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THE WELL-POSEDNESS OF THE CAUCHY PROBLEM FOR A SYSTEM OF THERMOELASTICITY WITH SINGULAR COEFFICIENTS

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Abstract. We consider the Cauchy problem for a system of thermoelasticity with some non-Lipschitz coefficients. We prove well-posedness of the corresponding Cauchy problem in some functional spaces.

1. Introduction and main results.

The various problems of thermoelasticity are reduced to the Cauchy problem or the mixed problem of hyperbolic-parabolic coupled systems. When coefficients are sufficiently smooth, these problems where studied by various authors [5], [8]-[20]. In the mentioned works the well-posedness of the corresponding Cauchy problem or the mixed problem, and the behavior of solutions, were investigated. In this paper, we consider the Cauchy problem:

$$\begin{cases} u_{tt} - a(t)\Delta u + div\theta = 0 \\ \theta_t - \Delta\theta + div\ u_t = 0 \end{cases} \quad t \in [0,T]\ ,\ x \in R^n \tag{1.1}$$

with initial conditions

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad \theta(0,x) = \theta_0(x). \tag{1.2}$$

It is known that if $a(t) \geq a_0 > 0$ satisfies the Lipschitz condition and $u_0 \in H^s(R^n)$, $u_1 \in H^{s-1}(R^n)$, $\theta_0 \in H^s(R^n)$, then problem (1.1)-(1.2) has a unique solution $u \in C\left([0,T]\,; H^s(R^n)\right) \bigcap C^1\left([0,T]\,; H^{s-1}(R^n)\right)$ $\theta \in C\left([0,T]\,, H^s(R^n)\right)$ [1]-[4].

If we reject the Lipschitz condition, then this result, generally speaking is not valid. In this paper our interest is to investigate the case when $a(t)$ is not from the class C^1 at $t = 0$. Naturally, to investigate such problem there must be some condition on the singularity order of $a'(t)$ at $t \to 0$. But then it is necessary to understand the relation between this singular behavior and the right classes of well-posedness. Indeed, the results for the hyperbolic equation with singular coefficient in this direction for example were obtained in the works [1]-[4], [7]. Studies of hyperbolic-parabolic coupled system in this direction are few. In the article [6], the loss of smoothness of the solutions to the one class system of thermoelasticity with logarithmical Lipschitz coefficient is investigated. In this

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paper we investigate the Cauchy problem for the system (1.1), (1.2) with singular coefficients $a(t)$.

At first we give some auxiliary notations. $\bigcap\limits_{\beta>0} H^{\beta}(R^n)$ is denoted by H^{∞}.

For $s \geq 1$ we'll denote by $\gamma_{\beta}^{(s)}$ the functional space with the norm $\|f\|_{\gamma_{\beta}^{s}} = \left\{\int_{R^n} \exp(\beta\,|\xi|^{\frac{1}{s}}) \left|\hat{f}(\xi)\right|^2 d\xi\right\}^{\frac{1}{2}}$, where $\hat{f}(\xi)$ is the Fourier transformation of $f(x)$, i.e. $\hat{f}(\xi) = F\left[f\right](\xi)$

By $\gamma^{(s)}$ we denote $\gamma^{(s)} = \bigcap\limits_{\beta>0} \gamma_{\beta}^{(s)}$.

Theorem 1.1. *Let*

$$a(t) \in C^1\left(0,T\right], \;\; a(t) \geq a_0, \;\; t \in \left(0,T\right], \tag{1.3}$$

$$t \cdot |a'(t)| \leq c, t \in (0,T] \tag{1.4}$$

where $a_0, c \in (0,\infty)$, *then Cauchy problem* (1.1), (1.2) *is well-posed in* $H^{\infty} \times H^{\infty} \times H^{\infty}$.

