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ON NONLINEAR BEAM EQUATION WITH INDEFINITE WEIGHT

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Abstract. We consider nonlinear fourth order problem with indefinite weight function, which attains both positive and negative values. We prove the existence of at least one negative and one positive solutions of this problem.

1. Introduction

We consider the following fourth order boundary value problem

$$u^{(4)}(t) = rg(t)f(u(t)),\ t \in (0,1), \tag{1.1}$$

$$u(0) = u(1) = u''(0) = u''(1) = 0, \tag{1.2}$$

where r is a real constant, the weight function $g \in C([0,1];\mathbb{R})$ and meas$\{t \in (0,1) : \sigma g(t) > 0\} > 0$ for each $\sigma \in \{+,-\}$, the function $f \in C(\mathbb{R};\mathbb{R})$ and satisfies the conditions: $tf(t) > 0$ for $t \in \mathbb{R}\backslash\{0\}$ and there exist $f_0, f_\infty \in (0,+\infty)$ such that

$$f_0 = \lim_{|t|\to 0} \frac{f(t)}{t},\ \ f_\infty = \lim_{|t|\to\infty} \frac{f(t)}{t} \ \text{ and } f_0 \neq f_\infty\,. \tag{1.3}$$

Problem (1.1)-(1.2), in particular, describes the deformation of an elastic beam with simple support at the end.

Note that the problem (1.1)-(1.2) is closely linked to the following linear eigenvalue problem

$$\begin{array}{l} u^{(4)}(t) = \lambda g(t)u(t),\ t \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0\,. \end{array} \tag{1.4}$$

In [13], it was shown that the problem (1.4) has exactly two principal eigenvalues, one positive and one negative, and the corresponding eigenfunctions do not change sign on $(0,1)$. But it should be noted that in the proof of this fact, the authors of [13] did not give a correct reference to the work [8]. However until recently there were no results on the multiplicities of the first m ($m > 4$) (for the definition of m, see [9]) eigenvalues and on the oscillatory properties for the corresponding eigenfunctions of the following eigenvalue problem

$$\begin{array}{l} u^{(4)}(t) + p(t)u(t) = \mu u(t),\ t \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0\,, \end{array} \tag{1.5}$$

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where $p \in C([0,1];\mathbb{R})$. In [9] it was shown that, the eigenvalues of problem (1.5) are real, and simple, except, possibly, the first m eigenvalues, and the corresponding eigenfunctions with numbers larger than m have the Sturm oscillation properties, i.e. the eigenfunction has only simple nodal zeros and the number of zeros of the eigenfunction is equal to the serial number of the corresponding eigenvalue increased by 1. But, in [13], the authors in proving Theorem 2.1 recall the work [8] and claim that the eigenfunction, corresponding to the first eigenvalue of the problem (1.5), has no zeros in the interval $(0, 1)$. Unfortunately in [8] oscillatory properties of eigenfunctions of the problem (1.4) were not studied. Recently, in the work [3, 4, 11] it was established that all eigenvalues of the problem (1.5) are simple and the corresponding eigenfunctions have the Sturm oscillation properties.

It should be noted also that in order to prove the existence of at least one solution of the problem (1.1)-(1.2) in the class of positive functions, in [13], the authors used global bifurcation results (see [13, p. 6598]) which also contains gaps. This result is similar to that for the nonlinear Sturm-Liouville problems which has been obtained by Rabinowitz [15]. In the nonlinear Sturm-Liouville problem considered in [15] nodal properties are preserved on the continuous branch of nontrivial solutions emanating from bifurcation points and this prevents the first alternative in part (ii) of [17; Lemma 2.6] occurring. But for the nonlinear fourth order problem nodal properties need not be preserved so we must consider this alternative. Therefore, in the study of nonlinear fourth-order problem there is a need to study the following questions: the construction of classes of functions that preserve the oscillation properties of eigenfunctions and their derivatives corresponding linear problem (1.4) such that if the solution of the nonlinear problem is contained on the boundary of this set, this decision must be identically zero (which means that continuous branch of solutions can not go from the boundary of the set). This question was solved in a recent paper [2], by means of which global bifurcation of solutions of the fourth-order nonlinear eigenvalue problem was studied.

The purpose of this paper is to prove the existence of at least one negative and one positive solutions of the problem (1.1)-(1.2).

