left|g_k(t)\right|dt.$$

The last equality in virtue of (4.7) and properties $g_k(t)\in\sigma_1(H)$, $g_k''(t)\in\sigma_1(H)$ gives that series denoted by d_3 converges. d_4 is also converges:

$$\sum_{k=N}^{\infty}\left|\frac{4ax_{k,0}\int_0^1 \cos^2\,(x_{k,0}t)\,g_k(t)\,dt}{2ax_{k,0}+a\sin\,2x_{k,0}+4x_{k,0}^3\sin^2\,x_{k,0}}-\int_0^1 g_k(t)\,dt\right|=$$

$$=\sum_{k=N}^{\infty}\left|\frac{2ax_{k,0}\int_0^1 (1+\cos\,2x_{k,0}t)\,g_k(t)\,dt}{2ax_{k,0}+a\sin\,2x_{k,0}+4x_{k,0}^3\sin^2\,x_{k,0}}-\int_0^1 g_k(t)\,dt\right|\leq$$

$$\leq\sum_{k=N}^{\infty}\left(1+O\left(\frac{1}{x_{k,0}}\right)\right)\int_0^1\left|\cos\,2x_{k,0}tg_k(t)\right|dt+\sum_{k=N}^{\infty}O\left(\frac{1}{x_{k,0}}\right)\int_0^1\left|g_k(t)\right|dt$$

this completes the proof of the lemma. □

Now let's calculate the value of series called the second regularized trace. For that we prove the following theorem.

Theorem 4.1 *Let $q(t)$ be an operator-function with properties 1-3, $L_0^{-1}QL_0$ be bounded operator in L_2 and $\gamma_k\sim gk^{\alpha}$ $g>0$, $\alpha>2$, then*

$$\sum_{n=1}^{\infty}\left(\lambda_n^{(2)}-\mu_n^{(2)}\right)=\frac{trq^2\left(0\right)}{4}+\frac{trAq\left(0\right)+tr\ Aq\left(1\right)}{2}-$$
$$-\frac{trq''\left(0\right)+trq''\left(1\right)}{8}-\int_0^1 trq^2\left(t\right)dt. \tag{4.12}$$

Proof. It follows from Lemma 4.1 and relations (4.2) and (4.3) that

$$\lim_{m\to\infty}\left(\sum_{n=1}^{n_m}\left(\lambda_n^2-\mu_n^2-\int_0^1 trq^2\left(t\right)dt\right)+\right.$$
$$\left.+\frac{1}{2\pi i}\int_{\Gamma_m}\sum_{k=2}^{N}\frac{\left(-1\right)^{k-1}}{k}tr\left[\left(L_0Q+QL_0+Q^2\right)R_0\left(\lambda\right)\right]^k d\lambda\right)=$$
$$=\sum_{k=1}^{N-1}\sum_{n=1}^{\infty}\left(x_{k,n}^2+\gamma_k\right)\frac{4ax_{k,n}\int_0^1\cos\,2x_{k,n}tf_k\left(t\right)dt}{2ax_{k,n}+a\sin\,2x_{k,n}+4x_{k,n}^3\sin^2\,x_{k,n}}+$$
$$+\sum_{k=N}^{\infty}\sum_{n=0}^{\infty}\left(x_{k,n}^2+\gamma_k\right)\frac{4ax_{k,n}\int_0^1\cos\,2x_{k,n}tf_k\left(t\right)dt}{2ax_{k,n}+a\sin\,2x_{k,n}+4x_{k,n}^3\sin^2\,x_{k,n}}+$$
$$+\sum_{k=1}^{N-1}\sum_{n=1}^{\infty}\left[\frac{2ax_{k,n}\int_0^1\left(1+\cos\,2x_{k,n}t\right)g_k\left(t\right)dt}{2ax_{k,n}+a\sin\,2x_{k,n}+4x_{k,n}^3\sin^2\,x_{k,n}}-\int_0^1 g_k\left(t\right)dt\right]+$$
$$+\sum_{k=N}^{\infty}\sum_{n=0}^{\infty}\left[\frac{2ax_{k,n}\int_0^1\left(1+\cos\,2x_{k,n}t\right)g_k\left(t\right)dt}{2ax_{k,n}+a\sin\,2x_{k,n}+4x_{k,n}^3\sin^2\,x_{k,n}}-\int_0^1 g_k\left(t\right)dt\right]. \tag{4.13}$$

