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THE ABSENCE OF GLOBAL SOLUTIONS OF A SYSTEM OF SEMILINEAR PARABOLIC EQUATIONS WITH A SINGULAR POTENTIAL

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Abstract. In the domain $Q_R^{'} = \{x; |x| > R\} \times (0, +\infty)$ we consider the following problem:

$$\begin{cases} \frac{\partial u}{\partial t} = \alpha_1 \Delta u + \frac{C_1}{|x|^2} u + |x|^{\sigma_1} v^{q_1}, \ u|_{t=0} = u_0(x) \geq 0 \\ \frac{\partial u}{\partial t} = \alpha_2 \Delta v + \frac{C_2}{|x|^2} u + |x|^{\sigma_2} v^{q_2}, \ v|_{t=0} = v_0(x) \geq 0, \end{cases}$$

where $\alpha_i > 0$, $\sigma_i > -2$, $0 \leq C_i \leq \alpha_i \left(\frac{n-2}{2}\right)^2$, $q_i > 1, i = 1, 2$. A sufficient condition on the absence of global solutions is obtained. The proof is based on the method of test functions.

1. Introduction

Let us introduce the following notations: $x = (x_1, ..., x_n) \in R^n$, $n \geq 3$, $r = |x| = \sqrt{x_1^2 + ... + x_n^2}$, $B_R = \{x; |x| < R\}$, $B_R^{'} = \{x; |x| > R\}$, $Q_R = B_R \times (0, +\infty)$, $Q_R^{'} = B_R^{'} \times (0, +\infty)$, $\nabla u = \left(\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}\right)$, and $C_{x,t}^{2,1}\left(Q_R^{'}\right)$ is the set of functions twice continuously differentiable with respect to x and continuously differentiable with respect to t in $Q_R^{'}$.

In the domain $Q_R^{'}$ we consider the system of equations

$$\begin{cases} \frac{\partial u}{\partial t} = \alpha_1 \Delta u + \frac{C_1}{|x|^2} u + |x|^{\sigma_1} v^{q_1} \\ \frac{\partial u}{\partial t} = \alpha_2 \Delta v + \frac{C_2}{|x|^2} u + |x|^{\sigma_2} v^{q_2} \end{cases} \tag{1.1}$$

with initial conditions

$$u|_{t=0} = u_0(x), \qquad v|_{t=0} = v_0(x), \tag{1.2}$$

where $u_0(x)$, $v_0(x) \in C\left(B_R^{'}\right)$, $u_0(x) \geq 0, v_0(x) \geq 0$, $\alpha_i > 0$, $\sigma_i > -2$, $0 \leq C_i \leq \alpha_i \left(\frac{n-2}{2}\right)^2$, $q_i > 1$, $i = 1, 2$.

We will study the existence of a global solution to problem (1.1), (1.2). Under a global solution of problem (1.1), (1.2) we will understand such pairs of functions (u, v) that $u \in C_{x,t}^{2,1}\left(Q_R^{'}\right)$, $v \in C_{x,t}^{2,1}\left(Q_R^{'}\right)$ and u, v satisfy system (1.1) at each point of $Q_R^{'}$ and initial conditions (1.2) for $t = 0$.

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The problems of non-existence of global solutions for different classes of differential equations and inequalities play a key role in theory and applications. Therefore, they are at constant attention of mathematicians and a great number of works were devoted to them. See the monograph [12] for the survey of such results.

In the classical paper [5] Fujita considered the following initial value problem

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + u^q, (x,t) \in R^n \times (0,+\infty), \\ \\ u|_{t=0} = u_0(x), x \in R^n, \end{cases} \tag{1.3}$$

and proved that positive global solutions of problem (1.3) do not exist for $1 < q < q^* = 1 + \frac{2}{n}$, and for $q > q*$ for small $u_0(x)$ there are positive global solutions. The case $q = q*$ was investigated in [7], [8] and it is proved that in this case there also do not exist positive global solutions. Pinsky [14] showed the existence and nonexistence of global solutions in $R^n \times (0,+\infty)$ to the equation $u_t - \Delta u = a(x)u^q$, where $q > 1$ and $a(x)$ behaves like $|x|^\sigma$ with $\sigma > -2$ for large $|x|$. The results of Fujita's work [5] aroused great interest in the problem of the absence of global solutions, and they were expanded in several directions. For example, instead of R^n, various bounded and unbounded domains are considered, or more general operators were considered than the Laplace operator and nonlinearities of a different type. A survey of such papers is available in [9], in the monograph [12] and in the book [15].

