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RIEMANN BOUNDARY VALUE PROBLEMS IN GENERALIZED WEIGHTED HARDY SPACES

BILAL T. BILALOV, ALI A. HUSEYNLI, AND FIDAN SH. SEYIDOVA

Abstract. Riemann boundary value problem of analytic function theory in weighted Hardy classes with variable summability index is considered in this work. The Fredholmness of this problem is investigated under certain conditions on coefficients and a weight. The general solution for homogeneous problem is obtained in weighted Hardy classes with variable summability index. In the case where the weight function satisfies the Muckenhoupt condition with variable summability index, the solvability of the non-homogeneous Riemann problem with the right side from the generalized weighted Lebesgue space is studied.

1. Introduction

The boundary value problems of analytic functions theory has a deep history. Well-known monographs like [12, 24, 13, 27, 20], have been dedicated to the boundary value problems of analytic functions. In the classical formulation these problems are well studied (see [24, 13, 27, 20]). In terms of L_p-metrics in Hardy and Smirnov classes, the theory of boundary value problems has been developed by various mathematicians and has been illuminated in monographs I.I. Danilyuk [12], G.S.Litvinchuk [20], etc. Since recently, there arose an interest in the study of the Riemann boundary value problems in different spaces of analytic functions (see e.g. [17, 22, 15, 21, 14, 23, 8, 9, 2, 3, 4, 5, 6, 7, 10], etc). All the above-cited works dealing with weighted spaces of analytic functions consider the spaces with power weights.

In the present paper, we consider the Riemann boundary value problem in the weighted Hardy space with general weight and piecewise Hölder coefficient. We will find a sufficient condition on the weight under which this problem is Noetherian. We will also calculate the index of the problem. The general solution for homogeneous problem is obtained in weighted Hardy classes with variable summability index. In the case where the weight function satisfies the Muckenhoupt condition with variable summability index, the solvability of the non-homogeneous Riemann problem with the right side from the generalized weighted Lebesgue space is studied. It should be noted that in [25] the Noetherness of Riemann problem is studied in generalized weighted Hardy classes with power weight.

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2. Needful information

Let C be the complex plane, $\omega \equiv \{z \in C : |z| < 1\}$ be the unit circle, and $\partial\omega \equiv \{z \in C : |z| = 1\}$ be the unit circumference. We will need some notion from the theory of generalized Lebesgue spaces.

Let $p : [-\pi, \pi] \to [1, +\infty)$ be some Lebesgue-measurable function. By $\mathscr{L}_0$ we denote the class of all functions measurable on $[-\pi, \pi]$ with respect to Lebesgue measure. Denote

$$I_p(f) \stackrel{def}{\equiv} \int_{-\pi}^{\pi} |f(t)|^{p(t)} dt.$$

Let

$$\mathscr{L} \equiv \{f \in \mathscr{L}_0 : I_p(f) < +\infty\}.$$

With respect to the usual linear operations of addition and multiplication by a number $\mathscr{L}$ is a linear space as $p^+ = \sup \underset{[-\pi,\pi]}{vrai}\, p(t) < +\infty$. With respect to the norm

$$\|f\|_{p(\cdot)} \stackrel{def}{\equiv} \inf \left\{ \lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \le 1 \right\},$$

$\mathscr{L}$ is a Banach space, and we denote it by $L_{p(\cdot)}$. Let

$$WL \stackrel{def}{\equiv} \{p : p(-\pi) = p(\pi); \exists C > 0, \quad \forall t_1, t_2 \in [-\pi, \pi] : |t_1 - t_2| \le \tfrac{1}{2} \Rightarrow \\ \Rightarrow |p(t_1) - p(t_2)| \le \tfrac{C}{-\ln|t_1 - t_2|} \Big\}.$$

Throughout this paper, $q(t)$ will denote the conjugate of a function $p(t)$: $\frac{1}{p(t)} + \frac{1}{q(t)} \equiv 1$. Denote $p^- = \inf \underset{[-\pi,\pi]}{vrai}\, p(t)$. The following generalized Hölder inequality is true

$$\int_{-\pi}^{\pi} |f(t)\, g(t)|\, dt \le c(p^-; p^+) \|f\|_{p(\cdot)} \|g\|_{q(\cdot)},$$

where $c(p^-; p^+) = 1 + \frac{1}{p^-} - \frac{1}{p^+}$. Directly from the definition we get the property, which will be used in the sequel.

Property 2.1. *If $|f(t)| \le |g(t)|$ a.e. on $(-\pi, \pi)$, then $\|f\|_{p(\cdot)} \le \|g\|_{p(\cdot)}$.*

We will need the following easy-to-prove statement.

