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SOME EXAMPLES OF EXTREMAL FUNCTIONS ON THE FOCK SPACE $F(\mathbb{C})$

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Abstract. In this work, we recall some properties of the Fock space $F(\mathbb{C})$ and the Hilbert space H (defined by the norm $\|f\|_H = \|zf\|_{F(\mathbb{C})}$). Next, we give the best approximation of the bounded operators $L : F(\mathbb{C}) \to H$. As applications, we come up with some results regarding the approximate formulas for the difference operator, the Dunkl difference operator and the primitive operator.

1. Introduction

Fock space $F(\mathbb{C})$ (called also Segal-Bargmann space [4]) is the Hilbert space of entire functions f on $\mathbb{C}$ such that

$$\|f\|^2_{F(\mathbb{C})} := \frac{1}{\pi}\int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} \mathrm{d}x\mathrm{d}y < \infty, \quad z = x + iy.$$

This space was introduced by Bargmann in [2] and it was the tool of many works [4, 3, 11]. Especially, the differential operator $D = \mathrm{d}/\mathrm{d}z$ and the multiplication operator M by z are densely defined, closed and adjoint operators on $F(\mathbb{C})$ (see [2]).
This space is the background of some applications in this work. Especially, we give an application of the theory of reproducing kernels to the Tikhonov regularization on $F(\mathbb{C})$.

Let H be the Hilbert space of entire functions f on $\mathbb{C}$ such that

$$\|f\|_H = \|zf\|_{F(\mathbb{C})} < \infty.$$

Let $L : F(\mathbb{C}) \to H$ be a bounded operator from $F(\mathbb{C})$ into H. For $\lambda > 0$, we define on the space $F(\mathbb{C})$, the new inner product by setting

$$\langle f, g\rangle_{\lambda,F(\mathbb{C})} = \lambda\langle f, g\rangle_{F(\mathbb{C})} + \langle Lf, Lg\rangle_H.$$

Building on the ideas of Saitoh [8, 10], Matsuura et al. [6] and Yamada et al. [12], and using the theory of reproducing kernels [1, 7], we give a best approximation of the operator L. More precisely, for all $\lambda > 0$, $h \in H$, the infimum

$$\inf_{f\in F(\mathbb{C})} \left\{\lambda\|f\|^2_{F(\mathbb{C})} + \|h - Lf\|^2_H\right\},$$

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is attained at one function $f^*_{\lambda,h}$, called the extremal function, and given by

$$f^*_{\lambda,h}(w) = \langle h, LK_{L,\lambda}(w,.)\rangle_H.$$

Next we show for $f^*_{\lambda,h}$ the following property:

$$|f^*_{\lambda,h}(w)| \leq \frac{e^{|w|^2/2}}{\sqrt{2\lambda}}\|h\|_H.$$

The contents of the paper are as follows. In section 2, we recall some properties of the Fock space $F(\mathbb{C})$. Next, we give an application of the theory of reproducing kernels to the Tikhonov regularization, which gives the best approximation of the operators $L : F(\mathbb{C}) \to H$. The last section is devoted to deduce the approximate formulas for the difference operator, the Dunkl difference operator and the primitive operator.

2. Extremal functions on the Fock space $F(\mathbb{C})$

We denote by

- $H(\mathbb{C})$ the space of entire functions on $\mathbb{C}$.
- $\mathrm{d}m(z)$, the measure defined on $\mathbb{C}$, by

$$\mathrm{d}m(z) := \frac{1}{\pi}e^{-|z|^2}\mathrm{d}x\mathrm{d}y, \quad z = x + iy.$$

- $L^2(m)$, the space of measurable functions f on $\mathbb{C}$ satisfying

$$\|f\|_{L^2(m)} := \Big[\int_{\mathbb{C}} |f(z)|^2 \mathrm{d}m(z)\Big]^{1/2} < \infty.$$

We define the pre Hilbert space $F(\mathbb{C})$, to be the space of functions in $H(\mathbb{C}) \cap L^2(m)$, equipped with the inner product

$$\langle f, g\rangle_{F(\mathbb{C})} = \int_{\mathbb{C}} f(z)\overline{g(z)}\mathrm{d}m(z),$$

and the norm

$$\|f\|_{F(\mathbb{C})} = \Big[\int_{\mathbb{C}} |f(z)|^2 \mathrm{d}m(z)\Big]^{1/2}.$$

The following properties are proved in [2].

