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SOLVABILITY OF BOUNDARY VALUE PROBLEMS FOR FOURTH ORDER DIFFERENTIAL-OPERATOR EQUATION WITH A PARAMETER

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Abstract. In a Hilbert space H, we consider a boundary value problem for a fourth order differential-operator equation with a quadratic complex parameter in the case when boundary conditions besides of the complex parameter, contain linear unbounded operators. Coercive solvability of the considered problem, in the space $L_p((0,1);H)$, $p\in(1,\infty)$, is proved. Application of abstract results to boundary value problems for fourth order elliptic partial differential equations in nonsmooth domains, is given.

1. Introduction

In paper [5], in a UMD Banach space E, the following boundary value problem was studied for a fourth order elliptic differential- operator equation with a complex parameter λ

$$\begin{aligned}(L(\lambda)u)(x) &:= \lambda^2 u(x) - \lambda(2u''(x) + A_2 u(x)) + u''''(x)\\ &+A_2 u''(x) + A_4 u(x) = f(x), \quad x\in(0,1), \end{aligned} \tag{1.1}$$

$$\begin{aligned} &L_k u := \alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(1) = \varphi_k, \quad k=1,2,\\ &L_k(\lambda)u := \alpha_k(u^{(m_k)}(0) - \lambda u^{(m_{k-2})}(0))\\ &+ \beta_k(u^{(m_k)}(1) - \lambda u^{(m_{k-2})}(1)) = \varphi_k, \quad k=3,4, \end{aligned} \tag{1.2}$$

where $0\le m_1$, $m_2\le 1$ are integers, $m_3=m_1+2$, $m_4=m_2+2$; α, β are some complex numbers; A_4 is an R -sectorial operator in E, A_2 is a linear unbounded operator in E which subordinates to the operator $A_4^{1/2}$ in a some sense . Under some conditions, imposed on the operator pencil $L_0(\mu)=\mu^4 I+\mu_2 A_2+A_4$ and on the coefficients α_k,β_k, for the problem (1.1), (1.2), for sufficiently large $|\lambda|$ from some angle, containing a positive axis, a theorem on isomorphism between the solutions and the right hand side of the problem (1.1),(1.2), in the space

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$L_p((0,1);E)$, $p \in (1,\infty)$ was proved. It was also established that, for the boundary value problem (1.1),(1.2), it holds coercive solvability with respect to u and λ. Note that Fredholm property of the boundary value problems (1.1),(1.2), for $\lambda = 0$, was studied in the monograph [9, chapter 5, section 5.6] and in the paper [4], in a Hilbert space H and a UMD Banach space E, respectively.

In the present paper, using the ideas and technique of paper [5], in a separable Hilbert space H, we study solvability of boundary value problems for the equation (1.1), in the case when $A_2 = 0$ and the boundary conditions have unbounded operators. So, in the paper, in a separable Hilbert space, we study solvability of the following boundary value problems:

$$(L(\lambda)u)(x) := \lambda^2 u(x) - 2\lambda u^{''}(x) + u^{''''}(x) + Au(x) = f(x), \qquad x \in (0,1), \quad (1.3)$$

$$\begin{gathered} L_1 u := u^{'}(1) + B_1 u(0) = \varphi_1, \\ L_2 u := u^{'}(0) = \varphi_2, \\ L_3(\lambda)u := u^{'''}(1) - \lambda u^{'}(1) + B_2((u''(0) - \lambda u(0)) = \varphi_3, \\ L_4(\lambda)u := u^{'''}(0) - \lambda u^{'}(0) = \varphi_4, \end{gathered} \quad (1.4)$$

where λ is a complex parameter; A, B_1, B_2 are linear unbounded operators in H, satisfying some conditions that will be formulated in the sequel when formulating a theorem.