Theorem 1.2. *Let* (1.3) *be satisfied. Suppose that there exist* $c_1, c_2 > 0$ *such that, for all* $t \in [0,T]$

$$t^q \left|a'(t)\right| \leq c_1, \tag{1.5}$$

$$t^p a(t) \leq c_2, \tag{1.6}$$

where $q > 1$ *and* $p \in [0,1)$ *with* $p < q-1$, *then the Cauchy problem* (1.1)-(1.2) *is* $\gamma^{(s)} \times \gamma^{(s)} \times \gamma^{(s)}$ *well-posed for all* $s < \frac{q-p}{q-1}$.

2. Proof of theorem 1.1.

The proof of theorem 1.1 and 1.2 is carried out by standard regularization method that based on some energetic estimation.

First of all, we observe that (1.4) implies that $a(\cdot) \in L_1(0,T)$, and there exist $c_1, c_2 > 0$ such that

$$|a(t)| \leq c_1 + c_2 \ln(1 + \frac{1}{t}), \tag{2.1}$$

We define for all $(t,\xi) \in [0,T] \times R^n \setminus \{0\}$

$$\tilde{a}(t,\xi) = \begin{cases} a(T) & if \quad T\,|\xi| \leq 1, \\ a(|\xi|^{-1}) & if \quad T\,|\xi| > 1 \;\; and \quad t\,|\xi| \leq 1, \\ a(t) & if \quad t\,|\xi| > 1, \end{cases} \tag{2.2}$$

For all $(t,\xi) \in [0,T] \times R^n \setminus \{0\}$ we define also

$$d(t,\xi) = \begin{cases} |\tilde{a}(t,\xi) - a(t)| \cdot |\xi| & if \quad t\,|\xi| \leq 1, \\ \frac{|a'(t)|}{a(t)} & if \quad t\,|\xi| > 1, \end{cases} \tag{2.3}$$

and $k_{\lambda}(t,\xi) = (1 + |\xi|^2)^{\lambda} \exp(-\int_0^t d(s,\xi)ds)$.

We denote by $v(t,\xi)$ and $w(t,\xi)$ a Fourier transformation of $u(t,x)$ and $\theta(t,x)$, respectively with respect to the variable x, i.e. $v(t,\xi) = F[u](t,\xi)$, $w(t,\xi) = F[\theta](t,\xi)$. Let's define the weighted energetic function in the following way

$$E_\lambda(t) = \int\limits_{R_n} E_\lambda(t,\xi)d\xi,$$

where

$$E_\lambda(t,\xi) = \left[|\dot{v}(t,\xi)|^2 + (1+\tilde{a}(t,\xi)|\xi|^2)\cdot|v(t,\xi)|^2 + \cdot|w(t,\xi)|^2\right]\cdot k_\lambda(t,\xi) \tag{2.4}$$

From (2.1) and (2.2) we deduced that for all $(t,\xi) \in [0,T] \times R^n \setminus \{0\}$

$$\tilde{a}(t,\xi) \le c_3 + c_4 \ln(1+|\xi|). \tag{2.5}$$

Lemma 2.1. *There exists $M > 0$, such that*

$$E_\lambda(t) \le ME_\lambda(0), t \in [0,T].$$

Proof. After simple transformations we get

$$\frac{dE_\lambda(t,\xi)}{dt} = \big[2Re\dot{v}(t,\xi).\overline{\dot{v}}(t,\xi) + 2Rew(t,\xi).\dot{\bar{w}}(t,\xi)$$
$$+\dot{\tilde{a}}(t,\xi)|\xi|^2\cdot|v(t,\xi)|^2 + \left(1+\tilde{a}(t,\xi)|\xi|^2\right)\ 2\cdot Rev(t,\xi).\dot{\bar{v}}(t,\xi)\Big]$$
$$\times k(t,\xi) - E_\lambda(t,\xi)\cdot d(t,\xi). \tag{2.6}$$

Since

$$Re\sum_{k=1}^{n} i\xi_k\left[w\cdot\dot{\bar{v}} + \dot{w}\cdot\bar{v}\right] = 0,$$

the equalities (1.1) and (2.6) show that

$$\frac{dE_\lambda(t,\xi)}{dt} = \Big[[1+(\tilde{a}(t,\xi)-a(t))|\xi|^2]\,|.v(t,\xi)|^2$$
$$+\,\dot{\tilde{a}}(t,\xi)\cdot|\xi|^2|v(t,\xi)|^2 - |\xi|^2\cdot|w(t,\xi)|^2\Big]\cdot k(t,\xi) - E_\lambda(t,\xi)d(t,\xi). \tag{2.7}$$