In Section 2, we show that there exist two positive and negative principal eigenvalues of linear problem obtained from (1.4) and the corresponding eigenfunctions have no zeros in the interval $(0, 1)$. In Section 3, a family of sets to exploit oscillatory properties of eigenfunctions and their derivatives of corresponding linear problem (1.4) is introduced and the existence of global continua of solutions of fourth order linearizable problems contained in these sets is proved. Here we give the application of global bifurcation technique to the study of positive solutions of the problem (1.1)-(1.2).

2. On the negative and positive principal eigenvalues of linear problem (1.4)

In this section we show the existence and present the basic properties of the principal eigenvalues of the linear problem (1.4).

Theorem 2.1. *The linear spectral problem* (1.4) *has two simple principal eigenvalues* λ_1^+ *and* λ_1^-, *and the corresponding eigenfunctions* $u_1^+(t)$ *and* $u_1^-(t)$ *have no zeros in the interval* $(0,1)$.

Proof. It follows from [12; Theorem 3.6 and Corollaries 3.7, 3.8] (in the case $p = 2$ and $N = 1$) that the linear spectral problem (1.4) has two sequences of real eigenvalues

$$0 < \lambda_1^+ \leq \lambda_2^+ \leq ... \leq \lambda_k^+ \mapsto +\infty,$$

and

$$0 > \lambda_1^- \geq \lambda_2^- \geq ... \geq \lambda_k^- \mapsto -\infty$$

and no other eigenvalues.

Define the linear differential operator $L : D(L) \to L_2(0,1)$ by

$$(Lu)(t) = u^{(4)}(t)$$

and

$$D(L) = \{u \in L_2(0,1) \, : \, u \in W_2^4(0,1), \, u(0) = u(1) = u''(0) = u''(1) = 0\}.$$

For a fixed $\lambda \in \mathbb{R}$ we consider the following eigenvalue problem

$$\begin{array}{l} u^{(4)}(t) - \lambda g(t)u(t) = \mu u(t), \ t \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{array} \tag{2.1}$$

By [3; Theorem 1] (see also [4]) the problem has a sequence of real and simple eigenvalues

$$\mu_1(\lambda) < \mu_2(\lambda) < ... < \mu_k(\lambda) \mapsto +\infty\,;$$

moreover, for each $k \in \mathbb{N}$ the eigenfunction $v_{k,\,\lambda}(t)$ corresponding to the eigenvalue $\mu_k(\lambda)$ has $k-1$ simple zeros in the interval $(0,1)$.

Let

$$S_\lambda = \left\{ \int_0^1 |v''|^2 dt - \lambda \int_0^1 g(t)|v|^2 dt \, : u \in D(L), \, \int_0^1 |v|^2 dt = 1 \right\}.$$

It is clear that S_λ is bounded below. It is shown in Courant and Hilbert [6] by variational arguments that $\mu_1(\lambda) = \inf S_\lambda$. Moreover, it follows by the above argument that the eigenfunction $v_{1,\,\lambda}(t)$ corresponding to $\mu_1(\lambda)$ does not vanish in $(0,1)$. Thus, clearly, λ is a principal eigenvalue of (1.4) if and only if $\mu_1(\lambda) = 0$.

For a fixed $v \in D(L)$ the mapping

$$\lambda \to \int_0^1 |v''|^2 dt - \lambda \int_0^1 g(t)|v|^2 dt$$

is an affine and so concave function. As the infimum of any collection of concave functions is concave, it follows that $\lambda \to \mu_1(\lambda)$ is a concave function. Also, by considering test functions $v_1, v_2 \in D(L)$ such that $\int_0^1 g(t)|v_1|^2 dt > 0$ and $\int_0^1 g(t)|v_2|^2 dt < 0$, it is easy to see that $\mu_1(\lambda) \to -\infty$ as $\lambda \to \pm\infty$. Thus $\mu_1(\lambda)$ is an increasing function until it attains its maximum, and is a decreasing function thereafter.

Then, as can be seen from the variational characterization of $\mu_1(\lambda)$ or the fact that L has a positive principal eigenvalue, $\mu_1(0) > 0$ and so $\mu_1(\lambda)$ must has a graph which intersects the real axis in two points, first of which is to the left, and second to the right from origin of coordinates. Hence, problem (1.4) has exactly two simple principal eigenvalues, one positive and one negative, which coincide with the λ_1^+ and λ_1^-, respectively.

Thus, the eigenvalue λ_1^+ (λ_1^-) is simple and the corresponding eigenfunction $u_1^+(t)$ ($u_1^-(t)$) do not vanish in the interval $(0,1)$. The proof of Theorem 2.1 is complete.