We first derive a formula for the fourth term on the right of (4.13). Compute the value of series

$$\sum_{n=0}^{\infty}\left[\frac{2ax_{k,n}}{2ax_{k,n}+a\sin\,2x_{k,n}+4x_{k,n}^3\sin^2\,x_{k,n}}-1\right]=$$
$$=\lim_{N\to\infty}\sum_{n=0}^{N-1}\left[\frac{2ax_{k,n}}{2ax_{k,n}+a\sin\,2x_{k,n}+4x_{k,n}^3\sin^2\,x_{k,n}}-1\right]. \tag{4.14}$$

For this at $N\to\infty$ we will investigate the asymptotics behavior of the following function

$$S_N\left(t\right)=\sum_{n=0}^{N-1}\left[\frac{2ax_{k,n}}{2ax_{k,n}+a\sin\,2x_{k,n}+4x_{k,n}^3\sin^2\,x_{k,n}}-1\right].$$

Express the $k-$th term of sum $S_N\left(t\right)$ as a residue at the pole $x_{k,n}$ of some function of complex variable z:

$$G\left(z\right)=-\frac{az}{\left(\frac{a\ ctg\,z}{z}-1-\left(z^2+\gamma_k\right)\right)z^2\sin^2\,z}. \tag{4.15}$$

This function has simple poles at the points $x_{k,n}$, πn and $z=0$.
Find the residue at $x_{k,n}$:

$$\underset{z=x_{k,n}}{res}\,G(z)=-\frac{ax_{k,n}}{x_{k,n}^2\sin^2\,x_{k,n}\left(\frac{a\;ctg\,z}{z}-1-(z^2+\gamma_k)\right)'_{z=x_{k,n}}}=$$

$$=\frac{2ax_{k,n}}{2ax_{k,n}+a\sin\,2x_{k,n}+4x_{k,n}^3\sin^2x_{k,n}}.$$

We have

$$\underset{z=\pi n}{res}\,G(z)=-\underset{z=\pi n}{res}\frac{az}{\left(\frac{a\cos\,z}{z}-\sin\,z-z^2\sin\,z-\gamma_k\sin\,z\right)z^2\sin z}=$$

$$=-\frac{a\pi n}{a\frac{\cos\,\pi n}{\pi n}\,(\pi n)^2\cos\,\pi n}=-1.$$

Take as a contour of integration the rectangle with vertices at $\pm iB$, $A_N\pm iB$, which has cut at $ix_{k,0}$ and will pas it by on the left, and the points $-ix_{k,0}$ and 0 on the right. Take also $B>x_{k,0}$. Then B will go to infinity and take $A_N=\pi N+\frac{\pi}{2}$. For such choice of A_N we have $x_{k,N-1}<A_N<x_{k,N}$ and the number of points inside of the contour of integration equals $N\left(k=\overline{0,N-1}\right)$.

One can easily show that inside this contour the function $\frac{a\;ctg\,z}{z}-1-\left(z^2+\gamma_k\right)$ has exactly N roots, so $x_{k,N-1}<A_N<x_{k,N}$.

Since $G(z)$ is an odd function of z, then the integrals along the part of contours on imaginary axis, and the integral along semicircles centered at $\pm ix_{k,0}$ vanish.

If $z=u+iv$, then for large v and for $u\geq 0$, $G(z)$ is of order $O\left(\frac{e^{2|v|t}}{|v|^3}\right)$ that is why for the given value of A_N the integrals along upper and lower sides of the contour also go to zero when $B\to\infty$.
So, we arrive at the following equality

$$S_N(t)=\frac{1}{2\pi i}\lim_{B\to\infty}\int_{A_N-iB}^{A_N+iB}G(z)\,dz+\frac{1}{2\pi i}\lim_{r\to 0}\int_{\substack{|z|=r\\ -\frac{\pi}{2}<\varphi<\frac{\pi}{2}}}G(z)\,dz. \quad (4.16)$$

As $N\to\infty$

$$\frac{1}{2\pi i}\lim_{B\to\infty}\int_{A_N-iB}^{A_N+iB}G(z)\,dz\sim\frac{1}{2\pi i}\int_{A_N-i\infty}^{A_N+i\infty}\frac{dz}{z^3\sin^2\,z}=$$

$$=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{dv}{(A_N+iv)^3\,(1-\cos\,(2A_N+2iv))}=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{dv}{(A_N+iv)^3\,(1+\cos\,2iv)}$$

$$=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{dv}{(A_N+iv)^3\,(1+\;ch\,2v)}\equiv K. \quad (4.17)$$