Statement 2.1. *Let $p \in WL$, $p(t) > 0, \forall t \in [-\pi, \pi]$; $\{\alpha_i\}_1^m \subset R$. The weighted function $\rho(t) = \prod_{i=1}^m |t - \tau_i|^{\alpha_i}$, belongs to the space $L_{p(\cdot)}$, if $\alpha_i > -\frac{1}{p(\tau_i)}$, $\forall i = \overline{1, m}$; where $-\pi = \tau_1 < \tau_2 < ... < \tau_m = \pi$.*

By S we denote the singular integral

$$Sf = \frac{1}{2\pi i} \int_{\partial\omega} \frac{f(\tau)}{\tau - t} d\tau, \quad t \in \partial\omega,$$

Let $\rho : [-\pi, \pi] \to (0, +\infty)$ be some weight function. Define the weighted classes

$$L_{p(\cdot),\rho(\cdot)} : L_{p(\cdot),\rho(\cdot)} \stackrel{def}{\equiv} \left\{ f : \ \rho f \in L_{p(\cdot)} \right\},$$

with a norm $\|f\|_{p(\cdot),\rho(\cdot)} \stackrel{def}{\equiv} \|\rho f\|_{p(\cdot)}$. The validity of the following statement is established in [16].

Statement 2.2. [16] *Let $p \in WL$, $1 < p^-$. Then, singular operator S is acting boundedly from $L_{p(\cdot),\rho(\cdot)}$ to $L_{p(\cdot),\rho(\cdot)}$ if and only if*

$$-\frac{1}{p(\tau_k)} < \alpha_k < \frac{1}{q(\tau_k)}, \quad k = \overline{1, m}. \tag{2.1}$$

We also will use the following variable Muckhenhoupt condition.

Definition 2.1. Given $p(\cdot) : [-\pi, \pi] \to [1, +\infty]$, and a weight function $\omega(\cdot)$. We say that $\omega \in A_{p(\cdot)}$ if

$$\sup_{I \subset [-\pi, \pi]} |I|^{-1} \|\omega(\cdot)\, \chi_I(\cdot)\|_{p(\cdot)} \left\|\omega^{-1}(\cdot)\, \chi_I(\cdot)\right\|_{q(\cdot)} < +\infty.$$

It should be noted that for more details on these and other facts one can see works [11, 19, 28, 26].

We will also need the weighted Hardy classes. By H_p^+ we denote the usual Hardy class of analytical functions in ω. Let $\mathscr{A}$ be the σ-algebra of Borel sets in $[-\pi, \pi]$ and $\rho-$ be a σ-finite measure on $\mathscr{A}$. $L_{p(\cdot);d\rho} \equiv L_{p(\cdot);d\rho}(-\pi, \pi)$ will denote the Banach space $\mathscr{A}$ of measurable functions on $[-\pi, \pi]$ furnished with the norm

$$\|f\|_{p(\cdot)} \stackrel{def}{\equiv} \inf \left\{ \lambda > 0 : I_{p(\cdot);d\rho}\left(\frac{f}{\lambda}\right) \le 1 \right\},$$

where

$$I_{p(\cdot);d\rho}(f) = \int_{-\pi}^{\pi} |f(t)|^{p(t)}\, d\rho(t).$$

Assume

$$\tilde{H} \equiv \left\{ f \in H_1^+ : \ f^+ \in L_{p(\cdot);d\rho} \right\},$$

where $f^+\left(e^{it}\right)$ are non-tangential boundary values of function f on $\partial\omega$. The norm in space $\tilde{H}$ is defined by

$$\|f\|_{\tilde{H}} = \left\|f^+\left(e^{it}\right)\right\|_{p(\cdot);d\rho}. \tag{2.2}$$

Let us show that if the continuous inclusion $L_{p(\cdot);d\rho} \subset L_1$ is true, then $\tilde{H}$ is a Banach space with respect to the norm (2.2), where $L_1 \equiv L_1(-\pi, \pi)$ is a Lebesgue space of summable functions on $(-\pi, \pi)$. Let $\{f_n\}_{n \in N} \subset \tilde{H}$ be some fundamental sequence, i.e.

$$\left\|f_n^+\left(e^{it}\right) - f_m^+\left(e^{it}\right)\right\|_{p(\cdot);d\rho} \to 0, \ \ n, m \to \infty,$$

where $f_n^+ = f_n/_{\partial\omega}$ are non-tangential boundary values of function f_n on $\partial\omega$. From completeness of $L_{p(\cdot);\rho}$ it follows that

$\exists \varphi \in L_{p(\cdot);\rho} : f_n^+\left(e^{it}\right) \to \varphi(t)$, $n \to \infty$, in $L_{p(\cdot);\rho}$.