Similar to what has been done in paper [5], in this paper, for the problem (1.3), (1.4), for sufficiently large $|\lambda|$ from some angle containing a positive semi-axis, we prove a theorem on isomorphism between the solutions and the right hand side of the problem (1.3), (1.4), in the space $L_p((0,1);E)$, $p \in (1,\infty)$. It was established that for the boundary value problem (1.3), (1.4), it holds coercive solvability with respect to u. Sufficient conditions were found for the operators in the equation and in the boundary conditions, providing coercive solvability of the problem (1.3), (1.4), in the space $L_p((0,1);E)$, $p \in (1,\infty)$ with respect to u. By means of special substitutions, which are in paper [5], our problem that is reduced to a boundary value problem for the second order elliptic differential-operator equation with a spectral parameter, wherein one of the boundary conditions contains an unbounded operator.

Solvability of boundary value problems for second order elliptic differential-operator equations with a spectral parameter, in the case when one of the boundary conditions contains a linear unbounded operator subordinated to the main operator of the equation, was studied in papers [2],[1] in Hilbert and UMD Bahach spaces, respectively.

Fredholm property of perturbed boundary value problems, corresponding to the boundary value problem (1.3), (1.4), for $\lambda = 0$, in a Hilbert space H, was studied in paper [3], and we will use some facts from this paper. Because of presence of unbounded operators in the boundary conditions, the obtained results enable us to study solvability of a new class of boundary value problems for fourth order elliptic partial differential equations with a quadratic complex parameter in non-smooth domains. One of such applications is given at the end of the paper. We now give some necessary definitions and notions used in the paper.

Definition 1.1 A linear closed operator A is said to be strongly positive in a Hilbert space H if the domain of definition $D(A)$ is dense in H, for some $\delta \in [0,\pi)$

all the points from the angle $|arg\lambda| > \delta$, including 0, belong to the resolvent set of the operator A, and the resolvent satisfies the estimate

$$\|(A-\lambda I)^{-1}\| \le C(1+|\lambda|)^{-1},$$

where I is the unit operator in H, $C = const > 0$.

The simplest example of strongly positive operators are self-adjoint positive-definite operators acting in a Hilbert space. Note that from the strong positivity of the operator A it follows strong positivity of the operator A^{α}, $\alpha \in (0,1)$. Let A be a strongly positive operator in H. As A^{-1} is bounded in H, then

$$H(A^n) := \left\{u : u \in D(A^n),\ \ \|u\|_{H(A^n)} = \|A^n u\|_H \right\},\ n \in N,$$

is a Hilbert space with norm equivalent to the norm of the graph of the operator A^n. If A is strongly positive in H, it is known that the operator $-A$ is a generating operator of the analytic, for $t > 0$, semigroup e^{-tA} and this semi-group exponentially decreases, i.e., there exist two numbers $C > 0$, $\sigma_0 > 0$ such that $\|e^{-tA}\| \le ce^{-\sigma_0 t}, 0 \le t < +\infty$. By [6, theorem 1.5.5], $-A^{1/2}$ generates an analytic semi-group for $t > 0$, decreasing at infinity.

Definition 1.2 [8, theorem 1.14.5]. Let A be a strongly-positive operator in a Hilbert space H. Then the interpolation space $(H(A^n),H)_{\theta,p}$, $0 < \theta < 1$, of Hilbert spaces $H(A^n)$ and H is defined by the equality

$$(H(A^n),H)_{\theta,p} := \left\{u : u \in H, \|u\|_{(H(A^n),H)_{\theta,p}} : = \int_0^{+\infty} t^{-1+n\theta p} \left\|A^n e^{-tA} u\right\|_H^p dt < \infty \right\}, n \in N,$$

Set $(H(A^n),H)_{0,p} := H(A^n)$ and $(H(A^n),H)_{1,p} := H$.