3. Global bifurcation of positive or negative solutions of problem (1.1)-(1.2)

Let us denote the boundary conditions (1.2) by $B.C.$. Let E denote the Banach space $C^3[0,1]\cap B.C.$ with the usual norm $||u||_3 = ||u||_\infty + ||u'||_\infty + ||u''||_\infty + ||u'''||_\infty$, where $||u||_\infty = \max\limits_{t\in[0,1]} |u(t)|$. Let

Let

$$S = S_1 \cup S_2,$$

where

$$S_1 = \{u \in E : u^{(i)}(t) \neq 0,\ t \in (0,1),\ i = 0,\ 1,\ 2,\ 3\}$$

and

$S_2 = \{u \in E :$ there exists $i_0 \in \{0,\ 1,\ 2,\ 3\}$ and $t_0 \in (0,1)$ such that $u^{(i_0)}(t_0) = 0$, and if $u(t_0)\,u''(t_0) = 0$, then $u'(t)\,u'''(t) < 0$ in a neighborhood of t_0, and if $u'(t_0)$ $u'''(t_0) = 0$, then $u(t)u''(t) < 0$ in a neighborhood of $t_0\}$.

Note that if $u \in S$ then the Jacobian

$$J = \rho^3 \cos\psi \sin\psi$$

(see [1, 2, 5, 10]) of the Prüfer-type transformation

$$\begin{cases} u(t) = \rho(t)\sin\psi(t)\cos\theta(t), \\ u'(t) = \rho(t)\cos\psi(t)\sin\varphi(t), \\ u''(t) = \rho(t)\cos\psi(t)\cos\varphi(t), \\ u'''(t) = \rho(t)\sin\psi(t)\sin\theta(t), \end{cases}$$

does not vanish in $(0,1)$.

For each $u \in S$ we define $\rho(u,t)$, $\theta(u,t)$, $\varphi(u,t)$ $w(u,t)$ to be the continuous functions on $[0,1]$ satisfying

$$\rho(u,t) = u^2(t) + u'^2(t) + u''^2(t) + u'''^2(t),$$

$$\theta(u,t) = \operatorname{arctg}\frac{u'''(t)}{u(t)},\ \ \theta(u,0) = -\pi/2\,,$$

$$\varphi(u,t) = \operatorname{arctg}\frac{u'(t)}{u''(t)},\ \ \varphi(u,0) = \pi/2\,,$$

$$w(u,t) = \operatorname{ctg}\psi(u,t) = \frac{u''(t)\sin\theta(u,t)}{u'''(t)\cos\varphi(u,t)},\ \ w(u,0) = 0\,,$$

and $\psi(u,t) \in (0,\pi/2)$, $t \in (0,1)$.

It is apparent that ρ, θ, $\varphi, w : S \times [0,1] \to \mathbb{R}$ are continuous.

For each $\nu \in \{+\,,\,-\}$ let S_1^ν denotes the subset of $u \in S$ such that

1) $\theta(u,1) = \pi/2$;

2) $\varphi(u,1) = \frac{3\pi}{2}$;

3) for a fixed u, as t increases from 0 to 1, the function $\theta(u,t)$ ($\varphi(u,t)$) strictly increasing takes values of $m\pi/2$, $m \in \mathbb{Z}$ ($s\pi$, $s \in \mathbb{Z}$); as t decreases from 1 to 0, the function $\theta(u,t)$ ($\varphi(u,t)$), strictly decreasing takes values of $m\pi/2$, $m \in \mathbb{Z}$ ($s\pi$, $s \in \mathbb{Z}$);

4) the function $\nu u(t)$ is positive in a neighborhood of $t = 0$.

Let $S_1 = S_1^+ \cup S_1^-$. By [2; Theorem 4.4], [5; Lemma 2.2, Theorems 5.1, 5.2, 6.1, 6.3-6.5] and the proof of Theorem 2.1 we have $u_1^+ \in S_1^+$, $u_1^- \in S_1^-$, i.e the sets S_1^+, S_1^- and S_1 are nonempty. It follows immediately from the definition of these sets that they are disjoint and open in E. Moreover, by [2; Lemma 2.2] if $u(t) \in \partial S_1^\nu \cap C^4[0,1]$, $\nu \in \{+\,,\,-\}$, then $u(t)$ has at least one zero of multiplicity 4 in $(0,1)$.

Remark 3.1. Without loss of generality, we assume that u_1^+ and u_1^- lies in S_1^+ and $||u_1^\pm||_3 = 1$.

