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APPROXIMATION OF ANALYTIC FUNCTIONS BY SEQUENCES OF LINEAR OPERATORS IN A POLYDISC

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Abstract. We consider the space of analytic functions in polydisc with the topology of compact convergence, and prove some theorems on the approximation and statistical approximation of functions in this space by the sequences of linear operators.

1. Introduction

Let $D_0 = \{z \in C : |z| < 1\}$ be a unit circle with center at the origin, and $D_0^m = D_0 \times ... \times D_0$ be a polydisc in space C^m. By $A(D_0^m)$ we denote the space of analytical functions in domain D_0^m, where convergence is meant as a uniform convergence in any closed domain lying inside D_0^m.

For linear operators defined on $A(D_0)$, A.D.Gadjiev [5] introduced the concept of k-positiveness and obtained Korovkin type approximation theorems for a sequence of k-positive linear operators. Various problems of approximation of analytic functions by k-positive linear operators have later been studied extensively in [1,3,6,7,9,11,12]. The approximation of analytic functions by linear operators without k-positivity has been considered in [8,10].

This paper is dedicated to the approximation of analytic functions in $A(D_0^m)$ by sequences of linear operators. The paper is structured as follows. In Section 2, we give necessary and sufficient conditions for convergence to zero of the sequence of functions in the space $A(D_0^m)$, and prove the theorems on the approximation of analytic functions by sequences of linear operators. In Section 3, we present similar results for statistical convergence.

2. Approximation of analytic functions by sequences of linear operators

It is known [see, for example, 2] that the system of functions

$$z^k = z_1^k ... z_m^k,$$

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where $z = (z_1, \ldots, z_m) \in C^m$, $k = (k_1, \ldots, k_m) \in Z_+^m$, $Z_+ = N \bigcup \{0\}$, forms a basis for $A(D_0^m)$, i. e. every function $f \in A(D_0^m)$ can be represented in the form

$$f(z) = \sum_{k \in Z_+^m} f_k z^k. \tag{2.1}$$

Note that the coefficients f_k, $k \in Z_+^m$ are defined by the formula

$$f_k = \frac{1}{(2\pi i)^m} \int_{\Gamma^m} f(z) z^{-k-1} dz$$

and satisfy the condition

$$\overline{\lim_{n\to\infty}} \left(\sum_{|k|=n} |f_k| \right)^{\frac{1}{n}} \le 1,$$

where $\Gamma^m = \left\{ z = (z_1, ..., z_m) : \ |z_i| = r, \ i = \overline{1, m} \right\}$, $0 < r < 1$, $|k| = k_1 + ... + k_m$.

First we prove the following theorem on the convergence to zero in $A(D_0^m)$ for the sequence of analytical functions.

Theorem 2.1. *The sequence $f_n(z)$ tends to zero in $A(D_0^m)$ if and only if the coefficients of expansion $f_n(z) = \sum_{k \in Z_+^m} f_k^{(n)} z^k$ for any $n \in N$, $k \in Z_+^m$ satisfy the condition*

$$\left| f_k^{(n)} \right| < \varepsilon_n \left(1 + \delta^{(n)} \right)^k,$$

where $\lim_{n\to\infty} \varepsilon_n = 0$, $\delta^{(n)} = \left(\delta_1^{(n)}, ..., \delta_m^{(n)} \right)$, $\lim_{n\to\infty} \delta_i^{(n)} = 0$, $i = \overline{1, m}$, $\left(1 + \delta^{(n)}\right)^k = \left(1 + \delta_1^{(n)}\right)^{k_1} ... \left(1 + \delta_m^{(n)}\right)^{k_m}$.

Proof. The sufficiency follows from the fact that for every $0 < r < 1$ we have

$$\|f_n\|_{A(D_0^m), r} \equiv \max_{\left\{ z \in D_0^m : |z_i| \le r, i = \overline{1,m} \right\}} |f_n(z)| < \frac{\varepsilon_n}{1 - r\left(1 + \delta^{(n)}\right)}, \tag{2.2}$$

where $\frac{1}{1 - r\left(1+\delta^{(n)}\right)} = \frac{1}{1 - r\left(1 + \delta_1^{(n)}\right)} \cdot \ldots \cdot \frac{1}{1 - r\left(1 + \delta_1^{(n)}\right)}$, and the right side of the inequality (2.2) tends to zero as $n \to \infty$.