Denote by $L_p((0,1);H)\,(1 < p < \infty)$ a Banach space (for $p = 2$, a Hilbert space) of functions $x \to u(x) : [0,1] \to H$, strongly measurable and summable in p-th degree, with the norm

$$\|u\|_{L_p(0,1);H)} := \left(\int_0^1 \|u(x)\|_H^p\, dx\right)^{1/p} < \infty,$$

and by $W_p^n((0,1);H(A^n),H) := \left\{u : A^n u, u^{(n)} \in L_p((0,1);H)\right\}$ denote a space of vector-functions with the norm

$$\|u\|_{W_p^n((0,1);H(A^n);H)} := \|A^n u\|_{L_p(0,1);H)} + \left\|u^{(n)}\right\|_{L_p(0,1);H)}.$$

It is known that [8, theorem 1.8.2] if $u \in W_p^n((0,1);H(A^n),H)$, then

$$u^{(j)}(\cdot) \in (H(A^n),H)_{\frac{j+1/p}{n},p}, \quad j = 0,...,n-1.$$

2. Theorem on isomorphism for boundary value problems for fourth order elliptic differential- operator equations with a quadratic complex parameter.

Let us consider, in a separable Hilbert space H, the boundary value problem (1.3), (1.4).

Theorem 2.1. *Let the following conditions be fulfilled:*

1) A is a self-adjoint, positive-definite operator in H;

2) A linear closed operator B_1 boundedly acts from $H(A)$ in to $H(A^{3/4})$ and from $H(A^{3/4})$ in to $H(A^{1/2})$;

3) A linear closed operator B_2 boundedly acts from $H(A^{1/2})$ in to $H(A^{1/4})$ and from $H(A^{1/4})$ in to H.

Then, for sufficiently large $|\lambda|$ from the angle $|arg\lambda| \le \varphi < \pi$, the operator $\mathbb{L}(\lambda) : u \to \mathbb{L}(\lambda)u := (L(\lambda)u, L_1u, L_2u, L_3(\lambda)u, L_4(\lambda)u)$ is an isomorphism from $W_p^4((0,1); H(A), H)$ onto

$$L_p((0,1);H) \dot{+} (H(A),H)_{\frac{1}{4}+\frac{1}{4p},p} \dot{+} (H(A),H)_{\frac{1}{4}+\frac{1}{4p},p}$$
$$\dot{+}(H(A),H)_{\frac{3}{4}+\frac{1}{4p},p} \dot{+} (H(A),H)_{\frac{3}{4}+\frac{1}{4p},p}$$

and for these values of λ, the following estimate holds for the solution of the problem (1.3), (1.4) [1]

$$|\lambda|^2 \|u\|_{L_p((0,1);H)} + |\lambda| \|u\|_{L_p((0,1);H(A^{1/2}))} + \left\|u''\right\|_{L_p((0,1);H(A^{1/2}))} + \left\|u''''\right\|_{L_p((0,1);H)}$$
$$+ \|Au\|_{L_p(0,1);H)} \le C \Bigg[|\lambda|^{\frac{1}{2}-\frac{1}{2p}} \|f\|_{L_p((0,1);H)} + \sum_{k=1}^{2} \Bigg(\|\varphi_k\|_{(H(A),H)_{\frac{1}{4}+\frac{1}{4p},p}}$$
$$+ \|\varphi_{k+2}\|_{(H(A),H)_{\frac{3}{4}+\frac{1}{4p},p}} \Bigg) + |\lambda|^{\frac{1}{2}-\frac{1}{2p}} \sum_{k=1}^{2} \Big(\|\varphi_k\|_{H(A^{1/2})} + \|\varphi_{k+2}\|_H \Big) \Bigg], \tag{2.1}$$

where the constant C does not depend on λ.