If $t|\xi| > 1$,by (2.2) and (2.3) we obtain that

$$\tilde{a}(t,\xi) - a(t) = 0, d(t,\xi) = \frac{|\dot{a}(t)|}{a(t)} \tag{2.8}$$

and hence

$$\dot{\tilde{a}}(t,\xi)|\xi|^2 k(t,\xi) - E_\lambda(t,\xi)d(t,\xi) \le (\dot{a}(t) - |\dot{a}(t)|)\ |\xi|^2\ k(t,\xi) \le 0. \tag{2.9}$$

By virtue of (1.1), (2.8) and (2.9), it follows from (2.7) that

$$\frac{dE_\lambda(t,\xi)}{dt} \le 0. \tag{2.10}$$

As $t|\xi| < 1$, then we have

$$2(\dot{\tilde{a}}(t,\xi) - a(t))Re\dot{v}(t,\xi)\bar{v}(t,\varsigma) - (\dot{\tilde{a}}(t,\xi) - a(t))[|\dot{v}(t,\xi)|^2 + |v(t,\xi)|^2] \le 0.$$

Therefore

$$\frac{dE_\lambda(t,\xi)}{dt} \le M\left[|\dot{v}(t,\xi)|^2 + |v(t,\xi)|^2\right]\cdot k(t,\xi) \le M\,E_\lambda(t,\xi). \tag{2.11}$$

By applying the Gronwall inequality, from (2.10), (2.11) we get $E_\lambda(t,\xi) \le M_T E(0,\xi)$, where $M_T = e^{MT}$. Thus

$$\mathrm{E}_\lambda(t) \le M_T \mathrm{E}(0) \tag{2.12}$$

□

Lemma 2.2. *For any $\lambda \ge 0$ and $\varepsilon > 0$*

$$E_\lambda(t) \le \Phi_{\lambda+\varepsilon}(t), \quad t \in [0,T] \tag{2.13}$$

where

$$\Phi_\alpha(t) = \int\limits_{R^n} (1+|\xi|^2)^\alpha \left[|\dot{v}(t,\xi)|^2 + (1+|\xi|^2)\,|v(t,\xi)|^2 + |w(t.\xi)|^2 \right] d\xi. \tag{2.14}$$

Proof. Inequality (2.13) is resulted from inequality (2.5). □

Lemma 2.3. *There exists $N > 0$ and $\Lambda > 0$ such that for all $t \in [0,T]$,*

$$E_\delta(t) \ge \Lambda \cdot \Phi_{\delta-N}(t).$$

Proof. Let $T\,|\xi| \le 1$. By the definition of $d(s,\xi)$ we have

$$\int\limits_0^t d(s,\xi)ds \le \int\limits_0^T |\tilde{a}(t,\xi) - a(t)| \cdot \frac{1}{T} dt \le |\tilde{a}(t,\xi)| + \frac{1}{T} \|a(\cdot)\|_{L_1(0,T)}\,. \tag{2.15}$$

Secondly, we consider the case $T\,|\xi| > 1$. From (2.2) and (2.15) we have that

$$\int\limits_0^t d(s,\xi)ds \le \int\limits_0^T d(s,\xi)d\xi \le \int\limits_0^{|\xi|^{-1}} |\tilde{a}(t,\xi) - a(t)| \cdot |\xi|\, dt$$

$$+ \int\limits_{|\xi|^{-1}}^T \frac{|\dot{a}(t,\xi)|}{a(t)} dt \le c_1 + c_2 \ln(1+|\xi|) + |\xi| \int\limits_0^{|\xi|^{-1}} a(t)dt$$

$$+ \int\limits_{|\xi|^{-1}}^T \frac{C}{t} dt \le c_3 + c_4 \ln(1+|\xi|). \tag{2.16}$$