To prove the necessity, we choose $\delta^{(n)} = \left(\delta_1^{(n)}, ..., \delta_m^{(n)} \right) \to (0, ..., 0)$ such that

$$\varepsilon_n = \max_{\left\{ z \in D_0^m : |z_i| = \frac{1}{1+\delta_i^{(n)}}, i = \overline{1,m} \right\}} |f_n(z)|$$

tends to zero as $n \to \infty$. Then the equality

$$f_k^{(n)} = \frac{1}{(2\pi i)^m} \int_{\Gamma^m} f(z) z^{-k-1} dz,$$

$$\Gamma^m = \left\{ z = (z_1, ..., z_m) : \ |z_i| = \frac{1}{1 + \delta_i^{(n)}}, \ i = \overline{1, m} \right\}$$

implies the estimate

$$\left| f_k^{(n)} \right| = \frac{1}{(2\pi)^m} \left| \int_{\Gamma^m} |f(z)| \left| z^{-k-1} \right| |dz| \right| \le \varepsilon_n \left(1 + \delta^{(n)} \right)^k.$$

Theorem is proved. □

Let the sequence $g=\{g_k\}_{k\in Z_+^m}$ of positive numbers satisfy the conditions

$$\forall k\in Z_+^m:\Delta_k(g)=\inf_{p\in Z_+^m,p\neq k}\left|\sqrt{g_k}-\sqrt{g_p}\right|>0,$$

$$\lim_{|k|\to\infty}\left(\Delta_k\left(g\right)\right)^{1/|k|}=1,\ \lim_{|k|\to\infty}\left(g_k\right)^{1/|k|}=1. \tag{2.3}$$

Definition 2.1. By $A_g\left(D_0^m\right)$ we denote the set of analytic functions $f\left(z\right)=\sum_{k\in Z_+^m}f_kz^k\in A\left(D_0^m\right)$ whose coefficients satisfy the following condition

$$|f_k|\leq M_fg_k \tag{2.4}$$

for every $k\in Z_+^m$, where M_f is a constant independent of k.

Now we consider the linear operators in $A\left(D_0^m\right)$. It follows from (2.1) that for any linear operator $T:A\left(D_0^m\right)\to A\left(D_0^m\right)$ the expansion

$$\left(Tf\right)\left(z\right)=\sum_{k\in Z_+^m}\left(\sum_{p\in Z_+^m}T_{k,p}f_p\right)z^k$$

is valid, where $f\left(z\right)=\sum_{k\in Z_+^m}f_kz^k$ and $T\left(z^p\right)=\sum_{k\in Z_+^m}T_{k,p}z^k$.

Theorem 2.2. *Let $T_n:A\left(D_0^m\right)\to A\left(D_0^m\right)$ be a sequence of linear operators*

$$\left(T_nf\right)\left(z\right)=\sum_{k\in Z_+^m}\left(\sum_{p\in Z_+^m}T_{k,p}^{(n)}f_p\right)z^k, \tag{2.5}$$

where $f\left(z\right)=\sum_{k\in Z_+^m}f_kz^k\in A\left(D_o^m\right)$. If there exist sequences ε_n and $\delta^{(n)}=\left(\delta_1^{(n)},...,\delta_m^{(n)}\right)$ satisfying the conditions

$$\lim_{n\to\infty}\varepsilon_n=0,\ \lim_{n\to\infty}\delta_i^{(n)}=0, i=\overline{1,m}, \tag{2.6}$$

such that the inequalities

$$\left|\sum_{p\in Z_+^m}T_{k,p}^{(n)}-1\right|<\varepsilon_n\left(1+\delta^{(n)}\right)^k, \tag{2.7}$$

$$\left|\sum_{p\in Z_+^m}\left|T_{k,p}^{(n)}\right|-1\right|<\varepsilon_n\left(1+\delta^{(n)}\right)^k, \tag{2.8}$$

$$\left|\sum_{p\in Z_+^m}\left|T_{k,p}^{(n)}\right|\sqrt{g_p}-\sqrt{g_k}\right|<\varepsilon_n\left(1+\delta^{(n)}\right)^k, \tag{2.9}$$

$$\left|\sum_{p\in Z_+^m}\left|T_{k,p}^{(n)}\right|g_p-g_k\right|<\varepsilon_n\left(1+\delta^{(n)}\right)^k, \tag{2.10}$$

hold, then for any function $f\in A_g\left(D_0^m\right)$ the sequence $T_nf\left(z\right)$ tends to $f\left(z\right)$ in $A\left(D_0^m\right)$.