Proof. By the substitution

$$v(x) := \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix} := \begin{pmatrix} u(x) \\ u''(x) - \lambda u(x) \end{pmatrix}$$

problem (1.3), (1.4) is reduced to the equivalent problem

$$v''(x) = \mathbb{A}v(x) + \lambda v(x) + F(x), \quad x \in (0,1), \tag{2.2}$$

$$v'(1) + \mathbb{B}v(0) = \Phi_1, \tag{2.3}$$
$$v'(0) = \Phi_2,$$

where

$$\mathbb{A} := \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}, \quad \mathbb{B} := \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}, \quad F(x) := \begin{pmatrix} 0 \\ f(x) \end{pmatrix},$$

$$\Phi_k := \begin{pmatrix} \varphi_k \\ \varphi_{k+2} \end{pmatrix}, \quad k = 1, 2.$$

We consider the operator $\mathbb{A}$ in the space $\mathbb{H} := H(A^{1/2}) \oplus H$.

[1] By virtue of [10, chapter 5, theorem 1.7 and corollary 1.9], the embedding $W_p^4((0,1); H(A), H) \subset W_p^2((0,1); H(A^{1/2}), H(A^{1/4}), H)$ is continuous. Therefore, $u'' \in L_p((0,1); H(A^{1/2}))$.

Let $D(\mathbb{A}) := H(A) \oplus H(A^{1/2})$. As is shown in the proof of theorem 5.6.1/2 from [9], the operator $\mathbb{A}$ is self-adjoint, positive- definite in $\mathbb{H}$, i.e. equation (2.2) is a second order elliptic differential- operator equation with a spectral parameter.

In paper [3], it is shown that the operator $\mathbb{B}$ boundedly acts from $\mathbb{H}(\mathbb{A})$ into $\mathbb{H}(\mathbb{A}^{1/2})$ and from $\mathbb{H}(\mathbb{A}^{1/2})$ in to $\mathbb{H}$. Then, by [2, theorem 2] (see also [1, theorem 5.1]), the operator.

$$P(\lambda) : v \to P(\lambda)v := (D^2 - (\mathbb{A} + \lambda I)v(x), v'(1) + \mathbb{B}v(0), v'(0)),$$

where $D := \frac{d}{dx}$, corresponding to the boundary value problem (2.2), (2.3), for sufficiently large $|\lambda|$ from the angle $|arg\lambda| \leq \varphi < \pi$, is an isomorphism from $W_p^2((0,1);\mathbb{H}(\mathbb{A}),\mathbb{H})$ onto $L_p((0,1);\mathbb{H}) \dot{+} (\mathbb{H}(\mathbb{A}),\mathbb{H})_{\frac{1}{2}+\frac{1}{2p},p} \dot{+} (\mathbb{H}(\mathbb{A}),\mathbb{H})_{\frac{1}{2}+\frac{1}{2p},p}$ and, for these λ, the following estimate for the solution of the problem (2.2), (2.3) is valid

$$|\lambda| \, \|v\|_{L_p((0,1);\mathbb{H})} + \left\|v''\right\|_{L_p((0,1);\mathbb{H})} + \|\mathbb{A}v\|_{L_p((0,1);\mathbb{H})}$$
$$\leq C \left[|\lambda|^{\frac{1}{2}-\frac{1}{2p}} \|F\|_{L_p((0,1);\mathbb{H})} + \sum_{k=1}^{2} \left(\|\Phi_k\|_{(\mathbb{H}(\mathbb{A}),\mathbb{H})_{\frac{1}{2}+\frac{1}{2p},p}} + |\lambda|^{\frac{1}{2}-\frac{1}{2p}} \|\Phi_k\|_{\mathbb{H}} \right) \right], \tag{2.4}$$

where $C > 0$ is a constant independent on λ. Further, we have

$$(\mathbb{H}(\mathbb{A}),\mathbb{H})_{\theta,p} = \left(H(A) \oplus H(A^{1/2}), H(A^{1/2}) \oplus H\right)_{\theta,p}$$
$$= \left(H(A), H(A^{1/2})\right)_{\theta,p} \dot{+} \left(H(A^{1/2}), H\right)_{\theta,p}, \quad \theta \in [0,1].$$

