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A NOTE ON THE COMPLETENESS AND MINIMALITY OF WEIGHTED TRIGONOMETRIC SYSTEMS

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Abstract. In this note an example of a weight function $\omega(\cdot) \in L_p(0,\pi)$, for which the system $\{\omega(t)\cos nt\}_{Z_+}$ is complete in $L_p(0,\pi)$ but neither this system nor a system obtained by elimination of any finite number of its elements is complete and at the same time minimal in $L_p(0,\pi)$, is given.

1. Introduction

The basis properties (completeness, minimality and basicity) of systems of the form $\{\prod_{j=1}^{r}|t-t_j|^{\alpha_j}e^{int}\}_{n\in Z}$, $\{\prod_{j=1}^{r}|t-t_j|^{\alpha_j}\cos nt\}_{n\in Z_+}$ and $\{\prod_{j=1}^{r}|t-t_j|^{\alpha_j}\sin nt\}_{n\in N}$, where $r\geq 1$, have been investigated in several papers (see, for example [1-7, 9-11, 13-17]). In all of these papers either the system itself or the system obtained from the original system by elimination of finite number of its terms is complete and minimal in the corresponding L_p space. Therefore it is natural to ask if there is a weight function $\omega(\cdot)\in L_p(0,\pi)$ for which the system $\{\omega(t)\cos nt\}_{Z_+}$ is complete but neither this system nor a system obtained from it by elimination of any finite number of its elements is complete and at the same time minimal in $L_p(0,\pi)$.

In this note an example is given which shows that an answer to this question is affirmative.

2. Auxiliary facts

We will use some auxiliary facts which are of some interest in its own too. In the sequel, by Z_+, we denote the set of nonnegative integers. The following lemma holds.

Lemma 2.1. *The system $\{\omega(t)cosnt\}_{n\in Z_+}$ is complete and minimal in $L_p(0,\pi)$, $1\leq p<\infty$, space if and only if $\omega(\cdot)\in L_p(0,\pi)$ and $\frac{1}{\omega(\cdot)}\in L_q(0,\pi)$, where $\frac{1}{p}+\frac{1}{q}=1$.*

Proof. Let the system $\{\omega(t)cosnt\}_{n\in Z_+}$ be complete and minimal in $L_p(0,\pi)$. Then it is evident that $\omega(\cdot)\in L_p(0,\pi)$. It follows from the minimality that

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$\{\omega(t)cosnt\}_{n\in Z_+}$ possesses a biorthogonal system; therefore, there is a function $b(\cdot)\in L_q$ for which

$$\int_0^{\pi} b(t)\omega(t)\cos ntdt = 0, \forall n\neq 0,$$

and

$$\int_0^{\pi} b(t)\omega(t)dt = 1.$$

Last two expressions imply that

$$b(t) = \frac{c}{\omega(t)}$$

for some nonzero number c. Therefore we obtain finally that $\frac{1}{\omega(\cdot)}\in L_q(0,\pi)$ (since $b(\cdot)\in L_q$).

Now, assume that $\omega(\cdot)\in L_p(0,\pi)$ and $\frac{1}{\omega(\cdot)}\in L_q(0,\pi)$. These facts imply that $\omega(t)\neq 0$ for almost all $t\in(0,\pi)$.

Let $f(t)$ be a function that is orthogonal to the given system:

$$\int_0^{\pi} f(t)\omega(t)\cos nt\ \ dt = 0, \forall n.$$

Since the Fourier coefficients of a summable function with respect to the cosine system is unique, we obtain from last equations that $f(t)\omega(t)=0$ a.e. Therefore $f(t)=0$ a.e. This proves the completeness of the system.

It is easy to see that under the conditions of the lemma, the system constructed by

$$b_n(t) = \frac{1}{\omega(t)}\cos nt$$

is a biorthogonal to the given system. This implies that the system $\{\omega(t)cosnt\}_{n\in Z_+}$ is minimal in $L_p(0,\pi)$.