Proof. From (2.7) - (2.10) we have

$$\sum_{p\in Z_+^m}\left|T_{k,p}^{(n)}\right|\left(\sqrt{g_p}-\sqrt{g_k}\right)^2 \le 4\varepsilon_n\left(1+\delta^{(n)}\right)^k\left(1+\sqrt{g_k}\right)^2. \qquad (2.11)$$

Consequently,

$$\sum_{p\in Z_+^m,\,p\neq k}\left|T_{k,p}^{(n)}\right| \le \frac{4\varepsilon_n\left(1+\delta^{(n)}\right)^k\left(1+\sqrt{g_k}\right)^2}{\Delta_k^2\left(g\right)}. \qquad (2.12)$$

For every function $f\left(z\right)=\sum_{k\in Z_+^m} f_k z^k \in A\left(D_o^m\right)$ we have

$$T_n f\left(z\right)-f\left(z\right)=\sum_{k\in Z_+^m}\left\{\sum_{p\in Z_+^m}T_{k,p}^{(n)}f_p-f_k\right\}z^k$$

$$=\sum_{k\in Z_+^m}\left\{\sum_{p\in Z_+^m}T_{k,p}^{(n)}-1\right\}f_k z^k$$

$$+\sum_{k\in Z_+^m}\left\{\sum_{p\in Z_+^m}T_{k,p}^{(n)}\left(f_p-f_k\right)\right\}z^k = J_n^{(1)}\left(z\right)+J_n^{(2)}\left(z\right). \qquad (2.13)$$

If $f\in A_g\left(D_o^m\right)$, then from (2.4), (2.7), (2.11) and (2.12) we have:

$$\left|J_n^{(1)}\left(z\right)\right| \le \sum_{k\in Z_+^m}\left|\sum_{p\in Z_+^m}T_{k,p}^{(n)}-1\right|\left|f_k\right|\left|z\right|^k \le M_f\varepsilon_n\sum_{k\in Z_+^m}\left(1+\delta^{(n)}\right)^k g_k\left|z\right|^k;$$

$$\left|J_n^{(2)}\left(z\right)\right| \le \sum_{k\in Z_+^m}\left\{\sum_{p\in Z_+^m}\left|T_{k,p}^{(n)}\right|\left|f_p-f_k\right|\right\}\left|z\right|^k$$

$$\le M_f\sum_{k\in Z_+^m}\left\{\sum_{p\in Z_+^m,p\neq k}\left|T_{k,p}^{(n)}\right|\left[g_p+g_k\right]\right\}\left|z\right|^k$$

$$\le M_f\sum_{k\in Z_+^m}\left\{\sum_{p\in Z_+^m,p\neq k}\left|T_{k,p}^{(n)}\right|\left[2\left(\sqrt{g_p}-\sqrt{g_k}\right)^2+3g_k\right]\right\}\left|z\right|^k$$

$$\le 4M_f\varepsilon_n\sum_{k\in Z_+^m}\left(2+\frac{3g_k}{\Delta_k^2\left(g\right)}\right)\left(1+\delta^{(n)}\right)^k\left(1+\sqrt{g_k}\right)^2\left|z\right|^k.$$

Hence, by virtue of (2.13), we obtain that the sequence $T_n f\left(z\right)-f\left(z\right)$ is uniformly converging to zero in every compact lying inside D_0^m. Theorem is proved. □

Now we state the following general result on the approximation in $A\left(D_0^m\right)$.

Theorem 2.3. *Let the sequences of positive numbers* $b = \{b_k\}_{k\in Z_+^m}$ *and* $g = \{g_k\}_{k\in Z_+^m}$ *satisfy* (2.3) *and* $T_n : A\left(D_0^m\right) \to A\left(D_0^m\right)$ *be linear operators defined by* (2.5). *If there exist sequences* ε_n *and* $\delta^{(n)} = \left(\delta_1^{(n)}, ..., \delta_m^{(n)}\right)$ *satisfying* (2.6) *such that the inequalities*

$$\left|\sum_{p\in Z_+^m} T_{k,p}^{(n)} g_p - g_k\right| < \varepsilon_n \left(1+\delta^{(n)}\right)^k, \tag{2.14}$$

$$\left|\sum_{p\in Z_+^m} \left|T_{k,p}^{(n)}\right| g_p - g_k\right| < \varepsilon_n \left(1+\delta^{(n)}\right)^k, \tag{2.15}$$

$$\left|\sum_{p\in Z_+^m} \left|T_{k,p}^{(n)}\right| g_p \sqrt{b_p} - g_k\sqrt{b_k}\right| < \varepsilon_n \left(1+\delta^{(n)}\right)^k, \tag{2.16}$$

$$\left|\sum_{p\in Z_+^m} \left|T_{k,p}^{(n)}\right| g_p b_p - g_k b_k\right| < \varepsilon_n \left(1+\delta^{(n)}\right)^k, \tag{2.17}$$

hold, then for any function $f \in A_g\left(D_0^m\right)$ *the sequence* $T_n f(z)$ *tends to* $f(z)$ *in* $A\left(D_0^m\right)$.