We have

$$\|f_n - f_m\|_{H_1^+} = \left\|f_n^+\left(e^{it}\right) - f_m^+\left(e^{it}\right)\right\|_{L_1} \le$$
$$\le M\left\|f_n^+\left(e^{it}\right) - f_m^+\left(e^{it}\right)\right\|_{p(\cdot);d\rho} \to 0\,,\ \ n,m \to \infty,$$

where M is a constant depending only on $p(\cdot)$ and $\rho(\cdot)$. This immediately implies that $\{f_n\}_{n\in N}$ is fundamental in H_1^+ and, consequently

$$\exists f \in H_1^+ : f_n \to f\,,\ n \to \infty,\ \text{in } H_1^+.$$

Thus

$$f_n^+\left(e^{it}\right) \to f^+\left(e^{it}\right)\,,\ n \to \infty,\ \ \text{in } L_1,\ \text{where } f^+ = f/_{\partial\omega}.$$

On the other hand, from

$$\left\|f_n^+\left(e^{it}\right) - \varphi(t)\right\|_{L_1} \le M\left\|f_n^+\left(e^{it}\right) - \varphi(t)\right\|_{p(\cdot);d\rho} \to 0\,,\ \ n \to \infty,$$

it follows that, $f_n^+\left(e^{it}\right) \to \varphi(t)$, $n \to \infty$, in L_1. Then it is absolutely clear that $f^+\left(e^{it}\right) = \varphi(t)$ a.e. on $(-\pi,\pi)$, and, as a result, $f_n \to f$, $n \to \infty$, in $\tilde{H}$, i.e. $\tilde{H}$ is a Banach space, and we denote it by $H^+_{p(\cdot),d\rho}$. Thus, we get the validity of

Theorem 2.1. *Let the embedding $L_{p(\cdot);d\rho} \subset L_1$, $1 \le p < +\infty$, be continuous. Then $H^+_{p(\cdot),d\rho}$ is a Banach space.*

Let us define the class ${}_mH^-_{p(\cdot)}$, where $m \in Z$ is some integer. So, let $\Phi(z)$ be an analytic function outside ω, which has finite order at infinity, i.e. let

$$\Phi(z) = \sum_{n=-\infty}^{n=k} a_n z^n\ \ ,\ \ \ k < +\infty,$$

be the Laurent series in a neighborhood of the infinitely remote point of the function $\Phi(z)$.

Under the order of the function $\Phi(z)$ at the infinitely remote point we will mean the largest number $n = k$ of these expansions, for which $a_n \ne 0$. In the case of $k = 0$ the function $\Phi(z)$ is bounded and different from zero at the point $z = \infty$; in the case of $k > 0$ it has a pole of order k; and in the case of $k < 0-$has a zero order $(-k)$. Thus, $\Phi(\cdot)$ has a form $\Phi(z) = \Phi_0(z) + \Phi_1(z)$, where $\Phi_0(z)$ $-$is a regular part, and $\Phi_1(z)$ is a main part of Laurent series of $\Phi(z)$ at the infinitely remote point. If in this case $k \le m$ and the function $\overline{\Phi_0\left(\frac{1}{\bar z}\right)}$ belongs to the class $H^+_{p(\cdot)}$, then we will say that the function $\Phi(\cdot)$ belongs to Hardy class ${}_mH^-_{p(\cdot)}$ outside ω. Denote

$$\tilde{H}^- \equiv \left\{f \in {}_mH_1^- : f^-(e^{it}) \in L_{p(\cdot),d\rho}\right\},$$

where $f^-\left(e^{it}\right) = f/_{\partial\omega}$ are non-tangential boundary values of f on $\partial\omega$ from the outside of ω. The norm in $\tilde{H}^-$is defined as follows

$$\|f\|_{\tilde{H}^-} \equiv \left\|f^-\left(e^{it}\right)\right\|_{p(\cdot);d\rho}\ ,\ \forall f \in \tilde{H}^-. \tag{2.3}$$

Similarly to the previous case, we can prove that for $L_{p(\cdot);d\rho} \subset L_1$, the class $\tilde{H}^-$ is a Banach space with respect to the norm (2.3), and we denote it by ${}_mH^-_{p(\cdot);d\rho}$. Thus, we have