The lemma is proved. □

Lemma 2.2. *If the system $\{\omega(t)cosnt\}_{n\in Z_+}$ becomes minimal in L_p by elimination of its terms with indices $k_1,...,k_N$ then the system $\{\omega(t)cosnt\}_{n\in Z_+/\{k_1,..,k_N\}}$ has a biorthogonal system $\{b_n(t)\}_{Z_+/\{k_1,..,k_N\}}$ which is of the following form*

$$b_n(t) = \frac{\cos nt + \xi_n^{k_1}\cos k_1t + ... + \xi_n^{k_N}\cos k_Nt}{\omega(t)}, \tag{2.1}$$

where $\xi_n^{k_1}$,..., $\xi_n^{k_N}$ are some complex numbers.

Proof. The fact that $\{\omega(t)cosnt\}_{n\in Z_+/\{k_1,..,k_N\}}$ has a biorthogonal system follows from its minimality. Denote the biorthogonal system by $\{b_n(t)\}_{Z_+/\{k_1,..,k_N\}}$. Take arbitrary natural number $n\neq k_1,..,k_N$. By the definition of the biorthogonal system

$$\int_0^{\pi} b_n(t)\omega(t)\cos ktdt = 0, \forall k\neq n, k_1,..,k_N \tag{2.2}$$

and

$$\int_0^\pi b_n(t)\omega(t)\cos ntdt = 1. \tag{2.3}$$

The relations (2.2) along with the fact that the Fourier coefficients of a summable function with respect to the cosine system is unique imply that there are some complex numbers α_n and $\xi_n^{k_1}$,..., $\xi_n^{k_N}$ such that

$$b_n(t)\omega(t) = \alpha_n \cos nt + \xi_n^{k_1}\cos k_1 t + ... + \xi_n^{k_N}\cos k_N t.$$

Hence,

$$b_n(t) = \frac{\alpha_n \cos nt + \xi_n^{k_1}\cos k_1 t + ... + \xi_n^{k_N}\cos k_N t}{\omega(t)}.$$

Substituting it into (2.3) and taking into account that $\{\cos nt\}$ is an orthonormal system, we find that $\alpha_n = 1$ for all n. This proves the relation (2.1).

The lemma is proved. □

Lemma 2.3. *Let $A_0, ..., A_N$ be some complex numbers and*

$$P(t) = A_0 + ... + A_N \cos Nt.$$

If

$$P^{(n)}(t_0) = 0, \forall n \in Z_+, \tag{2.4}$$

for some $t_0 \in [0, \pi]$, then

$$A_0 = ... = A_N = 0.$$

Proof. Since

$$P(t) = A_0 + ... + A_N \cos Nt.$$

is an entire function on the whole complex plane, the relations (2.4) imply that

$$P(z) \equiv 0, \forall z \in C,$$

and in particular

$$A_0 + ... + A_N \cos Nt = 0, \forall t \in (0, \pi). \tag{2.5}$$

Note that the cosine system $\{cosnt\}$ is orthonormal on the segment $[0, \pi]$ and hence linearly independent. Therefore the relations (2.5) imply that all coefficients must be zero: $A_0 = ... = A_N = 0$.

The lemma is proved. □

Lemma 2.4. *Let $\omega(t)$ be any continuous function defined on $[0, \pi]$ that is infinitely differentiable at zero, $\omega(t) \neq 0$, a.e. and*

$$\omega^{(n)}(0) = 0, \forall n \in Z_+.$$

Then $\frac{1}{\omega(\cdot)} \notin L_q(0, \pi)$ for any number $q \geq 1$.

Proof. Take an arbitrary number $q \geq 1$.

Let k_0 be any natural number such that $k_0 \cdot q > 1$. Application of L'Hospital's rule (see, for example [8, p. 316]) implies that

$$|\frac{\omega(t)}{t^{k_0}}| < 1, \forall t \in U_0,$$

where U_0 is some neighborhood of zero. These relations imply that

$$|\frac{1}{\omega(t)}| > \frac{1}{t^{k_0}}, \forall t \in U_0.$$

We obtain from the last inequality that $\frac{1}{\omega(\cdot)} \notin L_q(U_0)$ and hence $\frac{1}{\omega(\cdot)} \notin L_q(0,\pi)$ since $\frac{1}{t^{k_0}} \notin L_q(U_0)$.