(a) If $f, g \in F(\mathbb{C})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then

$$\langle f, g\rangle_{F(\mathbb{C})} = \sum_{n=0}^{\infty} n! a_n \overline{b_n}.$$

(b) The function K given for $w, z \in \mathbb{C}$, by

$$K(w, z) = e^{\overline{w}z},$$

is the reproducing kernel for the Fock space $F(\mathbb{C})$, that is,

(i) for every $w \in \mathbb{C}$, the function $z \to K(w, z)$ belongs to $F(\mathbb{C})$;

(ii) for all $w \in \mathbb{C}$ and $f \in F(\mathbb{C})$, we have

$$\langle f, K(w, .)\rangle_{F(\mathbb{C})} = f(w). \tag{2.1}$$

(c) If $f \in F(\mathbb{C})$, then

$$|f(w)| \leq e^{|w|^2/2}\|f\|_{F(\mathbb{C})}, \quad w \in \mathbb{C}. \tag{2.2}$$

(d) The space $F(\mathbb{C})$ equipped with the inner product $\langle ., . \rangle_{F(\mathbb{C})}$ is a Hilbert space; and the set $\left\{\frac{z^n}{\sqrt{n!}}\right\}_{n\in\mathbb{N}}$ forms a Hilbertian basis for the space $F(\mathbb{C})$.

We define the pre Hilbert space H as the space of entire functions, equipped with the inner product

$$\langle f, g\rangle_H = \int_{\mathbb{C}} f(z)\overline{g(z)}|z|^2 \mathrm{d}m(z).$$

The space H satisfies the following properties.

(a) If $f, g \in H$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then

$$\langle f, g\rangle_H = \sum_{n=0}^{\infty}(n+1)!a_n\overline{b_n},$$

$$\|f\|_H^2 = \sum_{n=0}^{\infty}(n+1)!|a_n|^2.$$

(b) The space H is a subspace of the Fock space $F(\mathbb{C})$, and

$$\|f\|_{F(\mathbb{C})} \leq \|f\|_H.$$

(c) The space H equipped with the inner product $\langle ., .\rangle_H$ is a Hilbert space; and the set $\left\{\frac{z^n}{\sqrt{(n+1)!}}\right\}_{n\in\mathbb{N}}$ forms an Hilbert basis for the space H.

Let $\lambda > 0$ and let $L : F(\mathbb{C}) \to H$ be a bounded linear operator from $F(\mathbb{C})$ into H. We denote by $\langle ., .\rangle_{\lambda,F(\mathbb{C})}$ the inner product defined on the space $F(\mathbb{C})$ by

$$\langle f, g\rangle_{\lambda,F(\mathbb{C})} := \lambda\langle f, g\rangle_{F(\mathbb{C})} + \langle Lf, Lg\rangle_H \ ,$$

and the norm $\|f\|_{\lambda,F(\mathbb{C})} := \sqrt{\langle f, f\rangle_{\lambda,F(\mathbb{C})}}$.

Let $f \in F(\mathbb{C})$. From relation (2.2), we have

$$|f(w)| \leq \frac{e^{|w|^2/2}}{\sqrt{\lambda}}\|f\|_{\lambda,F(\mathbb{C})}, \quad w \in \mathbb{C}.$$

Then, the map $f \to f(w)$, $w \in \mathbb{C}$ is a continuous linear functional on $(F(\mathbb{C}), \langle ., .\rangle_{\lambda,F(\mathbb{C})})$. Thus from [1], $(F(\mathbb{C}), \langle ., .\rangle_{\lambda,F(\mathbb{C})})$ has a reproducing kernel denoted by $K_{L,\lambda}(w, z)$.

By using the theory of extremal function and reproducing kernel of Hilbert space [7, 10] we establish the extremal function associated to the operator L.