Theorem 2.2. *Let $L_{p(\cdot);d\rho} \subset L_1$, $1 \le p < +\infty$, be true. Then the space defined above ${}_mH^-_{p(\cdot);d\rho}$ is a Banach space.*

3. General solution of homogeneous problem

In the sequel, we will assume that the measure $\rho(\cdot)$ is absolutely continuous with respect to a Lebesgue measure on $[-\pi,\pi]$ and $d\rho(x) = \vartheta^{p(x)}(x)\,dx$. Corresponding spaces are denoted by $L_{p(\cdot);\vartheta}$ and $H^{\pm}_{p(\cdot);\vartheta}$, respectively. Consider the following homogeneous Riemann problem

$$F^+(\tau) - G(\tau)F^-(\tau) = 0,\ \ \tau \in \partial\omega, \tag{3.1}$$

where $G : \partial\omega \to C$ is some complex-valued function (coefficient of the problem). By solution of the problem (3.1) we mean a pair of analytic functions

$$\left(F^+(z); F^-(z)\right) \in H^+_{p(\cdot);\vartheta} \times_m H^-_{p(\cdot);\vartheta},$$

whose non-tangential boundary values on $\partial\omega$ a.e. satisfy the relation (3.1), where $p(\cdot) \in WL\ \wedge p^- > 1$ and $m < 0$ is some fixed integer. In the sequel, we will assume that G satisfies the following conditions:

α) $G^{\pm 1} \in L_\infty(\partial\omega)$;

β) $\theta(t) \equiv \arg G\left(e^{it}\right)$ *is a piecewise Hölder function on* $[-\pi,\pi]$ *and* $-\pi < s_1 < ... < s_r < \pi$ *are the corresponding points of discontinuity.*

Consider the following functions:

$$X_1(z) = \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\left|G\left(e^{it}\right)\right| \frac{e^{it}+z}{e^{it}-z}dt\right\},$$

$$X_2(z) = \exp\left\{\frac{i}{4\pi}\int_{-\pi}^{\pi} \theta(t)\frac{e^{it}+z}{e^{it}-z}dt\right\},$$

which are analytic inside and outside ω, respectively. Let

$$Z_k(z) \equiv \begin{cases} X_k(z), |z| < 1, \\ \left[X_k(z)\right]^{-1}, |z| > 1,\ k = 1,2\ . \end{cases}$$

Denote

$$Z(z) \equiv Z_1(z)\, Z_2(z)\,.$$

As usual, by $Z^{\pm}(\tau)$ we denote non-tangential boundary values of $Z(z)$ on $\partial\omega$ from inside and outside ω, respectively. Using Sohockii-Plemelj formulas, we directly obtain

$$\left|G(e^{it})\right| = \frac{Z_1^+(e^{it})}{Z_1^-(e^{it})}, \exp i\theta\,(t) = \frac{Z_2^+(e^{it})}{Z_2^-(e^{it})}.$$

Consequently, we have

$$G(e^{it}) = \frac{Z^+(e^{it})}{Z^-(e^{it})}, \text{ a.e. on } (-\pi, \pi). \tag{3.2}$$

$Z(z)$ will be called the canonical solution of the problem (3.1). Taking into account the relation (3.2), from (3.1) we get

$$\frac{F^+(\tau)}{Z^+(\tau)} = \frac{F^-(\tau)}{Z^-(\tau)}, \quad \text{a.e.} \quad \tau \in \partial\omega.$$

Assume

$$\Phi(z) \equiv \frac{F(z)}{Z(z)}.$$

Due to the fact that $Z(z)$ has neither zeros and nor poles at $z \notin \partial\omega$, it is clear that the functions $\Phi(z)$ and $F(z)$ have the same order at infinity. By definition of solution, we have $F \in H_1^{\pm}$. Let us express the function $\theta(t)$ as the sum

$$\theta(t) = \theta_0(t) + \theta_1(t),$$

where θ_0 is a continuous part, and θ_1 is a jump function defined by

$$\theta_1(-\pi+0) = 0\,, \;\; \theta_1(t) = \sum_{-\pi<s_k<t} h_k + \left[\theta(t) - \theta(t-0)\right],$$

where

$$h_k = \theta(s_k+0) - \theta(s_k-0), k = \overline{1,r},$$

are jumps of the function $\theta(\cdot)$ at s_k. Following [12], we denote

$$h_0 = h_0^{(1)} - h_0^{(0)},$$

where

$$h_0^{(1)} = \theta_1(-\pi+0) - \theta_1(\pi-0), h_0^{(0)} = \theta_0(-\pi-0) - \theta_0(\pi-0).$$