By [8,theorem 1.3.3./(b) and formulas 1.15.4/(2) and 1.15.2/(4)] we have

$$\left(H(A), H(A^{1/2})\right)_{\theta,p} = \left(H(A^{1/2}), H(A)\right)_{1-\theta,p}$$
$$= (H, H(A))_{1-\frac{\theta}{2},p} = (H(A), H)_{\frac{\theta}{2},p};$$
$$\left(H(A^{1/2}), H\right)_{\theta,p} = \left(H, H(A^{1/2})\right)_{1-\theta,p}$$
$$= (H, H(A))_{\frac{1}{2}(1-\theta),p} = (H(A), H)_{\frac{1}{2}+\frac{\theta}{2},p}.$$

Hence, for $\theta = \frac{1}{2} + \frac{1}{2p}$, we have

$$\left(H(A), H(A^{1/2})\right)_{\frac{1}{2}+\frac{1}{2p},p} = (H(A), H)_{\frac{1}{4}+\frac{1}{4p},p};$$
$$\left(H(A^{1/2}), H\right)_{\frac{1}{2}+\frac{1}{2p},p} = (H(A), H)_{\frac{3}{4}+\frac{1}{4p},p}.$$

Thus,

$$(\mathbb{H}(\mathbb{A}),\mathbb{H})_{\frac{1}{2}+\frac{1}{2p},p} = (H(A), H)_{\frac{1}{4}+\frac{1}{4p},p} \dot{+} (H(A), H)_{\frac{3}{4}+\frac{1}{4p},p}. \tag{2.5}$$

Using (2.5), rewrite inequality (2.4) in the form

$$|\lambda| \left(\|u\|_{L_p((0,1);H(A^{1/2}))} + \left\|u'' - \lambda u\right\|_{L_p((0,1);H)} \right) + \left\|u''\right\|_{L_p((0,1);H(A^{1/2}))}$$
$$+ \left\|u'''' - \lambda u''\right\|_{L_p((0,1);H)} + \left\|u'' - \lambda u\right\|_{L_p((0,1);H(A^{1/2}))} + \|Au\|_{L_p((0,1);H)}$$

$$\leq C\left[|\lambda|^{\frac{1}{2}-\frac{1}{2p}}\|f\|_{L_p((0,1);H)}+\sum_{k=1}^{2}\left(\|\varphi_k\|_{(H(A),H)_{\frac{1}{4}+\frac{1}{4p},p}}\right.\right.$$

$$\left.\left.+\|\varphi_{k+2}\|_{(H(A),H)_{\frac{3}{4}+\frac{1}{4p},p}}\right)+|\lambda|^{\frac{1}{2}-\frac{1}{2p}}\sum_{k=1}^{2}\left(\|\varphi_k\|_{H(A^{1/2})}+\|\varphi_{k+2}\|_{H}\right)\right], \tag{2.6}$$

where $C>0$ is a constant independent on λ. The estimate (2.6) is valid for sufficiently large $|\lambda|$ from the angle $|\arg\lambda|\leq\varphi<\pi$, uniform with respect to λ. Using the technique of the proof in [9, theorem 3.2.1] and the theorem on the Fourier multiplicators in Hilbert spaces, we can show that, for any $\varepsilon>0$ there exists $C_\varepsilon>0$ such that

$$\|u''\|_{L_p((0,1);H)}+|\lambda|\,\|u\|_{L_p((0,1);H)}$$

$$\leq C_\varepsilon\left\|u''-\lambda u\right\|_{L_p((0,1);H)},\quad |\arg\lambda|<\pi-\varepsilon, \tag{2.7}$$