For a fixed $r > 0$ we consider the following nonlinear eigenvalue problem

$$\begin{aligned} &u^{(4)}(t) = \lambda r g(t) f(u(t)),\ t \in (0,1), \\ &u(0) = u(1) = u''(0) = u''(1) = 0. \end{aligned} \tag{3.1}$$

Lemma 3.1 [1, 2]. *If $(\lambda, u) \in \mathbb{R} \times E$ is a solution of (3.1) and $u \in \partial S_1^\nu$, $\nu \in \{+\,,\,-\}$, then $u \equiv 0$.*

Let $\mathfrak{L}$ denote the closure of the set of nontrivial solutions of (1.1)-(1.2).

Theorem 3.1. *Let $r > 0$ be fixed. Then for each $k \in \{-1,\, 1\}$ and each $\nu \in \{-\,,\,+\}$ there exists a continuum $\mathfrak{L}_k^\nu$ of solutions of problem (3.1) in $\left(\mathbb{R}^{\operatorname{sgn} k} \times S_1^\nu\right) \cup \left\{\left(\frac{\lambda_1^{\operatorname{sgn} k}}{r f_0}, 0\right)\right\} \cup \left\{\left(\frac{\lambda_1^{\operatorname{sgn} k}}{r f_\infty}, \infty\right)\right\}$ which meets $\left(\frac{\lambda_1^{\operatorname{sgn} k}}{r f_0}, 0\right)$ and $\left\{\left(\frac{\lambda_1^{\operatorname{sgn} k}}{r f_\infty}, \infty\right)\right\}$ in $\mathbb{R} \times E$, where $\mathbb{R}^{\operatorname{sgn} k} = \{\varkappa \in \mathbb{R} : 0 < \varkappa \operatorname{sgn} k \leq +\infty\}$.*

Proof. By virtue of (1.5) there exists functions $\alpha \in C(\mathbb{R},\mathbb{R})$ and $\beta \in C(\mathbb{R},\mathbb{R})$ such that

$$f(u) = f_0 u + \alpha(u),\ \ f(u) = f_\infty u + \beta(u), \tag{3.2}$$

and

$$\lim_{|u| \to 0} \frac{\alpha(u)}{u} = 0, \quad \lim_{|u| \to +\infty} \frac{\beta(u)}{u} = 0. \tag{3.3}$$

It follows from (3.2) that the problem (3.1) can be rewritten in the following form

$$\begin{aligned} &u^{(4)}(t) = \lambda r f_0 g(t) u(t) + \lambda r g(t) \alpha(u(t)),\ t \in (0,1), \\ &u(0) = u(1) = u''(0) = u''(1) = 0. \end{aligned} \tag{3.4}$$

or

$$\begin{aligned} &u^{(4)}(t) = \lambda r f_\infty g(t) u(t) + \lambda r g(t) \beta(u(t)),\ t \in (0,1), \\ &u(0) = u(1) = u''(0) = u''(1) = 0. \end{aligned} \tag{3.5}$$

Since $\lambda = 0$ is not an eigenvalue of the linear problem (1.5) for $p \equiv 0$ (i.e. of operator L) it follows that the problem (3.4) and (3.5) are equivalent to following integral equations

$$u(t) = \lambda r f_0 \int_0^1 K(t,s)g(s)u(s)ds + \lambda r \int_0^1 K(t,s)g(s)\alpha(u(s))ds, \qquad (3.6)$$

$$u(t) = \lambda r f_\infty \int_0^1 K(t,s)g(s)u(s)ds + \lambda r \int_0^1 K(t,s)g(s)\beta(u(s))ds, \qquad (3.7)$$

respectively, where $K(t,s)$ is a Green's function of differential expression $\ell(u) = u^{(4)}$ with respect to the $B.C.$.

Define $\mathcal{L} : E \to E$ by

$$(\mathcal{L}u)(t) = \int_0^1 K(t,s)g(s)u(s)ds$$

$\mathcal{F} : \mathbb{R} \times E \to E$ by

$$(\mathcal{F}(u))(t) = \int_0^1 K(t,s)g(s)\alpha(u(s))ds.$$

and $\mathcal{G} : \mathbb{R} \times E \to E$ by

$$(\mathcal{G}(u))(t) = \int_0^1 K(t,s)g(s)\beta(u(s))ds.$$

It is easily seen that the operator $\mathcal{L}$ is compact in E and the operators $\mathcal{F} : \mathbb{R} \times E \to E$ and $\mathcal{G} : \mathbb{R} \times E \to E$ are completely continuous. Thus problems (3.6) (or (3.4)) and (3.7) (or (3.5)) can be rewritten in the following equivalent forms