Let

$$U_0(t) \equiv \left\{\sin\left|\frac{t-\pi}{2}\right|\right\}^{-\frac{h_0^{(0)}}{2\pi}} \exp\left\{-\frac{1}{4\pi}\int_{-\pi}^{\pi}\theta_0(s) ctg\frac{t-s}{2}ds\right\}.$$

As shown in [12], from condition α) it follows that the following inequality is true

$$\left\|Z_1^-\left(e^{it}\right)\right\|_\infty^{\pm 1} < +\infty,$$

where $\|\cdot\|_\infty$ denotes the norm of $L_\infty(-\pi,\pi)$. Using the results of [12] again, we find that the boundary values of $Z_2(z)$ can be represented as

$$\left|Z_2^-\left(e^{it}\right)\right| = U_0(t)\, U(t) \left|\sin\frac{t-\pi}{2}\right|^{-\frac{h_0}{2\pi}},$$

where

$$U(t) = \prod_{k=1}^{r}\left|\sin\frac{t-s_k}{2}\right|^{\frac{-h_k}{2\pi}}.$$

Using this notation, we can write the boundary values of $|Z(\tau)|$ as follows

$$\left|Z^{-}\left(e^{it}\right)\right| = U_0\left(t\right)\left|Z_1^{-}\left(e^{it}\right)\right|\left|\sin\frac{t-\pi}{2}\right|^{-\frac{h_0}{2\pi}} U\left(t\right).$$

According to the results of [12], the functions $U_0^{\pm 1}\left(t\right)$ belong to $L_p\left(-\pi,\pi\right)$, $\forall p \in (0,+\infty)$, and, moreover, for sufficiently small values of $\delta > 0$ we have $Z^{\pm 1}\left(z\right) \in H_\delta^+$. As a result, we obtain that the function $\Phi(z)$ belongs to the Hardy class H_μ^+ for some $\mu > 0$. It is absolutely obvious that if $\Phi^+(e^{it}) \in L_1\left(-\pi,\pi\right)$, then $\Phi \in H_1^+$. Since we have $\Phi^+\left(\tau\right) = \Phi^-\left(\tau\right)$ a.e. $\tau \in \partial\omega$, it is clear that it suffices to prove that $\Phi^-(e^{it})$ belongs to $L_1\left(-\pi,\pi\right)$. We have

$$\left|\Phi^{-}\left(e^{it}\right)\right| = \left|F^{-}\left(e^{it}\right)\right|\left|Z^{-}\left(e^{it}\right)\right|^{-1}, \quad \text{a.e. } t \in \left(-\pi,\pi\right).$$

By definition of solution, $F^-\left(e^{it}\right) \in L_{p(\cdot);\vartheta}$. Therefore, if

$$\left|Z^{-}\left(e^{it}\right)\right|^{-1} \in L_{q(\cdot);\vartheta^{-1}},$$

then $\Phi^-\left(\tau\right) \in L_1$, which follows directly from the generalized Hölder's inequality in $L_{p(\cdot)}$. So assume that the following inequality is fulfilled

$$\int_{-\pi}^{\pi}\left|w\left(t\right)\vartheta^{-1}\left(t\right)\right|^{q(t)}dt < +\infty, \tag{3.3}$$

where

$$w\left(t\right) = \left|\sin\frac{t-\pi}{2}\right|^{\frac{h_0}{2\pi}}\prod_{k=1}^{r}\left|\sin\frac{t-s_k}{2}\right|^{\frac{h_k}{2\pi}}. \tag{3.4}$$

If the condition (3.3) holds, then $\left|Z^-\left(\tau\right)\right|^{-1}$ belongs to $L_{q(\cdot);\vartheta^{-1}}$, and, as a result, $\Phi^-\left(\tau\right) \in L_1$, so $F^- \in L_{p(\cdot),\vartheta}$ and $\Phi^-\left(e^{it}\right) = F^-\left(e^{it}\right)\left(Z^-\left(e^{it}\right)\right)^{-1}$.Then, by Smirnov theorem,$\Phi \in H_1^{\pm}$, and, as a result, it follows from the uniqueness theorem that $\Phi\left(z\right)$ is a polynomial of order $k \le m$ (because $\Phi^+\left(\tau\right) = \Phi^-\left(\tau\right)$ a.e. $\tau \in \partial\omega$), i.e. $\Phi\left(z\right) \equiv P_k\left(z\right)$, where $P_k\left(z\right)$ is a polynomial of order $k \le m$. Thus