$$\|u''\|_{L_p((0,1);H(A^{1/2}))}+|\lambda|\,\|u\|_{L_p((0,1);H(A^{1/2}))}$$

$$\leq C_\varepsilon\left\|u''-\lambda u\right\|_{L_p((0,1);H(A^{1/2}))},\quad |\arg\lambda|<\pi-\varepsilon, \tag{2.8}$$

where C_ε is independent on λ. These inequalities hold also in the angle $|\arg\lambda|\leq\varphi<\pi$ as $\varphi<\pi$. By (2.7), (2.8) for λ from the angle $|\arg\lambda|\leq\varphi<\pi$, the left hand side of inequality (2.6) is greater than

$$C_0\left(|\lambda|\,\|u\|_{L_p((0,1);H(A^{1/2}))}+|\lambda|\,\|u''\|_{L_p((0,1);H)}\right.$$

$$\left.|\lambda|^2\,\|u\|_{L_p((0,1);H)}+\|u''\|_{L_p((0,1);H(A^{1/2}))}\right)$$

i.e.,

$$|\lambda|\,\|u\|_{L_p((0,1);H(A^{1/2}))}+|\lambda|\,\|u''\|_{L_p((0,1);H)}+|\lambda|^2\,\|u\|_{L_p((0,1);H)}$$

$$\|u''\|_{L_p((0,1);H(A^{1/2}))}\leq C\left[|\lambda|^{\frac{1}{2}-\frac{1}{2p}}\|f\|_{L_p((0,1);H)}+\sum_{k=1}^{2}\left(\|\varphi_k\|_{(H(A),H)_{\frac{1}{4}+\frac{1}{4p},p}}\right.\right.$$

$$\left.\left.+\|\varphi_{k+2}\|_{(H(A),H)_{\frac{3}{4}+\frac{1}{4p},p}}+|\lambda|^{\frac{1}{2}-\frac{1}{2p}}\sum_{k=1}^{2}\left(\|\varphi_k\|_{H(A^{1/2})}+\|\varphi_{k+2}\|_{H}\right)\right)\right]. \tag{2.9}$$

From (2.6) and (2.9), we have

$$\|u''''\|_{L_p((0,1);H)}+\|Au\|_{L_p((0,1);H)}\leq\left\|u''''-\lambda u''\right\|_{L_p((0,1);H)}+|\lambda|\left\|u''\right\|_{L_p((0,1);H)}$$

$$+\|Au\|_{L_p((0,1);H)}\leq C\left[|\lambda|^{\frac{1}{2}-\frac{1}{2p}}\|f\|_{L_p((0,1);H)}+\sum_{k=1}^{2}\left(\|\varphi_k\|_{(H(A),H)_{\frac{1}{4}+\frac{1}{4p},p}}\right.\right.$$

$$\left.\left.+\|\varphi_{k+2}\|_{(H(A),H)_{\frac{3}{4}+\frac{1}{4p},p}}\right)+|\lambda|^{\frac{1}{2}-\frac{1}{2p}}\sum_{k=1}^{2}\left(\|\varphi_k\|_{H(A^{1/2})}+\|\varphi_{k+2}\|_{H}\right)\right]. \tag{2.10}$$

From (2.9) and (2.10) it follows the estimate (2.1). Theorem 2.1 is proved.

3. Application of abstract results to elliptic partial differential equations

In the square $\Omega = [0,1] \times [0,1]$, let us consider a boundary value problem for fourth order elliptic equations with a parameter.

$$L(\lambda)u := \lambda^2 u(x,y) - 2\lambda \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^4 u(x,y)}{\partial x^4}$$

$$+\sum_{k=0}^{2} (-1)^k \frac{\partial^k}{\partial y^k} \left(a_{2-k}(y) u_y^{(k)}(x,y) \right) = f(x,y), \tag{3.1}$$

$$(L_1 u)(y) := \frac{\partial u(1,y)}{\partial x} + b_1(y) \frac{\partial u(0,y)}{\partial y} = \varphi_1(y), \quad y \in [0,1],$$