$$u = \lambda r f_0 \mathcal{L}u + \lambda r \mathcal{F}(u) \qquad (3.8)$$

and

$$u = \lambda r f_\infty \mathcal{L}u + \lambda r \mathcal{G}(u). \qquad (3.9)$$

By (1.3) we have

$$\mathcal{F}(u) = o(||u||_3) \text{ as } ||u||_3 \to 0, \qquad (3.10)$$

and

$$\mathcal{G}(u) = o(||u||_3) \text{ as } ||u||_3 \to +\infty\,. \qquad (3.11)$$

By virtue of (3.10) and (3.11) the linearization of (3.8) at $u = 0$ and of (3.9) at $u = \infty$ are the spectral problems

$$u = \lambda r f_0 \mathcal{L}u \qquad (3.12)$$

and

$$u = \lambda r f_\infty \mathcal{L}u, \qquad (3.13)$$

respectively. Obviously, the problem (3.12) and (3.13) are equivalent to the spectral problems

$$\begin{aligned} &u^{(4)}(t) = \lambda r f_0 g(t)u(t), \ t \in (0,1), \\ &u(0) = u(1) = u''(0) = u''(1) = 0\,, \end{aligned} \qquad (3.14)$$

and

$$u^{(4)}(t) = \lambda r f_\infty g(t) u(t),\ t \in (0,1),$$
$$u(0) = u(1) = u''(0) = u''(1) = 0, \tag{3.15}$$

respectively.

The principal eigenvalues $\frac{\lambda_1^{\operatorname{sgn} k}}{r f_0}$, $k \in \{-\,,+\}$, of problem (3.14) are the characteristic values of problem (3.12) and are simple. Hence all the conditions of Theorem 1.3 from [15] are satisfied and there exists a continua $\mathfrak{L}_{\lambda_1^{\operatorname{sgn} k}} \equiv \mathfrak{L}_k,\ k \in \{-\,,+\}$ of problem (3.8), as in Theorem 1.3 in [15]. It follows by [15; Lemma 1.24] that if $(\lambda, u) \in \mathfrak{L}_k$ and is in a small neighborhood of a point $\left(\frac{\lambda_1^{\operatorname{sgn} k}}{r f_0}, 0\right)$ then

$$u = \tau u_1^{\operatorname{sgn} k} + w, \text{ where } w = o(|\tau|) \text{ as } \tau \to 0. \tag{3.16}$$

Since S_1 open subset in E and $u_1^{\operatorname{sgn} k} \in S_1$, then we have the following relations

$$(\lambda, u) \in \mathbb{R} \times S_1 \quad \text{and} \quad \left(\mathfrak{L}_k \backslash \left\{\left(\lambda_1^{\operatorname{sgn} k}\big/ r f_0, 0\right)\right\}\right) \cap B_\xi\left(\lambda_1^{\operatorname{sgn} k}\big/ r f_0\right) \subset \mathbb{R} \times S_1, \tag{3.17}$$

for all small positive ξ, where $B_\xi\left(\lambda_1^{\operatorname{sgn} k}\big/ r f_0\right)$ is a open ball in $\mathbb{R} \times E$ of radius ξ centered at $\left(\lambda_1^{\operatorname{sgn} k}\big/ r f_0, 0\right)$. It follows from Lemma 3.1 that

$$\left(\mathfrak{L}_k \backslash \left\{\left(\lambda_1^{\operatorname{sgn} k}\big/ r f_0, 0\right)\right\}\right) \cap (\mathbb{R} \times \partial S_1) = \emptyset \tag{3.18}$$

which implies that

$$\mathfrak{L}_k \subset (\mathbb{R} \times S_1) \cup \left\{\left(\lambda_1^{\operatorname{sgn} k}\big/ r f_0, 0\right)\right\}.$$

Moreover, it should be noted that the sets $\mathfrak{L}_{-1}$ and $\mathfrak{L}_1$ do not intersect, since $\lambda = 0$ is not an eigenvalue of the operator L by virtue of which $\mathfrak{L} \cap (\{0\} \times E \backslash \{0\}) = \emptyset$. Consequently,

$$\mathfrak{L}_k \subset \left(\mathbb{R}^{\operatorname{sgn} k} \times S_1\right) \cup \left\{\left(\lambda_1^{\operatorname{sgn} k}\big/ r f_0, 0\right)\right\}$$

and alternative (ii) of Theorem 1.3 from [15] is not possible.