$$F\left(z\right) \equiv Z\left(z\right)P_k\left(z\right). \tag{3.5}$$

Assume that the inequalities

$$h_k < 2\pi\,,\ k = \overline{0,r}, \tag{3.6}$$

are fulfilled. Then we have $Z^{\pm}(e^{it}) \in L_1\left(-\pi,\pi\right)$ and, again by Smirnov theorem, we obtain $Z\left(z\right) \in H_1^{\pm}$. So, $Z\left(\cdot\right)$ has a zero order at infinity, then it follows directly from (3.5) that $F \in H_1^{\pm}$. Let the following inequality also be fulfilled

$$\int_{-\pi}^{\pi}\left|w^{-1}\left(t\right)\vartheta\left(t\right)\right|^{p(t)}dt < +\infty. \tag{3.7}$$

Then from the expression for boundary values $Z^-\left(e^{it}\right)$ it follows $Z^-\left(e^{it}\right) \in L_{p(\cdot);\vartheta}$, which means that $F^-\left(e^{it}\right) \in L_{p(\cdot);\vartheta}$. As a result, we obtain that if $m \ge 0$ and $k \le m$, then it is true

$$\left(F^+\left(z\right);F^-\left(z\right)\right) \in H^+_{p(\cdot);\vartheta} \times_m H^-_{p(\cdot);\vartheta}.$$

Thus, if $m \geq 0$, then the expression (3.5) is the general solution of the homogeneous problem (3.1) in the classes $H^{+}_{p(\cdot);\vartheta} \times_m H^{-}_{p(\cdot);\vartheta}$, where $P_m(\cdot)$ is an arbitrary polynomial of order $k \leq m$. So the following theorem is true.

Theorem 3.1. *Let $p(\cdot) \in WL \wedge p^- > 1$ and the coefficient $G(\tau)$ of the problem* (3.1) *satisfy the conditions α) and β). Suppose that the jumps of functions $\theta(t) \equiv \arg G\left(e^{it}\right)$ satisfy the relations* (3.3), (3.6), (3.7), *where $w(t)$ is defined by* (3.4). *Then, under $m \geq 0$, the general solution of the homogeneous problem* (3.1) *in classes $H^{+}_{p(\cdot);\vartheta} \times_m H^{-}_{p(\cdot);\vartheta}$ can be represented in the form of* (3.5), *where $Z(z)$ is a canonical solution, and $P_k(z)$ is an arbitrary polynomial of order $k \leq m$. If $m \leq -1$, then under the conditions* (3.3), (3.6), (3.7), *the problem* (3.1) *in $H^{+}_{p(\cdot);\vartheta} \times_m H^{-}_{p(\cdot);\vartheta}$ has only trivial, i.e. zero solution.*

4. General solution of nonhomogeneous problem

Consider an nonhomogeneous Riemann problem

$$F^{+}(\tau) - G(\tau) F^{-}(\tau) = g(\tau), \tau \in \partial\omega, \tag{4.1}$$

where $g \in L_{p(\cdot);\vartheta}$ is a given function. By the solution of problem (4.1) we mean a pair

$$\left(F^{+}(z); F^{-}(z)\right) \in H^{+}_{p(\cdot);\vartheta} \times_m H^{-}_{p(\cdot);\vartheta},$$

for which the boundary values $F^{\pm}(\tau)$ satisfy the relation (4.1) a.e. on $\partial\omega$. It is clear that the general solution of (4.1) can be represented as

$$F(z) = F_0(z) + F_1(z),$$

where $F_0(z)$ is a general solution of the corresponding homogeneous problem (3.1), and $F_1(z)$ is some particular solution of the nonhomogeneous problem (4.1). We will construct a particular solution of (4.1). Let $Z(z)$ be a canonical solution of (3.1). Consider the piecewise analytic function

$$F_1(z) \equiv \frac{Z(z)}{2\pi} \int_{-\pi}^{\pi} \frac{g\left(e^{i\sigma}\right)}{Z^{+}\left(e^{i\sigma}\right)} \frac{d\sigma}{1 - ze^{-i\sigma}}. \tag{4.2}$$

Applying Sokhotskii-Plemelj formulas to (4.2), we obtain that the boundary values $F_1^{\pm}(\tau)$ satisfy (4.1) a.e. on $\partial\omega$. We have

$$F_1^{+}\left(e^{it}\right) = \frac{1}{2} g\left(e^{it}\right) + \frac{Z^{+}\left(e^{it}\right)}{2\pi} \int_{-\pi}^{\pi} \frac{g\left(e^{i\sigma}\right)}{Z^{+}\left(e^{i\sigma}\right)} \frac{d\sigma}{1 - e^{i(t-\sigma)}}.$$