Theorem 2.1. (See [5, 6, 9]). *For any $h \in H$ and for any $\lambda > 0$, there exists a unique function $f^*_{\lambda,h}$, where the infimum*

$$\inf_{f\in F(\mathbb{C})}\left\{\lambda\|f\|^2_{F(\mathbb{C})} + \|h - Lf\|^2_H\right\}$$

is attained. Moreover, the extremal function $f^*_{\lambda,h}$ is given by

$$f^*_{\lambda,h}(w) = \langle h, LK_{L,\lambda}(w, .)\rangle_H. \tag{2.3}$$

Theorem 2.2. *For any $h \in H$ and for any $\lambda > 0$, the extremal function $f^*_{\lambda,h}$ satisfies the following inequality*

$$|f^*_{\lambda,h}(w)| \leq \frac{e^{|w|^2/2}}{\sqrt{2\lambda}} \|h\|_H.$$

Proof. Let $f \in F(\mathbb{C})$. We have

$$\begin{aligned} f(w) &= \lambda\langle f, K_{L,\lambda}(w,.)\rangle_{F(\mathbb{C})} + \langle Lf, LK_{L,\lambda}(w,.)\rangle_H \\ &= \langle f, (\lambda I + L^*L)K_{L,\lambda}(w,.)\rangle_{F(\mathbb{C})}. \end{aligned}$$

Thus, by (2.1) we obtain

$$(\lambda I + L^*L)K_{L,\lambda}(w,.) = K(w,.).$$

Furthermore the precedent relation implies that

$$\lambda^2\|K_{L,\lambda}(w,.)\|^2_{F(\mathbb{C})} + 2\lambda\|LK_{L,\lambda}(w,.)\|^2_H + \|L^*LK_{L,\lambda}(w,.)\|^2_{F(\mathbb{C})} = \|K(w,.)\|^2_{F(\mathbb{C})}.$$

From this relation and using the fact that

$$\|K(w,.)\|^2_{F(\mathbb{C})} = K(w,w) = e^{|w|^2},$$

we obtain

$$\|LK_{L,\lambda}(w,.)\|_H \leq \frac{e^{|w|^2/2}}{\sqrt{2\lambda}}.$$

From (2.3) we have

$$|f^*_{\lambda,h}(w)| \leq \|h\|_H\|LK_{L,\lambda}(w,.)\|_H \leq \frac{e^{|w|^2/2}}{\sqrt{2\lambda}}\|h\|_H,$$

which completes the proof of the theorem. □

3. Applications

3.1. The difference operator. Let L be the operator defined on $F(\mathbb{C})$ by

$$Lf(z) := \frac{1}{z}(f(z) - f(0)). \tag{3.1}$$

Theorem 3.1. (i) *The operator L maps continuously from $F(\mathbb{C})$ into H.*

(ii) *For $w, z \in \mathbb{C}$ we have*

$$K_{L,\lambda}(w,z) = \frac{1}{\lambda} + \frac{1}{\lambda+1}(e^{\overline{w}z} - 1), \tag{3.2}$$

$$LK_{L,\lambda}(w,.)(z) = \frac{1}{\lambda+1}\frac{e^{\overline{w}z} - 1}{z}. \tag{3.3}$$

(iii) *If $h \in H$ we have*

$$f^*_{\lambda,h}(w) = \frac{1}{\lambda+1}wh(w), \tag{3.4}$$

$$Lf^*_{\lambda,h}(w) = \frac{1}{\lambda+1}h(w). \tag{3.5}$$

(iv) *If $f \in F(\mathbb{C})$ we have*

$$f^*_{\lambda,Lf}(w) = \frac{1}{\lambda+1}(f(w) - f(0)). \tag{3.6}$$

Proof. (i) Let $f \in F(\mathbb{C})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then $Lf(z) = \sum_{n=0}^{\infty} a_{n+1} z^n$ and

$$\|Lf\|_H^2 = \sum_{n=0}^{\infty} (n+1)!|a_{n+1}|^2 = \sum_{n=1}^{\infty} n!|a_n|^2 \leq \|f\|_{F(\mathbb{C})}^2.$$

(ii) Let $f, g \in F(\mathbb{C})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$. Then

$$\langle f, g\rangle_{\lambda, F(\mathbb{C})} = \lambda a_0 \overline{b_0} + (\lambda+1)\sum_{n=1}^{\infty} n! a_n \overline{b_n}.$$

Thus it gives (3.2) and (3.3).

(iii) Let $h \in H$ with $h(z) = \sum_{n=0}^{\infty} c_n z^n$. From (3.3) we obtain

$$f^*_{\lambda,h}(w) = \langle h, LK_{L,\lambda}(w,.)\rangle_H = \frac{1}{\lambda+1}\sum_{n=0}^{\infty} c_n w^{n+1} = \frac{1}{\lambda+1} w h(w).$$

By (3.1) and (3.4) we obtain the relation (3.5).