Now using Dancer's construction (see [7]) we decompose $\mathfrak{L}_k$, $k \in \{-1\,,1\}$, into two subcontinua $\mathfrak{L}_k^-$ and $\mathfrak{L}_k^+$, with meets $\left(\frac{\lambda_1^{\operatorname{sgn} k}}{r f_0}, 0\right)$ in $(\mathbb{R} \times S_1^-) \cup \left\{\left(\lambda_1^{\operatorname{sgn} k}\big/ r f_0, 0\right)\right\}$ and $(\mathbb{R} \times S_1^+) \cup \left\{\left(\lambda_1^{\operatorname{sgn} k}\big/ r f_0, 0\right)\right\}$, respectively. Note that if $\tau \in \mathbb{R}^\nu \backslash \{0\}$ then $\tau u_1^{\operatorname{sgn} k} \in S_1^\nu$. Hence by (3.16) and (3.17) it follows that the following inclusions

$$\left(\mathfrak{L}_k^- \backslash \left\{\left(\lambda_1^{\operatorname{sgn} k}\big/ r f_0, 0\right)\right\}\right) \cap B_\xi\left(\lambda_1^{\operatorname{sgn} k}\big/ r f_0\right) \subset \mathbb{R} \times S_1^-$$

and

$$\left(\mathfrak{L}_k^+ \backslash \left\{\left(\lambda_1^{\operatorname{sgn} k}\big/ r f_0, 0\right)\right\}\right) \cap B_\xi\left(\lambda_1^{\operatorname{sgn} k}\big/ r f_0\right) \subset \mathbb{R} \times S_1^+$$

are valid for all small positive ξ. Moreover, by the relation (3.18) we have

$$(\mathfrak{L}_k^\nu \backslash \left\{\left(\lambda_1^{\operatorname{sgn} k}\big/ r f_0, 0\right)\right\}) \cap (\mathbb{R} \times \partial S_1^\nu) = \emptyset \text{ for each } \nu \in \{-\,,+\},$$

which implies that

$$\mathfrak{L}_k^\nu \backslash \left\{\left(\lambda_1^{\operatorname{sgn} k}\big/ r f_0, 0\right)\right\} \subset \mathbb{R} \times S_1^\nu.$$

This means that $\mathfrak{L}_k^\nu \backslash \left\{\left(\lambda_1^{\operatorname{sgn} k}\big/ rf_0, 0\right)\right\}$ cannot leave $\mathbb{R}^{\operatorname{sgn} k} \times S_1^\nu$ outside of a neighborhood of $\left(\frac{\lambda_1^{\operatorname{sgn} k}}{rf_0}, 0\right)$. Note also that $S_1^- \cap S_1^+ = \emptyset$. Then it follows from [7; Theorem 2] that for each $k \in \{1, -1\}$ both the sets $\mathfrak{L}_k^+$ and $\mathfrak{L}_k^-$ are unbounded in $\mathbb{R}^{\operatorname{sgn} k} \times E$.

On the other hand, by the discussion above and [14; Theorem 2.4] (see also [16]) for each $k \in \{-1, 1\}$ there exists an unbounded component $\mathcal{D}_{\frac{\lambda_1^{\operatorname{sgn} k}}{rf_\infty}} \equiv \mathcal{D}_k \subset \mathbb{R}^{\operatorname{sgn} k} \times E$ of $\mathfrak{L}$ which contains $\left(\frac{\lambda_1^{\operatorname{sgn} k}}{rf_\infty}, \infty\right)$ and if $\Lambda \subset \mathbb{R}^{\operatorname{sgn} k}$ is an interval such that $\Lambda \cap \sigma(L, r) = \frac{\lambda_1^{\operatorname{sgn} k}}{rf_\infty}$, (where $\sigma(L, r)$ is a set of eigenvalues of problem (3.1) for $f(s) \equiv s$) and $\mathcal{M}$ is a neighborhood of $\left(\frac{\lambda_1^{\operatorname{sgn} k}}{rf_\infty}, \infty\right)$ whose projection on $\mathbb{R}^{\operatorname{sgn} k}$ lies in Λ and whose projection on E is bounded away from 0, then either

(i) $\mathcal{D}_k \backslash \mathcal{M}$ is bounded in $\mathbb{R}^{\operatorname{sgn} k} \times E$, in which case $\mathcal{D}_k \backslash \mathcal{M}$ meets $\mathbb{R}^{\operatorname{sgn} k}$, or

(ii) $\mathcal{D}_k \backslash \mathcal{M}$ is unbounded; if additionally $\mathcal{D}_k \backslash \mathcal{M}$ has a bounded projection on $\mathbb{R}^{\operatorname{sgn} k}$, then $\mathcal{D}_k \backslash \mathcal{M}$ contains $\left(\frac{\lambda_m^{\operatorname{sgn} k}}{rf_\infty}, \infty\right)$, where $m \in \mathbb{N}$ and $m > 1$.