Hence, from (2.4) and (2.5) we get

$$e^{-\int_0^t d(s,\xi)ds} \ge c_5 (1+|\xi|)^{-c_4}. \tag{2.17}$$

Hence there exist $N > 0$ and $c > 0$ such that for all $t \in [0,1]$

$$E_\lambda(t) \ge c\Phi_{\lambda-N}(t)$$

From lemmas 2.1-2.3 it follows that the Cauchy problem (1.1), (1.2) is well-posed in $H^\infty \times H^\infty \times H^\infty$. □

3. Proof of theorem 1.2.

Let $1 < s < \frac{q-p}{q-1}$. At first we introduce some notation :

$$\tilde{a}_1(t,\xi) = \begin{cases} a(T) & if \quad T\,|\xi|^{\frac{1}{s(q-1)}} \le 1, \\ a(|\xi|^{-\frac{1}{s(q-1)}}) & if \quad T\,|\xi|^{\frac{1}{s(q-1)}} > 1, t\,|\xi|^{\frac{1}{s(q-1)}} \le 1, \\ a(t) & if \quad t\,|\xi|^{\frac{1}{s(q-1)}} > 1\ , \end{cases}$$

$$d_1(t,\xi) = \begin{cases} |\tilde{a}(t,\xi) - a(t)| \cdot |\xi| & if \quad t\,|\xi|^{\frac{1}{s(q-1)}} \le 1\ , \\ \frac{|\dot{\tilde{a}}(t,\xi|}{a(t)} & if \quad t\,|\xi|^{\frac{1}{s(q-1)}} > 1 \end{cases}$$

and

$$\phi_{\lambda,\beta}(t,\xi) = e^{-\int_0^t d_1(s,\xi)ds + \beta\,|\xi|^{\frac{1}{\lambda}}},$$

where β is a positive constant.

We also introduce the notation

$$H_{\lambda,\beta}(t) = \int\limits_{R^n} H_{\lambda,\beta}(s,\xi)ds$$

where

$$H_{\lambda,\beta}(s,\xi) = \left[|\dot{v}(t,\xi)|^2 + (1 + \tilde{a}(t,\xi)\,|\xi|^2)\,|v(t,\xi|^2 + |w(t,\xi)|^2\right]\phi_{\lambda,\beta}(t,\xi).$$

Using (2.6) we get that

$$H_{\lambda,\beta}(t) \le M \int\limits_{R^n} e^{\beta'|\xi|^{\frac{1}{\lambda}}} \left[|\dot{v}(t,\xi)|^2 + (1 + |\xi|^2)(|v(t,\xi)|^2 + |w(t,\xi)|^2\right] d\xi. \qquad (3.1)$$

On the other hand

$$\int\limits_0^t d_1(s,\xi)ds \le \delta(|\xi|^{\frac{1}{\lambda}} + 1), i = 1, 2. \qquad (3.2)$$

Thus, we have

$$\phi_{\lambda,\beta}(t,s) \ge e^{-2\delta + (\beta - 2\delta)\,|\xi|^{\frac{1}{\lambda}}}$$

and

$$H_{\lambda,\beta}(t) \ge e^{-2\delta} \int\limits_{R^n} e^{\beta'|\xi|^{\frac{1}{\lambda}}} \left[|\dot{v}(t,\xi)|^2 + (1 + |\xi|^2)(|v(t,\xi)|^2 + |w(t,\xi)|^2)\right] d\xi, \quad (3.3)$$

where $\beta' = \beta - 2\delta$.

In the same way as lemma 2.1, we can show that there exists $M > 0$ such that

$$H_{\lambda,\beta}(t) \le M H_{\lambda,\beta}(0), t \in [0,T]\,. \qquad (3.4)$$

From (3.1)-(3.4) it follows that the Cauchy problem (1.1), (1.2) is well-posed in $\gamma^{(s)} \times \gamma^{(s)} \times \gamma^{(s)}$.

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