(iv) The relation (3.6) follows also by (3.1) and (3.4). □

3.2. The Dunkl difference operator. Let L be the operator defined on $F(\mathbb{C})$ by

$$Lf(z) := \frac{1}{2z}(f(z) - f(-z)). \tag{3.7}$$

Theorem 3.2. (i) *The operator L maps continuously from $F(\mathbb{C})$ into H.*

(ii) *For $w, z \in \mathbb{C}$ we have*

$$K_{L,\lambda}(w,z) = \sum_{n=0}^{\infty} \frac{2(\overline{w}z)^n}{[2\lambda+1-(-1)^n]n!} = \frac{\cosh(\overline{w}z)}{\lambda} + \frac{\sinh(\overline{w}z)}{\lambda+1}, \tag{3.8}$$

$$LK_{L,\lambda}(w,.)(z) = \sum_{n=0}^{\infty} \frac{[1+(-1)^n](\overline{w})^{n+1} z^n}{[2\lambda+1+(-1)^n](n+1)!} = \frac{\sinh(\overline{w}z)}{(\lambda+1)z}. \tag{3.9}$$

(iii) *If $h \in H$ we have*

$$f^*_{\lambda,h}(w) = \frac{w}{2(\lambda+1)}[h(w) + h(-w)], \tag{3.10}$$

$$Lf^*_{\lambda,h}(w) = \frac{1}{2(\lambda+1)}[h(w) + h(-w)]. \tag{3.11}$$

(iv) *If $f \in F(\mathbb{C})$ we have*

$$f^*_{\lambda,Lf}(w) = \frac{1}{2(\lambda+1)}[f(w) - f(-w)]. \tag{3.12}$$

Proof. (i) Let $f \in F(\mathbb{C})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then

$$Lf(z) = \frac{1}{2}\sum_{n=0}^{\infty}[1+(-1)^n] a_{n+1} z^n$$

and

$$\|Lf\|_H^2 = \frac{1}{4}\sum_{n=0}^{\infty}(n+1)![1+(-1)^n]^2 |a_{n+1}|^2 \leq \|f\|_{F(\mathbb{C})}^2.$$

(ii) Let $f, g \in F(\mathbb{C})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$. Then

$$\langle f, g\rangle_{\lambda,F(\mathbb{C})} = \frac{1}{2}\sum_{n=0}^{\infty}[2\lambda + 1 - (-1)^n]n!a_n\overline{b_n}.$$

Thus it gives (3.8) and (3.9).

(iii) Let $h \in H$ with $h(z) = \sum_{n=0}^{\infty} c_n z^n$. From (3.9) we obtain

$$f^*_{\lambda,h}(w) = \langle h, LK_{L,\lambda}(w, .)\rangle_H = \sum_{n=0}^{\infty} \frac{[1 + (-1)^n]c_n}{2\lambda + 1 + (-1)^n} w^{n+1}.$$

From this relation we obtain (3.10) and (3.11).

(iv) The relation (3.12) follows directly by (3.7) and (3.10). □

3.3. The primitive operator. Let L be the operator defined for $f \in F(\mathbb{C})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, by

$$Lf(z) := \int_{[0,z]} f(w)\mathrm{d}w = \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} z^n, \tag{3.13}$$

where $[0, z] = \{tz,\ t \in [0, 1]\}$ is the segment in $\mathbb{C}$.

Theorem 3.3. (i) *The operator L maps continuously from $F(\mathbb{C})$ into H.*

(ii) *For $w, z \in \mathbb{C}$ we have*

$$K_{L,\lambda}(w, z) = \sum_{n=0}^{\infty} \frac{(n + 1)(\overline{w}z)^n}{[(\lambda + 1)(n + 1) + 1]n!}, \tag{3.14}$$

$$LK_{L,\lambda}(w, .)(z) = \sum_{n=1}^{\infty} \frac{(\overline{w})^{n-1} z^n}{[(\lambda + 1)n + 1](n - 1)!}. \tag{3.15}$$

(iii) *If $h \in H$ with $h(z) = \sum_{n=0}^{\infty} c_n z^n$ we have*

$$f^*_{\lambda,h}(w) = \sum_{n=0}^{\infty} \frac{(n + 1)(n + 2)c_{n+1}}{(\lambda + 1)(n + 1) + 1} w^n, \tag{3.16}$$

$$Lf^*_{\lambda,h}(w) = \sum_{n=1}^{\infty} \frac{(n + 1)c_n}{(\lambda + 1)n + 1} w^n. \tag{3.17}$$