Then,

$$|K|=\left|\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{dv}{(A_N+iv)^3\,(1+ch\,2v)}\right|<\int_{-\infty}^{+\infty}\frac{dv}{\sqrt{\left(A_N^2+v^2\right)^3}}=$$

$$=2\int_0^{+\infty}\frac{dv}{\sqrt{\left(A_N^2+v^2\right)^3}}<\frac{2}{A_N}\int_0^{+\infty}\frac{dv}{\sqrt{A_N^2+v^2}}=$$

$$= \frac{2}{A_N} \ln \left| \frac{v}{A_N} + \sqrt{\frac{v^2}{A_N^2} + 1} \right|_0^{A_N} = \frac{const}{A_N}. \tag{4.18}$$

Therefore,

$$\int_0^1 S_N(t)\, g_k(t)\, dt = \frac{1}{2\pi i} \int_0^1 g_k(t)\, dt \int\limits_{A_N - i\infty}^{A_N + i\infty} G(z)\, dz +$$

$$+ \frac{1}{2\pi i} \lim_{r \to 0} \int_0^1 g_k(t)\, dt \int\limits_{\substack{|z| = r \\ -\frac{\pi}{2} < \varphi < \frac{\pi}{2}}} G(z)\, dz \tag{4.19}$$

We get

$$\frac{1}{2\pi i} \int\limits_{\substack{|z| = r \\ -\frac{\pi}{2} < \varphi < \frac{\pi}{2}}} G(z)\, dz =$$

$$= -\frac{1}{2\pi i} \int\limits_{\substack{|z| = r \\ -\frac{\pi}{2} < \varphi < \frac{\pi}{2}}} \frac{adz}{\frac{a}{2} \sin 2z - (1 + z^2 + \gamma_k)\, z \sin^2 z} \sim$$

$$\sim -\frac{1}{2\pi i} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{aire^{i\varphi} d\varphi}{are^{i\varphi} - \left(1 + (re^{i\varphi})^2 + \gamma_k\right)(re^{i\varphi})^3} \underset{r \to 0}{\longrightarrow} -\frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi = -\frac{1}{2} \tag{4.20}$$

So, using (4.16), (4.17), (4.18) and (4.20) in (4.19) we have

$$\lim_{N \to \infty} \int_0^1 S_N(t)\, g_k(t)\, dt = -\frac{1}{2} \int_0^1 g_k(t)\, dt. \tag{4.21}$$

Now let us derive calculations for

$$T_N(t) = \sum_{n=0}^{N-1} \frac{2ax_{k,n} \cos 2x_{k,n} t}{2ax_{k,n} + a \sin 2x_{k,n} + 4x_{k,n}^3 \sin^2 x_{k,n}}.$$

Consider the function of complex variable

$$F(z) = \frac{-az \cos 2zt}{\left(\frac{a \; ctg\, z}{z} - 1 - (z^2 + \gamma_k)\right) z^2 \sin^2 z}.$$

This function has simple poles at the points $x_{k,n}$, πn and $z = 0$:

$$\operatorname*{res}_{z = x_{k,n}} F(z) = \frac{2ax_{k,n} \cos(2x_{k,n} t)}{2ax_{k,n} + a \sin 2x_{k,n} + 4x_{k,n}^3 \sin^2 x_{k,n}}$$

and

$$\operatorname*{res}_{z = \pi n} F(z) = -\cos 2\pi n t.$$

Again take as a contour of integration the above considered contour. One can show that as $N \to \infty$

$$\frac{1}{2\pi i}\int_{A_N-i\infty}^{A_N+i\infty}\frac{az\cos 2zt\,dz}{\left(\frac{a\ ctg\,z}{z}-1-\left(z^2+\gamma_k\right)\right)z^2\sin^2 z}\sim\frac{const}{A_N} \tag{4.22}$$

From here we get

$$\lim_{N\to\infty}\int_0^1 g_k(t)\int_{A_N-i\infty}^{A_N+i\infty}F(z)\,dzdt=0. \tag{4.23}$$

By virtue of (4.23)

$$\lim_{N\to\infty}\int_0^1 T_N(t)\,g_k(t)\,dt=\lim_{N\to\infty}\int_0^1 M_N(t)\,g_k(t)\,dt+$$

$$+\frac{1}{2\pi i}\lim_{r\to 0}\int_0^1 g_k(t)\,dt\int\limits_{\substack{|z|=r\\ -\frac{\pi}{2}<\varphi<\frac{\pi}{2}}}F(z)\,dz \tag{4.24}$$

Here

$$M_N(t)=\sum_{n=1}^{N}\cos 2\pi nt.$$

The first term in (4.24) is equal to

$$\lim_{N\to\infty}\int_0^1 M_N(t)\,g_k(t)\,dt=\frac{g_k(0)+g_k(1)}{4}$$

and the second term in (4.24) as $r\to 0$ goes to $-\frac{1}{2}\int_0^1 g_k(t)\,dt$.
Other words

$$\lim_{N\to\infty}\int_0^1 T_N(t)\,g_k(t)\,dt=\frac{g_k(0)+g_k(1)}{4}-\frac{1}{2}\int_0^1 g_k(t)\,dt. \tag{4.25}$$