The lemma is proved. □

Remark 2.1. Actually, it is easy to see from the proof of Lemma 2.4 that, under the conditions of Lemma 2.4, $\frac{1}{\omega(\cdot)} \notin L_q(0,\alpha)$, for any $\alpha \in (0,\pi]$.

Lemma 2.5. *Let $\omega(t)$ be any continuous function defined on $[0,\pi]$ that is infinitely differentiable at zero, $\omega(t) \neq 0$, a.e. and*

$$\omega^{(n)}(0) = 0, \forall n \in Z_+.$$

Then the relation

$$\frac{A_1 \cos k_1 t + ... + A_N \cos k_N t}{\omega(t)} \in L_q$$

is possible if and only if $A_1 = ... = A_N = 0$.

Proof. Assume that

$$\frac{A_1 \cos k_1 t + ... + A_N \cos k_N t}{\omega(t)} \in L_q \tag{2.6}$$

for some choice of numbers $A_1, ..., A_N$.

Denote

$$P(t) = A_1 \cos k_1 t + ... + A_N \cos k_N t.$$

The relation (2.6) implies that $P(0) = 0$. Indeed, if $P(0) \neq 0$, then continuity of the function $P(t)$ would imply the existence of numbers $M > 0$ and $\alpha \in (0,\pi)$ for which $|P(t)| > M$ for all $t \in (0,\alpha)$. But this relation along with (2.6) implies that $\frac{1}{\omega(\cdot)} \in L_q(0,\alpha)$ which is impossible (see Remark 2.1).

Now, if at least one of the numbers $A_1, ..., A_N$ was different from zero, Lemma 2.3 would imply that there is a natural number n_0 such that

$$P(0) = ... = P^{(n_0-1)}(0) = 0, P^{(n_0)}(0) \neq 0. \tag{2.7}$$

Let k_0 be any natural number satisfying the relation $k_0 \cdot q > 1$. We put the function

$$\frac{A_1 \cos k_1 t + ... + A_N \cos k_N t}{\omega(t)}$$

in the following form:

$$\frac{A_1 \cos k_1 t + ... + A_N \cos k_N t}{\omega(t)} = \frac{\frac{P(t)}{t^{n_0}}}{t^{k_0} \frac{\omega(t)}{t^{n_0+k_0}}}. \tag{2.8}$$

Applying L'Hospital's rule and taking into account (2.7) and using the definition of $\omega(t)$, we obtain that

$$\lim_{t\to 0} \frac{P(t)}{t^{n_0}} = P^{(n_0)}(0) \neq 0,$$

and

$$\lim_{t\to 0} \frac{\omega(t)}{t^{n_0+k_0}} = 0.$$

Therefore there is a number $\alpha \in (0, \pi)$ such that

$$|\frac{P(t)}{t^{n_0}}| > \frac{|P^{(n_0)}(0)|}{2},$$

and

$$|\frac{\omega(t)}{t^{n_0+k_0}}| < \frac{|P^{(n_0)}(0)|}{2}$$

for all $t \in (0, \alpha)$. These facts along with (2.8) imply that

$$|\frac{A_1 \cos k_1 t + ... + A_N \cos k_N t}{\omega(t)}| > \frac{1}{t^{k_0}}, \tag{2.9}$$

for all $t \in (0, \alpha)$. Since $k_0 \cdot q > 1$, $\frac{1}{t^{k_0}} \notin L_q(0, \alpha)$. Therefore (2.9) implies that

$$\frac{A_1 \cos k_1 t + ... + A_N \cos k_N t}{\omega(t)} \notin L_q(0, \alpha)$$

and hence

$$\frac{A_1 \cos k_1 t + ... + A_N \cos k_N t}{\omega(t)} \notin L_q(0, \pi)$$

which contradicts to our assumption (2.6). The obtained contradiction proves the lemma.

The lemma is proved. □

3. Main result and its proof

The aim of this note is the proof of the following theorem.

Theorem 3.1. *Let $\omega(\cdot)$ be any continuous function defined on $[0, \pi]$ that is infinitely differentiable at zero, $\omega(t) \neq 0$, a.e. and*

$$\omega^{(n)}(0) = 0, \forall n \in Z_+.$$

Then the system $\{\omega(t) \cos nt\}_{n\in Z_+}$ is complete but is not minimal and can not be made complete and at the same time minimal in $L_p(0, \pi)$ by elimination of finite number of its terms.