Proof. From (2.14) - (2.17) we have

$$\sum_{p\in Z_+^m} \left|T_{k,p}^{(n)}\right| g_p \left(\sqrt{b_p}-\sqrt{b_k}\right)^2 \le 4\varepsilon_n \left(1+\delta^{(n)}\right)^k \left(1+\sqrt{b_k}\right)^2. \tag{2.18}$$

Consequently,

$$\sum_{p\in Z_+^m,\, p\neq k} \left|T_{k,p}^{(n)}\right| g_p \le \frac{4\varepsilon_n \left(1+\delta^{(n)}\right)^k \left(1+\sqrt{b_k}\right)^2}{\Delta_k^2(b)}. \tag{2.19}$$

For every function $f(z) = \sum_{k\in Z_+^m} f_k z^k \in A\left(D_o^m\right)$ we have

$$T_n f(z) - f(z) = \sum_{k\in Z_+^m} \left\{\sum_{p\in Z_+^m} T_{k,p}^{(n)} f_p - f_k\right\} z^k$$

$$= \sum_{k\in Z_+^m} \left\{\sum_{p\in Z_+^m} T_{k,p}^{(n)} g_p - g_k\right\} \frac{f_k}{g_k} z^k$$

$$+ \sum_{k\in Z_+^m} \left\{\sum_{p\in Z_+^m} T_{k,p}^{(n)} \left(\frac{f_p}{g_p} - \frac{f_k}{g_k}\right) g_p\right\} z^k = \tilde{J}_n^{(1)}(z) + \tilde{J}_n^{(2)}(z). \tag{2.20}$$

If $f \in A_g\left(D_o^m\right)$, then from (2.4), (2.14), (2.18) and (2.19) we have:

$$\left|\tilde{J}_n^{(1)}(z)\right| \le \sum_{k\in Z_+^m} \left|\sum_{p\in Z_+^m} T_{k,p}^{(n)} g_p - g_k\right| \left|\frac{f_k}{g_k}\right| |z|^k \le M_f \varepsilon_n \sum_{k\in Z_+^m} \left(1+\delta^{(n)}\right)^k |z|^k;$$

$$\left|\tilde{J}_n^{(2)}(z)\right| \leq \sum_{k\in Z_+^m}\left\{\sum_{p\in Z_+^m}\left|T_{k,p}^{(n)}\right|\left|\frac{f_p}{g_p}-\frac{f_k}{g_k}\right|g_p\right\}|z|^k$$

$$\leq 2M_f\sum_{k\in Z_+^m}\left\{\sum_{p\in Z_+^m,p\neq k}\left|T_{k,p}^{(n)}\right|g_p\right\}|z|^k$$

$$\leq 8M_f\varepsilon_n\sum_{k\in Z_+^m}\frac{\left(1+\delta^{(n)}\right)^k\left(1+\sqrt{b_k}\right)^2|z|^k}{\Delta_k^2(b)}.$$

Hence, by virtue of (2.20), we obtain that the sequence $T_nf(z)-f(z)$ is uniformly converging to zero in every compact lying inside D_0^m. Theorem is proved. □

3. Statistical approximation of analytic functions by sequences of linear operators

Using the methods of [11], it is not difficult to obtain statistical analogues of the above theorems. Let us first recall

Definition 3.1. [4]. A sequence x_n is said to be statistically convergent to the number x if for every $\varepsilon>0$

$$\lim_{n\to\infty}\frac{|\{p\leq n:\ |x_p-x|>\varepsilon\}|}{n}=0,$$

where $|\{i\leq n:\ |x_p-x|>\varepsilon\}|$ is the number of all $p\leq n$ for which $|x_p-x|>\varepsilon$. In this case we write $st.-\lim\limits_{n\to\infty}x_n=x$.

First we prove the following theorem on the statistically convergence to zero in $A(D_0^m)$ for the sequence of analytical functions.