$$(L_2 u)(y) := \frac{\partial u(0,y)}{\partial x} = \varphi_2(y), \quad y \in [0,1],$$

$$(L_3(\lambda)u(y) := \frac{\partial^3 u(1,y)}{\partial x^3} - \lambda \frac{\partial u(1,y)}{\partial x} + b_2(y) \frac{\partial}{\partial y} \left(\frac{\partial^2 u(0,y)}{\partial x^2} - \lambda u(0,y) \right) = \varphi_3(y),$$

$$(L_4(\lambda)u(y) := \frac{\partial^3 u(0,y)}{\partial x^3} - \lambda \frac{\partial u(0,y)}{\partial x} = \varphi_4(y), \quad y \in [0,1], \tag{3.2}$$

$$(P_l u)(x) = u_y^{(l)}(x,0) = 0, (Q_l u)(x) = u_y^{(l)}(x,1) = 0, \quad x \in [0,1], \quad l = 0,1, \tag{3.3}$$

where λ is a complex parameter, $a_{2-k}(y), b_1(y), b_2(y)$ are some continuous functions.

Denote the interpolation spaces of Sobolev space by $B_{q,p}^s(0,1) := \left(W_q^{s_0}(0,1), W_q^{s_1}(0,1)\right)_{\theta,p}$ where $0 \leq s_0, s_1$ are integers, $0 < \theta < 1$, $1 < q < \infty$, $1 < p < \infty$ and $s = (1\text{-}\theta)s_0 + \theta s_1$. In particular, $W_q^{s_0}(0,1) := B_{q,q}^s(0,1) := \left(W_q^{s_0}(0,1), W_q^{s_1}(0,1)\right)_{\theta,p}$ if $0 < s \neq$ to integer.

Theorem 3.1. *Let the following conditions be fulfilled:*

1) $a_{2-k}(y)$ $(k = 0,1,2)$ *are real, k times continuously differentiable on* $[0,1]$ *functions,* $a_{2-k}(y) > 0$ *on* $[0,1]$*;*

2) $b_1(y) \in C^3[0,1]$, $\quad b_2(y) \in C^1[0,1]$.

Then, for sufficiently large $|\lambda|$ *from the angle* $|\arg \lambda| \leq \varphi < \pi$, *the operator* $\mathbb{L}(\lambda) : u \to \mathbb{L}(\lambda)u = (L(\lambda)u, L_1 u, L_2 u, L_3(\lambda)u, L_4(\lambda)u)$ *is an isomorphism from* $W_p^4((0,1); W_2^4(0,1); P_l u = 0, Q_l u = 0,\ l = 0,1)$ *onto*

$L_p((0,1); L_2(0,1)) \dotplus B_{2,p}^{3-\frac{1}{p}}((0,1); P_l u = Q_l u = 0,\ l = 0,1) \dotplus$
$\dotplus B_{2,p}^{3-\frac{1}{p}}((0,1); P_l u = Q_l u = 0, l = 0,1) \dotplus B_{2,p}^{1-\frac{1}{p}}((0,1); P_0 u = Q_0 u = 0)$
$\dotplus B_{2,p}^{1-\frac{1}{p}}((0,1);\ P_0 u = Q_0 u = 0)$ *and, for these values of* λ, *the following estimate holds for the solution* $u(x,y)$ *of the problem* (3.1)-(3.3)

$$|\lambda|^2 \|u\|_{L_p((0,1);L_2(0,1))} + |\lambda| \|u\|_{L_p\left((0,1);W_2^2(0,1)\right)} + \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{L_p\left((0,1);W_2^2(0,1)\right)}$$

$$+ \left\| \frac{\partial^4 u}{\partial x^4} \right\|_{L_p((0,1);L_2(0,1))} + \left\| \frac{\partial^4 u}{\partial y^4} \right\|_{L_p((0,1);L_2(0,1))}$$