Let

$$f(\sigma) = g\left(e^{i\sigma}\right) \left[Z^{+}\left(e^{i\sigma}\right)\right]^{-1}.$$

Consequently

$$\tilde{F}_1^{+}(t) = \frac{1}{2} f(t) + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(\sigma)\, d\sigma}{1 - e^{i(t-\sigma)}},$$

where

$$\tilde{F}_1(t) \equiv F_1^+\left(e^{it}\right)\left[Z^+\left(e^{it}\right)\right]^{-1}.$$

It is absolutely clear that $f(\cdot)$ belongs to the weighted space $L_{p(\cdot);\nu}$, where

$$\nu(t) = w^{-1}(t)\,\vartheta(t).$$

It is known that the singular operator

$$S(f) = \int_{-\pi}^{\pi} \frac{f(\sigma)\,d\sigma}{1-e^{i(t-\sigma)}},$$

is bounded in $L_{p(\cdot);\nu}$, $1 < p^- \le p^+ < +\infty$, $p \in WL$, if and only if the weight $\nu(\cdot)$ satisfies the Muckenhoupt condition $A_{p(\cdot)}$, i,e, $\nu \in A_{p(\cdot)}$ (see, for example, [11, 16]). Consequently, if $\nu \in A_{p(\cdot)}$, then $\tilde{F}_1 \in L_{p(\cdot);\nu}$. It is not difficult to see that

$$\tilde{F}_1 \in L_{p(\cdot);\nu} \Leftrightarrow F_1 \in L_{p(\cdot);\vartheta}.$$

The condition $\nu \in A_{p(\cdot)}$ implies $\nu \in L_1$ and $\nu^{-1} \in L_{q(\cdot)}$. Then from Hölder's inequality it follows

$$\int_{-\pi}^{\pi} |f(t)|\,dt = \int_{-\pi}^{\pi} |f(t)|\,\nu(t)\,\nu^{-1}(t)\,dt \le c_{p(\cdot)} \left\| |f(\cdot)|\ \nu(\cdot)\right\|_{p(\cdot)} \left\|\ \nu^{-1}(\cdot)\right\|_{q(\cdot)}.$$

Thus, f belongs to L_1, and, as a result, the Cauchy-Lebesgue type integral

$$\int_{-\pi}^{\pi} \frac{f(\sigma)\,d\sigma}{1-ze^{-i\sigma}},$$

belongs to the Hardy class $H^{\pm}_{p_1}$, for $\forall p_1 : 0 < p_1 < 1$, (see, for example, [18]). From expression for $F_1(z)$ it directly follows that it belongs to the space H^+_μ for sufficiently small $\mu > 0$. As we have already established, $F_1^+\left(e^{it}\right) \in L_{p(\cdot);\vartheta}$. Then it is evident that if $\vartheta^{-1} \in L_{q(\cdot)}$, then $F_1^+\left(e^{it}\right) \in L_1$. Then, by Smirnov theorem [18], the function $F_1^+(z)$ belongs to the class H_1^+. As a result, we obtain $F_1^+(z) \in H^+_{p(\cdot);\vartheta}$. Consider $F_1^-(z)$, and let all the above conditions be satisfied. Similar reasoning yields $F_1^-(z) \in_{-1} H^-_{p(\cdot);\vartheta}$. It is absolutely clear that $F_1^-(\infty) = 0$. Therefore

$$F_1^-(z) \in_m H^-_{p(\cdot);\vartheta},\ \forall m \ge -1.$$

Let $m < -1$. In this case, in order for the inclusion $F_1^-(z) \in_m H^-_{p(\cdot);\vartheta}$, to be valid, it is necessary that the following orthogonality conditions hold

$$\int_{-\pi}^{\pi} \frac{g\left(e^{i\sigma}\right)}{Z^+\left(e^{i\sigma}\right)} e^{in\sigma} d\sigma = 0\,,\ n = \overline{1,-m}. \tag{4.3}$$

Thus, under these conditions, the expression (4.2) is a particular solution of (4.1) in classes $H^+_{p(\cdot);\vartheta} \times_m H^-_{p(\cdot);\vartheta}$. If the conditions (3.3), (3.6) and (3.7) are fulfilled, then the general solution of the corresponding homogeneous problem has the form (3.5), where $Z(z)$ is a canonical solution, and $P_m(z)$ is an arbitrary polynomial of order $\le m$. For $m \le -1$ it is clear that $P_m(z) \equiv 0$. In this case, the problem (4.2) is uniquely solvable. So we have the following