Theorem 3.1. *The sequence $f_n(z)$ statistically tends to zero in $A(D_0^m)$ if and only if the coefficients of expansion $f_n(z)=\sum_{k\in Z_+^m}f_k^{(n)}z^k$ for any $n\in N$, $k\in Z_+^m$ satisfy the condition*

$$\left|f_k^{(n)}\right|<\varepsilon_n\left(1+\delta^{(n)}\right)^k,$$

where $st.-\lim\limits_{n\to\infty}\varepsilon_n=0$, $\delta^{(n)}=\left(\delta_1^{(n)},...,\delta_m^{(n)}\right)$, $\lim\limits_{n\to\infty}\delta_i^{(n)}=0$, $i=\overline{1,m}$.

Proof. The sufficiency follows from the inequality (2.2). We prove the necessity. The conditions of the theorem imply that for any $\delta>0$ and $\varepsilon>0$

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{p\leq n\ :\ \|f_p\|_{A\left(D_0^m\right),\frac{1}{1+\delta}}>\varepsilon\right\}\right|=0. \tag{3.1}$$

Take $\delta=\varepsilon=\frac{1}{q}$, $q=1,2,3,....$ It follows from (3.1) that there exists a sequence $n_1<n_2<n_3<....<n_q<...$ such that if $n\geq n_q$, then

$$\frac{1}{n}\left|\left\{p\leq n\ :\ \|f_p\|_{A\left(D_0^m\right),\frac{q}{1+q}}>\frac{1}{q}\right\}\right|<\frac{1}{q}. \tag{3.2}$$

Denote $\delta_1^{(p)} = ... = \delta_m^{(p)} = \frac{1}{q}$, $\varepsilon_p = \|f_p\|_{A\left(D_0^m\right), \frac{q}{1+q}}$ for $p \in \overline{n_q, n_{q+1}-1}$, $q = 1, 2, 3, ...$ According to the definition, it follows from (3.2) that $st. - \lim\limits_{n\to\infty} \varepsilon_n = 0$ and $\lim\limits_{n\to\infty} \delta_i^{(n)} = 0$, $i = \overline{1,m}$. Then for any $n \in N$, $k \in Z_+^m$ the equality

$$f_k^{(n)} = \frac{1}{(2\pi i)^m} \int_{\Gamma^m} f(z)\, z^{-k-1} dz,$$

$$\Gamma^m = \left\{ z = (z_1, ..., z_m) : \ |z_i| = \frac{1}{1+\delta_i^{(n)}},\ i = \overline{1,m} \right\}$$

implies the estimate

$$\left| f_k^{(n)} \right| = \frac{1}{(2\pi)^m} \left| \int_{\Gamma^m} |f(z)| \left| z^{-k-1} \right| |dz| \right| \le \varepsilon_n \left(1 + \delta^{(n)} \right)^k.$$

Theorem is proved. □

Theorem 3.2. *Let the sequences of positive numbers* $g = \{g_k\}_{k\in Z_+^m}$ *satisfy* (2.3) *and* $T_n : A(D_0^m) \to A(D_0^m)$ *be linear operators defined by* (2.5). *If there exist sequences* ε_n *and* $\delta^{(n)} = \left(\delta_1^{(n)}, ..., \delta_m^{(n)}\right)$ *satisfying the conditions*

$$st. - \lim_{n\to\infty} \varepsilon_n = 0,\ \lim_{n\to\infty} \delta_i^{(n)} = 0, i = \overline{1,m}, \tag{3.3}$$

such that the inequalities (2.7)-(2.10) *hold, then for any function* $f \in A_g(D_0^m)$ *the sequence* $T_n f(z)$ *statistically tends to* $f(z)$ *in* $A(D_0^m)$.

The proof is similar to the one of Theorem 2.2 and uses Theorem 3.1.

Theorem 3.3. *Let the sequences of positive numbers* $b = \{b_k\}_{k\in Z_+^m}$ *and* $g = \{g_k\}_{k\in Z_+^m}$ *satisfy* (2.3) *and* $T_n : A(D_0^m) \to A(D_0^m)$ *be linear operators defined by* (2.5). *If there exist sequences* ε_n *and* $\delta^{(n)} = \left(\delta_1^{(n)}, ..., \delta_m^{(n)}\right)$ *satisfying* (3.3) *such that the inequalities* (2.14)-(2.17) *hold, then for any function* $f \in A_g(D_0^m)$ *the sequence* $T_n f(z)$ *statistically tends to* $f(z)$ *in* $A(D_0^m)$.

The proof is similar to the one of Theorem 2.3 and uses Theorem 3.1.

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