From (4.21) and (4.25), we have

$$\sum_{k=N}^{\infty}\sum_{n=0}^{\infty}\left(\frac{2ax_{k,n}\int_0^1\left(1+\cos 2x_{k,n}t\right)g_k(t)\,dt}{2ax_{k,n}+a\sin 2x_{k,n}+4x_{k,n}^3\sin^2 x_{k,n}}-\int_0^1 g_k(t)\,dt\right)=$$

$$=\sum_{k=N}^{\infty}\frac{g_k(0)+g_k(1)}{4}-\sum_{k=N}^{\infty}\int_0^1 g_k(t)\,dt=\sum_{k=N}^{\infty}\frac{g_k(0)}{4}-\sum_{k=N}^{\infty}\int_0^1 g_k(t)\,dt \tag{4.26}$$

(4.26) is followed from

$$g_k(1)=\left(q^2(1)\,\varphi_k,\varphi_k\right)=\left(q(1)\,\varphi_k,q(1)\,\varphi_k\right)=0.$$

By similar way we will have

$$\sum_{k=1}^{N-1}\sum_{n=1}^{\infty}\left(\frac{2ax_{k,n}\int_0^1\left(1+\cos 2x_{k,n}t\right)g_k(t)\,dt}{2ax_{k,n}+a\sin 2x_{k,n}+4x_{k,n}^3\sin^2 x_{k,n}}-\int_0^1 g_k(t)\,dt\right)=$$

$$= \sum_{k=1}^{N-1} \frac{g_k(0)}{4} - \sum_{k=1}^{N-1} \int_0^1 g_k(t)\,dt. \tag{4.27}$$

Combining (4.26) and (4.27) the sum of two last series in (4.13) gives

$$\sum_{k=1}^{N-1} \frac{g_k(0)}{4} - \sum_{k=1}^{N-1} \int_0^1 g_k(t)\,dt + \sum_{k=N}^{\infty} \frac{g_k(0)}{4} -$$

$$- \sum_{k=N}^{\infty} \int_0^1 g_k(t)\,dt = \frac{trq^2(0)}{4} - \int_0^1 trq^2(t)\,dt. \tag{4.28}$$

Note that in [19] the following is calculated

$$\sum_{k=N}^{\infty} \sum_{n=0}^{\infty} \int_0^1 \frac{2ax_{k,n} \cos\, 2x_{k,n} t f_k(t)\,dt}{2ax_{k,n} + a \sin\, 2x_{k,n} + 4x_{k,n}^3 \sin^2\, x_{k,n}} +$$

$$+ \sum_{k=1}^{N-1} \sum_{n=1}^{\infty} \int_0^1 \frac{2ax_{k,n} \cos\, 2x_{k,n} t f_k(t)\,dt}{2ax_{k,n} + a \sin\, 2x_{k,n} + 4x_{k,n}^3 \sin^2\, x_{k,n}} =$$

$$= \frac{1}{4} \sum_{k=1}^{\infty} \left[\sum_{n=0}^{\infty} \cos\, n \cdot 0 \cdot \frac{2}{\pi} \int_0^{\pi} \cos\, nz f_k\left(\frac{z}{\pi}\right) dz + \right.$$

$$\left. + \sum_{n=0}^{\infty} \cos\, n \cdot \pi \cdot \frac{2}{\pi} \int_0^{\pi} \cos\, nz f_k\left(\frac{z}{\pi}\right) dz \right]. \tag{4.29}$$

From (4.29) we get

$$\sum_{k=1}^{N-1} \sum_{n=1}^{\infty} \gamma_k \frac{4ax_{k,n} \int_0^1 \cos\, 2x_{k,n} t f_k(t)\,dt}{2ax_{k,n} + a \sin\, 2x_{k,n} + 4x_{k,n}^3 \sin^2\, x_{k,n}} +$$

$$+ \sum_{k=N}^{\infty} \sum_{n=0}^{\infty} \gamma_k \frac{4ax_{k,n} \int_0^1 \cos\, 2x_{k,n} t f_k(t)\,dt}{2ax_{k,n} + a \sin\, 2x_{k,n} + 4x_{k,n}^3 \sin^2 x_{k,n}} =$$