$$\leq C\left[|\lambda|^{\frac{1}{2}-\frac{1}{2p}}\|f\|_{L_p((0,1);H)}+\sum_{k=1}^{2}\left(\|\varphi_k\|_{B_{2,p}^{3-\frac{1}{p}}(0,1)}+\|\varphi_{k+2}\|_{B_{2,p}^{1-\frac{1}{p}}(0,1)}\right)\right.$$

$$\left.+|\lambda|^{\frac{1}{2}-\frac{1}{2p}}\sum_{k=1}^{2}\left(\|\varphi_k\|_{W_2^2(0,1)}+\|\varphi_{k+2}\|_{L_2(0,1)}\right)\right], \tag{3.4}$$

where the constant C does not depend on the parameter λ.

Proof. Denote $H := L_2(0,1)$. In the space $L_2(0,1)$, define the operators A, B_1, B_2 by the following equalities.

$$D(A) := W_2^4\left((0,1); P_l u = Q_l u = 0,\ l = 0,1\right), Au := \sum_{k=0}^{2}(-1)^k \frac{d^k}{dy^k}\left(a_{2-k}(y)u^{(k)}\right), \tag{3.5}$$

$$D(B_1) := W_2^1(0,1), \quad B_1 u := b_1(y)u'(y); \tag{3.6}$$

$$D(B_2) := W_2^1(0,1), \quad B_2 u := b_2(y)u'(y). \tag{3.7}$$

Then, problem (3.1)-(3.3) is reduced to the boundary value problem

$$\lambda^2 u(x) - 2\lambda u''(x) + u''''(x) + Au(x) = f(x), \quad x \in (0,1), \tag{3.8}$$

$$\begin{gathered} u'(1) + B_1 u(0) = \varphi_1, \\ u'(0) = \varphi_2, \\ u'''(1) - \lambda u'(1) + B_2\left((u''(0) - \lambda u(0)\right) = \varphi_3, \\ u'''(0) - \lambda u'(0) = \varphi_4, \end{gathered} \tag{3.9}$$

where $u(x) := u(x,\cdot)$, $f(x) := f(x,\cdot)$ are the functions with the values in the Hilbert space $H = L_2(0,1)$, and $\varphi_k = \varphi_k(\cdot)$. Obviously, the proof of the theorem is reduced to verification of the conditions of theorem 1. It is known that (see for instance [7]) the operator A defined by the equality (3.5) is a self-adjoint, positive-definite operator in $L_2(0,1)$ with a discrete spectrum, i.e. the first condition of theorem 1 for problem (3.8) and (3.9) is satisfied. Fulfillment of the second and third conditions of theorem 1 for the operators B_1 and B_2 defined by equalities (3.6) and (3.7), respectively, was shown in paper [3]. Theorem 3.1 is proved.

Remark. In the formulation of theorem 2, $B_{2,p}^{3-\frac{1}{p}}((0,1); P_l u = Q_l u = 0, l = 0,1)$ denotes a set of the functions from the Besov space $B_{2,p}^{3-\frac{1}{p}}(0,1)$, satisfying boundary conditions $P_l u = Q_l u = 0, l = 0,1$, wherein the order of $P_l u$ and $Q_l u$ is less than $3-\frac{1}{p}$.

By [8, section 4.3.3], $B_{2,p}^{3-\frac{1}{p}}((0,1); P_l u = Q_l u = 0, l = 0,1)$ is defined as the interpolation space $(H(A), H)_{\frac{1}{4}+\frac{1}{4p},p}$, where

$H(A) = W_2^4((0,1); P_l u = Q_l u = 0, l = 0,1)$ and $H = L_2(0,1)$.

In a similar way, $B_{2,p}^{1-\frac{1}{p}}((0,1); P_0 u = Q_0 u = 0)$ is defined as the interpolation space $(H(A), H)_{\frac{3}{4}+\frac{1}{4p},p}$.

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