Moreover, $\mathcal{D}_k$, $k \in \{-, +\}$, can be decomposed into two subcontinua $\mathcal{D}_k^-$, $\mathcal{D}_k^+$ and there exists a neighborhood $Q \subset \mathcal{M}$ of $\left(\frac{\lambda_1^{\operatorname{sgn} k}}{rf_\infty}, \infty\right)$ such that $(\lambda, u) \in \mathcal{D}_k^- (\mathcal{D}_k^+) \cap Q$ and $(\lambda, u) \neq \left(\frac{\lambda_1^{\operatorname{sgn} k}}{rf_\infty}, \infty\right)$ implies

$$(\lambda, u) = (\lambda, \tau u_1^{\operatorname{sgn} k} + w),$$

where

$$\tau < 0 \ (\tau > 0) \text{ and } |\lambda - \lambda_1^k| = o(1), \ w = o(|\tau|) \text{ at } |\tau| = \infty.$$

Consequently,

$$\text{if } (\lambda, u) \in \mathcal{D}_k^\nu \backslash Q, \text{ then } (\lambda, u) \in \mathbb{R}^{\operatorname{sgn} k} \times S_1^\nu.$$

Let

$$(\lambda_n, u_n) \in \mathfrak{L}_k^\nu \text{ and } |\lambda_n| + ||u_n||_3 \to \infty \text{ as } n \to \infty.$$

We note that $\lambda_n \operatorname{sgn} k > 0$ for all $n \in \mathbb{N}$, since $\mathfrak{L} \cap (\{0\} \times E \backslash \{0\}) = \emptyset$. As in the proof of Theorem 1.1 from [13] we can prove that there exists a positive constant M such that

$$|\lambda_n| \leq M, \ n \in \mathbb{N},$$

which implies that

$$||u_n||_3 \to \infty \text{ as } n \to \infty.$$

It is obvious that

$$u_n = \lambda_n r f_\infty \mathcal{L} u_n + \lambda_n r \mathcal{G}(u_n). \tag{3.19}$$

Let $v_n = \frac{u_n}{||u_n||_3}$. Then v_n satisfies the relations

$$v_n = \lambda_n r f_\infty \mathcal{L} v_n + \lambda_n r \frac{\mathcal{G}(u_n)}{||u_n||_3} \tag{3.20}$$

By virtue of the completely continuity of the operators $\mathcal{L}$ and $\mathcal{G}$, and the boundedness of $\{\lambda_n\}_{n=1}^\infty$ it follows that there exists a subsequence of the sequence

$\{(\lambda_n, v_n)\}_{n=1}^{\infty}$ (which we will relabel as $\{(\lambda_n, v_n)\}_{n=1}^{\infty}$) which is convergent to $(\tilde{\lambda}, v)$ in $\mathbb{R}^{\operatorname{sgn} k} \times E$, with $||v||_3 = 1$, $v \in S_1^{\nu}$ and

$$v = \tilde{\lambda} r f_{\infty} \mathcal{L} v. \tag{3.21}$$

Hence by Theorem 2.1 we have

$$\tilde{\lambda} = \frac{\lambda_1^{\operatorname{sgn} k}}{r f_{\infty}}.$$

Thus

$$(\lambda_n, u_n) \to \left(\frac{\lambda_1^{\operatorname{sgn} k}}{r f_{\infty}}, \infty \right) \text{ as } n \to \infty .$$

Then it follows from [14; Corollary of Theorem 2.4] that $\mathcal{D}_k^{\nu}$ contains a subcontinuum $\mathfrak{D}_k^{\nu}$ lying in $\mathbb{R} \times S_1^{\nu}$ which is bounded and intersects the line $\mathcal{R} = \{(\lambda, 0) \in \mathbb{R} \times E\}$ of trivial solutions at $\left(\frac{\lambda_1^{\operatorname{sgn} k}}{r f_0}, 0 \right)$. Consequently, $\mathfrak{L}_k^{\nu} = \mathfrak{D}_k^{\nu}$. The proof of this theorem is complete.