(iv) *If $f \in F(\mathbb{C})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we have*

$$f^*_{\lambda,Lf}(w) = \sum_{n=0}^{\infty} \frac{(n + 2)a_n}{(\lambda + 1)(n + 1) + 1} w^n. \tag{3.18}$$

Proof. (i) Let $f \in F(\mathbb{C})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then $Lf(z) = \sum_{n=1}^{\infty}(a_{n-1}/n)z^n$ and

$$\|Lf\|^2_H = \sum_{n=1}^{\infty} \frac{(n + 1)!}{n^2} |a_{n-1}|^2 = \sum_{n=0}^{\infty} \frac{(n + 2)!}{(n + 1)^2} |a_n|^2 \leq 2\|f\|^2_{F(\mathbb{C})}.$$

(ii) Let $f, g \in F(\mathbb{C})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$. Then

$$\langle f, g\rangle_{\lambda,F(\mathbb{C})} = \sum_{n=0}^{\infty} \Big(\lambda + \frac{n + 2}{n + 1}\Big) n!a_n\overline{b_n}.$$

Thus it gives (3.14) and (3.15).

(iii) Let $h \in H$ with $h(z) = \sum_{n=0}^{\infty} c_n z^n$. From (3.15) we obtain

$$f^*_{\lambda,h}(w) = \langle h, LK_{L,\lambda}(w,.)\rangle_H = \sum_{n=1}^{\infty} \frac{n(n+1)c_n}{(\lambda+1)n+1} w^{n-1}.$$

From this relation we obtain (3.16) and (3.17).

(iv) Let $f \in F(\mathbb{C})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and let $h(z) = Lf(z)$.Thus, the relation (3.18) follows directly by (3.13) and (3.16). □

Theorem 3.4. (i) *If* $\lambda > 0$ *and* $h \in H$, *then*

$$\|f^*_{\lambda,h}\|_{F(\mathbb{C})} \leq \frac{1}{2\sqrt{\lambda}} \|h\|_H,$$

$$\|Lf^*_{\lambda,h} - (h - h_0)\|_H \leq \frac{\sqrt{\lambda}}{2} \|h\|_H, \tag{3.19}$$

where $h_0(w) = h(0)$.

(ii) *If* $f \in F(\mathbb{C})$, *then*

$$\|f^*_{\lambda,Lf} - f\|_{F(\mathbb{C})} \leq \frac{\sqrt{\lambda}}{2} \|f\|_{F(\mathbb{C})}. \tag{3.20}$$

Proof. (i) Let $\lambda > 0$ and $h \in H$ with $h(z) = \sum_{n=0}^{\infty} c_n z^n$. From (3.16) we have

$$\begin{aligned}
\|f^*_{\lambda,h}\|^2_{F(\mathbb{C})} &= \sum_{n=0}^{\infty} n! \frac{(n+1)^2(n+2)^2}{[(\lambda+1)(n+1)+1]^2} |c_{n+1}|^2 \\
&\leq \frac{1}{4\lambda} \sum_{n=1}^{\infty} (n+1)! |c_n|^2 \\
&\leq \frac{1}{4\lambda} \|h\|^2_H.
\end{aligned}$$

On the other hand from (3.17) we have

$$Lf^*_{\lambda,h}(w) - (h(w) - h(0)) = \sum_{n=1}^{\infty} \frac{-\lambda n c_n}{(\lambda+1)n+1} w^n.$$

Thus,

$$\|Lf^*_{\lambda,h} - (h - h_0)\|^2_H = \sum_{n=1}^{\infty} (n+1)! \frac{\lambda^2 n^2 |c_n|^2}{[(\lambda+1)n+1]^2} \leq \frac{\lambda}{4} \|h\|^2_H,$$

which yields (3.19).

(ii) Let $f \in F(\mathbb{C})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then from (3.18), we have

$$f^*_{\lambda,Lf}(w) - f(w) = \sum_{n=0}^{\infty} \frac{-\lambda(n+1)a_n}{(\lambda+1)(n+1)+1} w^n.$$

Consequently, we have

$$\|f^*_{\lambda,Lf} - f\|^2_{F(\mathbb{C})} = \sum_{n=0}^{\infty} n! \frac{\lambda^2(n+1)^2|a_n|^2}{[(\lambda+1)(n+1)+1]^2} \leq \frac{\lambda}{4} \|f\|^2_{F(\mathbb{C})},$$

which gives (3.20). □

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