Theorem 4.1. *Let the coefficient $G(\tau)$ of the problem (4.1) satisfy the conditions $\alpha);\beta)$. Assume that the jumps of the function $\theta(t) \equiv \arg G(e^{it})$ satisfy the relations (3.3), (3.6), (3.7), where the weight function $w(t)$ is defined by the expression (3.4). Let $\nu \in A_{p(\cdot)}$, $p(\cdot) \in WL \wedge p^- > 1$, where $\nu(t) = w^{-1}(t)\vartheta(t)$ and $\vartheta^{-1} \in L_{q(\cdot)}$. Then the nonhomogeneous Riemann problem (4.1) is solvable in classes $H^+_{p(\cdot);\vartheta} \times_m H^-_{p(\cdot);\vartheta}$, if the orthogonality conditions (4.3) hold. For $m \geq 0$ the general solution of (4.1) can be represented as*

$$F(z) = Z(z) P_m(z) + F_1(z), \tag{4.4}$$

where $Z(z)$ is a canonical solution of the homogeneous problem, $P_m(z)$ is an arbitrary polynomial of order $\leq m$, and $F_1(z)$ is defined by the expression (4.2). Moreover, for $m \leq -1$ the problem (4.1) is uniquely solvable, and for $m = -1$ it has a solution for $\forall g \in L_{p(\cdot);\vartheta}$.

5. Special case

Consider the special case

$$\vartheta(t) = \prod_{k=1}^{m} |t - t_k|^{\alpha_k}, \tag{5.1}$$

where $\{t_k\}_1^m \subset (-\pi, \pi)$ are different points. For simplicity, we assume

$$\{s_k\}_1^r \bigcap \{t_k\}_1^m = \emptyset. \tag{5.2}$$

In this case, the weight $\nu(t)$ has the following form

$$\nu(t) \equiv \left|\sin\frac{t-\pi}{2}\right|^{-\frac{h_0}{2\pi}} \prod_{k=1}^{r} \left|\sin\frac{t-s_k}{2}\right|^{-\frac{h_k}{2\pi}} \prod_{k=1}^{m} |t-t_k|^{\alpha_k}.$$

It is known that $\nu \in A_{p(\cdot)}$ if and only if (see, for example [16])

$$-\frac{1}{p(s_k)} < -\frac{h_k}{2\pi} < \frac{1}{q(s_k)}, \quad k = \overline{0,r};$$

$$-\frac{1}{p(t_k)} < \alpha_k < \frac{1}{q(t_k)}, \quad k = \overline{1,m}. \tag{5.3}$$

Consequently

$$-\frac{1}{q(s_k)} < \frac{h_k}{2\pi} < \frac{1}{p(s_k)}, \quad k = \overline{0,r}. \tag{5.4}$$

From this relation it directly follows that the conditions (3.3), (3.6), and (3.7) are fulfilled. It is obvious that the relation $\vartheta \in L_1$ also holds. As a result, from Theorem 4.1 we obtain the following

Corollary 5.1. *Let $p(\cdot) \in WL \wedge p^- > 1$ and the coefficient $G(\tau)$ satisfy the conditions $\alpha);\beta)$ and let (5.2)-(5.4) hold. Then the general solution of the Riemann problem (4.1) in classes $H^+_{p(\cdot);\vartheta} \times_m H^-_{p(\cdot);\vartheta}$ is given by (4.4), where the weight ϑ has the form (5.1). For $m < 0$, the orthogonality conditions (4.3) are necessary for solvability. Then the problem is uniquely solvable and $P_m(z) \equiv 0$. For $m \geq 0$, the* *homogeneous problem has mlinearly independent solutions, while the nonhomogeneous problem is solvable for* $\forall g \in L_{p(\cdot);\vartheta}$. *For* $m = -1$, *the nonhomogeneous problem is uniquely solvable for* $\forall g \in L_{p(\cdot);\vartheta}$.

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Bilal T. Bilalov
Institute of Mathematics and Mechanics of NAS of Azerbaijan, B.Vahabzadeh str. 9, AZ1141, Baku, Azerbaijan
E-mail address: b_bilalov@mail.ru

Ali A. Huseynli
Department of Mathematics, Khazar University, Mehseti str. 41, AZ1096, Baku, Azerbaijan
Institute of Mathematics and Mechanics of NAS of Azerbaijan, B.Vahabzadeh str. 9, AZ1141, Baku, Azerbaijan.
E-mail address: alihuseynli@gmail.com

Fidan Sh. Seyidova
Ganja State University
E-mail address: fidanseyidova207@gmail.com

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