Corollary 3.1. *Let $r < 0$ be fixed. Then for each $k \in \{-1, 1\}$ and each $\nu \in \{-, +\}$ there exists a continuum $\mathfrak{F}_k^{\nu}$ of solutions of problem*

$$\begin{aligned} &u^{(4)}(t) = -\lambda r g(t) f(u(t)), \ t \in (0,1), \\ &u(0) = u(1) = u''(0) = u''(1) = 0. \end{aligned} \tag{3.22}$$

in $\left(\mathbb{R}^{\operatorname{sgn}(-k)} \times S_1^{\nu}\right) \cup \left\{ \left(\frac{\lambda_1^{\operatorname{sgn} k}}{-r f_0}, 0 \right) \right\} \cup \left\{ \left(\frac{\lambda_1^{\operatorname{sgn} k}}{-r f_{\infty}}, \infty \right) \right\}$ *which meets* $\left(\frac{\lambda_1^{\operatorname{sgn} k}}{-r f_0}, 0 \right)$ *and* $\left\{ \left(\frac{\lambda_1^{\operatorname{sgn} k}}{-r f_{\infty}}, \infty \right) \right\}$ *in* $\mathbb{R} \times E$.

Theorem 3.2. *Let*

$$r \in \left(\frac{\lambda_1^-}{f_0}, \frac{\lambda_1^-}{f_{\infty}} \right) \cup \left(\frac{\lambda_1^+}{f_{\infty}}, \frac{\lambda_1^+}{f_0} \right)$$

in the case $f_0 < f_{\infty}$, or

$$r \in \left(\frac{\lambda_1^-}{f_{\infty}}, \frac{\lambda_1^-}{f_0} \right) \cup \left(\frac{\lambda_1^+}{f_0}, \frac{\lambda_1^+}{f_{\infty}} \right)$$

in the case $f_{\infty} < f_0$. Then the problem (1.1)-(1.2) *has at least one negative and one positive solutions.*

Proof. Let $f_0 < f_{\infty}$ and $r \in \left(\frac{\lambda_1^-}{f_0}, \frac{\lambda_1^-}{f_{\infty}} \right) \cup \left(\frac{\lambda_1^+}{f_{\infty}}, \frac{\lambda_1^+}{f_0} \right)$. Then

$$0 < \frac{\lambda_1^-}{-r f_0} < 1 < \frac{\lambda_1^-}{-r f_{\infty}} \quad \text{if} \quad r < 0, \tag{3.23}$$

$$0 < \frac{\lambda_1^+}{r f_{\infty}} < 1 < \frac{\lambda_1^+}{r f_0} \quad \text{if} \quad r > 0. \tag{3.24}$$

In the case $r < 0$ by Corollary 3.1 it follows that for each $\nu \in \{-, +\}$ the continuum $\mathfrak{F}_{-1}^{\nu}$ of solutions of problem (3.22) joins $\left(\frac{\lambda_1^-}{-r f_0}, 0 \right)$ and $\left(\frac{\lambda_1^-}{-r f_{\infty}}, \infty \right)$ in $\mathbb{R}^+ \times S_1^{\nu}$, and in the case $r > 0$ it follows by Theorem 3.1 that for each $\nu \in \{-, +\}$ the continuum $\mathfrak{L}_1^{\nu}$ of solutions of problem (3.1) joins $\left(\frac{\lambda_1^+}{r f_0}, 0 \right)$ and $\left(\frac{\lambda_1^+}{r f_{\infty}}, \infty \right)$ in $\mathbb{R}^+ \times S_1^{\nu}$. Then by (3.23) ((3.24)) for each $\nu \in \{-, +\}$ the continuum $\mathfrak{F}_{-1}^{\nu}$ ($\mathfrak{L}_1^{\nu}$)

crosses the hyperplane $\{1\} \times S_1^{\nu}$ in $\mathbb{R} \times E$. Hence there exists a solution w^{ν} of problem (1.1)-(1.2) which is contained in S_1^{ν}.

The case $f_\infty < f_0$ and $r \in \left(\frac{\lambda_1^-}{f_\infty}, \frac{\lambda_1^-}{f_0}\right) \cup \left(\frac{\lambda_1^+}{f_0}, \frac{\lambda_1^+}{f_\infty}\right)$ is considered similarly. The proof of Theorem 3.2 is complete.

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