Proof. Let $f(\cdot) \in L_q$ be a function that is orthogonal to the given system:

$$\int_0^\pi f(t)\omega(t)\cos ntdt = 0, \forall n.$$

Since $\omega(\cdot) \in L_p$ and $f(\cdot) \in L_q$, $f(\cdot)\omega(\cdot) \in L_1$. Using this fact and the fact that the Fourier coefficients of a summable function with respect to the cosine system is unique, we obtain from last equations that $f(t)\omega(t) = 0$ a.e. and hence $f(t) = 0$ a.e. Therefore the system $\{\omega(t)\cos nt\}_{n\in Z_+}$ is complete in L_p space.

Applying Lemma 2.4 and Lemma 2.1, we obtain that the system $\{\omega(t)\cos nt\}_{n\in Z_+}$ is not minimal in $L_p(0,\pi)$.

Consider the system $\{\omega(t)cosnt\}_{n\in Z_+/\{k_1,..,k_N\}}$ which is obtained from the original system $\{\omega(t)\cos nt\}_{n\in Z_+}$ by elimination of finite number of its terms with indices $k_1, ..., k_N$, where $k_1, ..., k_N$ are any nonnegative integers. Let $f(\cdot) \in L_q$ be a function that is orthogonal to the considered system:

$$\int_0^\pi f(t)\omega(t)\cos ntdt = 0, \forall n \neq k_1, ..., k_N.$$

Since $\omega(\cdot) \in L_p$ and $f(\cdot) \in L_q$, $f(\cdot)\omega(\cdot) \in L_1$. Using this fact and the fact that the Fourier coefficients of a summable function with respect to the cosine system is unique, we obtain that

$$f(t) = \frac{A_1\cos k_1t + ... + A_N\cos k_Nt}{\omega(t)},$$

where $A_1, ..., A_N$ are some constants. Since $f(\cdot) \in L_q$, Lemma 2.5 implies that $A_1 = ... = A_N = 0$ and hence $f(t) \equiv 0$. Thus the system $\{\omega(t)cosnt\}_{n\in Z_+/\{k_1,..,k_N\}}$ is complete in $L_p(0,\pi)$.

Now, if the system $\{\omega(t)cosnt\}_{n\in Z_+/\{k_1,..,k_N\}}$ is minimal, then by Lemma 2.2, it has a biorthogonal system $\{b_n(\cdot)\} \subset L_q$ that is of the form

$$b_n(t) = \frac{\cos nt + \xi_n^{k_1}\cos k_1t + ... + \xi_n^{k_N}\cos k_Nt}{\omega(t)},$$

where $\xi_n^{k_1}$,..., $\xi_n^{k_N}$ are some complex numbers. But as just was mentioned, this is impossible by Lemma 2.5. This reasoning shows that the system of elements $\{\omega(t)cosnt\}_{n\in Z_+/\{k_1,..,k_N\}}$ is not minimal.

The theorem is proved. □

It is easy to see that if $\omega(t) = 0$ on a set of a positive measure then the system $\{\omega(t)cosnt\}_{n\in Z_+}$ is not minimal and can not be made complete and minimal by elimination of finite number of elements. But in this case the original system $\{\omega(t)cosnt\}_{n\in Z_+}$ itself is not complete in L_p space.

Remark 3.1. Since the natural number N and indices $k_1, ..., k_N$ in the proof of the theorem are taken to be arbitrary numbers, nonminimality of the system $\{\omega(t)cosnt\}_{n\in Z_+/\{k_1,..,k_N\}}$ follows also directly from its completeness.

Remark 3.2. The set of functions $\omega(t)$ that are continuous on $[0,\pi]$, infinitely differentiable at zero, $\omega(t) \neq 0$, a.e. and

$$\omega^{(n)}(0) = 0, \forall n \in Z_+$$

is not empty. For example, the following function

$$\omega(t) = \begin{cases} e^{-\frac{1}{t^2}}, & \text{if } t \neq 0; \\ 0, & \text{if } t = 0. \end{cases}$$

satisfies all of these conditions (see, for example [12, p.121]).

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