$$= \frac{1}{4} \sum_{k=1}^{\infty} 2\gamma_k \left[\sum_{n=0}^{\infty} \cos\, n \cdot 0 \cdot \frac{2}{\pi} \int_0^{\pi} \cos\, nz f_k\left(\frac{z}{\pi}\right) dz + \right.$$

$$\left. + \sum_{n=0}^{\infty} \cos\, n \cdot \pi \cdot \frac{2}{\pi} \int_0^{\pi} \cos\, nz f_k\left(\frac{z}{\pi}\right) dz \right] =$$

$$= \frac{trAq(0) + trAq(1)}{2} \tag{4.30}$$

and by using condition 2) we have (Note that in this case we consider as the function of complex variable $H(z) = \frac{-2az \cos\, 2zt}{\left(\frac{a\ ctg\, z}{z} - 1 - (z^2 + \gamma_k)\right) \sin^2\, z,}$ whose residues at the poles πn and $x_{k,n}$ are equal to $-2(\pi n)^2 \cos\, 2\pi nt$ and $\frac{4ax_{k,n}^3 \cos\, 2x_{k,n} t}{2ax_{k,n} + a \sin\, 2x_{k,n} + 4x_{k,n}^3 \sin^2\, x_{k,n}}$, respectively.)

$$\sum_{k=N}^{\infty}\sum_{n=0}^{\infty}\frac{4ax_{k,n}^{3}\int_0^1\cos\,2x_{k,n}tf_k\left(t\right)dt}{2ax_{k,n}+a\sin\,2x_{k,n}+4x_{k,n}^{3}\sin^2\,x_{k,n}}=$$

$$=\sum_{k=N}^{\infty}\sum_{n=0}^{\infty}\int_0^1 2\left(\pi n\ \right)^2\cos\,2\pi ntf_k\left(t\right)dt=$$

$$=-\sum_{k=N}^{\infty}\sum_{n=0}^{\infty}\int_0^1\pi n\sin\,2\pi ntf_k'\left(t\right)dt=-\sum_{k=N}^{\infty}\sum_{n=0}^{\infty}\frac{1}{2}\int_0^1\cos\,2\pi ntf_k''\left(t\right)dt=$$

$$=-\sum_{k=N}^{\infty}\sum_{n=0}^{\infty}\frac{1}{2\pi}\int_0^{\pi}\cos\,2nzf_k''\left(\frac{z}{\pi}\right)dz=$$

$$=-\frac{1}{8}\sum_{k=N}^{\infty}\sum_{n=0}^{\infty}\left[\cos\,n\cdot 0\cdot\frac{2}{\pi}\int_0^{\pi}\cos\,nzf_k''\left(\frac{z}{\pi}\right)dz+\right.$$

$$\left.+\cos\,n\cdot\pi\cdot\frac{2}{\pi}\int_0^{\pi}\cos\,nzf_k''\left(\frac{z}{\pi}\right)dz\right]=-\sum_{k=N}^{\infty}\frac{q_k''\left(0\right)+q_k''\left(1\right)}{8}\tag{4.31}$$

By similar way we will get

$$\sum_{k=1}^{N-1}\sum_{n=1}^{\infty}\frac{4ax_{k,n}^{3}\int_0^1\cos\,2x_{k,n}tf_k\left(t\right)dt}{2ax_{k,n}+a\sin\,2x_{k,n}+4x_{k,n}^{3}\sin^2 x_{k,n}}=-\sum_{k=1}^{N-1}\frac{q_k''\left(0\right)+q_k''\left(1\right)}{8}\tag{4.32}$$

From (4.31) and (4.32) we have

$$\sum_{k=1}^{N-1}\sum_{n=1}^{\infty}\frac{4ax_{k,n}^{3}\int_0^1\cos\,2x_{k,n}tf_k\left(t\right)dt}{2ax_{k,n}+a\sin\,2x_{k,n}+4x_{k,n}^{3}\sin^2 x_{k,n}}+$$

$$+\sum_{k=N}^{\infty}\sum_{n=0}^{\infty}\frac{4ax_{k,n}^{3}\int_0^1\cos\,2x_{k,n}tf_k\left(t\right)dt}{2ax_{k,n}+a\sin\,2x_{k,n}+4x_{k,n}^{3}\sin^2 x_{k,n}}=$$

$$=-\sum_{k=1}^{N-1}\frac{q_k''\left(0\right)+q_k''\left(1\right)}{8}-\sum_{k=N}^{\infty}\frac{q_k''\left(0\right)+q_k''\left(1\right)}{8}=-\frac{trq''\left(0\right)+trq''\left(1\right)}{8}.\tag{4.33}$$

Combining (4.28), (4.30) and (4.33) we get the formula (4.12). Theorem is proved. □

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