Another possibility of extending Fujita's result is to investigate a system of Fujitatype reaction-diffusion equations, and this is the direction we take. In the paper [3] Escobedo and Herrero considered the problem(1.1),(1.2) on $R^n \times (0,+\infty)$ for $\alpha_1 = \alpha_2 = 1, \sigma_1 = \sigma_2 = 0, C_1 = C_2 = 0$ and obtained the condition of existence of global and non-global solutions. In the paper [4] Fila, Levin, Uda considered the problem(1.1),(1.2) on $R^n \times (0,+\infty)$ for $0 \le \alpha_1 \le 1, \alpha_2 = 1, \sigma_1 = \sigma_2 = 0, C_1 = C_2 = 0$ and studied the existence of nonnegative global and non-global solutions. In case $\alpha_1 = \alpha_2 = 1, C_1 = C_2 = 0$ Mochizuki and Huang [13] showed the existence and nonexistence of global solutions and the asymptotic behavior of the global solution of system (1.1) on $R^n \times (0,+\infty)$. Levin [10] studied nonnegative solutions of the initial boundary value problem for the system (1.1) for $\alpha_1 = \alpha_2 = 1, \sigma_1 = \sigma_2 = 0, C_1 = C_2 = 0$ in domain $D \times (0,+\infty)$, where D is a cone or the exterior of a bounded domain. Other related results can be found, for example in [1], [2], [6], [16], and references therein. In the present paper we consider problem (1.1),(1.2) in the domain $Q_R' = \{x; |x| > R\} \times (0,+\infty)$ for $\alpha_i \le 0, \sigma_i \ge -2, 0 \le C_i \le \alpha_i \left(\frac{n-2}{2}\right)^2, i = 1,2$ and using the technique of test functions, developed by Mitidieri and Pohozaev in the papers [11], [12], find an exponent of absence of a global solution.

2. The main result and its proof

Denote:

$$D_1 = \sqrt{\left(\frac{n-2}{2}\right)^2 - \frac{C_1}{\alpha_1}}, \quad D_2 = \sqrt{\left(\frac{n-2}{2}\right)^2 - \frac{C_2}{\alpha_2}},$$

$$\lambda_1^+ = -\frac{n-2}{2} + D_1, \quad \lambda_1^- = -\frac{n-2}{2} - D_1,$$
$$\lambda_2^+ = -\frac{n-2}{2} + D_2, \quad \lambda_2^- = -\frac{n-2}{2} - D_2,$$
$$\theta_1 = \frac{\sigma_1 + 2 + q_1(\sigma_2 + 2)}{q_1 q_2 - 1} - \lambda_1^+ - n,$$
$$\theta_2 = \frac{\sigma_2 + 2 + q_2(\sigma_1 + 2)}{q_1 q_2 - 1} - \lambda_2^+ - n.$$

Let us consider the functions

$$\xi_1(x) = |x|^{\lambda_1^+} - |x|^{\lambda_1^-},$$
$$\xi_2(x) = |x|^{\lambda_2^+} - |x|^{\lambda_2^-}.$$

It is easy to verify that $\xi_1(x)$, $\xi_2(x)$ are the solutions of the equations

$$\alpha_1 \Delta u + \frac{C_1}{|x|^2} u = 0, \quad \alpha_2 \Delta u + \frac{C_2}{|x|^2} u = 0$$

in B_1'.

The following theorem is the basic result of this paper.

Theorem 2.1. *Let* $n > 2, \alpha_i > 0, \sigma_i > -2, 0 \le C_i \le \alpha_i(\frac{n-2}{2})^2$ *and* $max(\theta_1, \theta_2) \ge 0, i = 1, 2$. *If* $u(x,t) \ge 0, v(x,t) \ge 0$ *is the solution of problem (1.1),(1.2), then* $u(x,t) \equiv 0, v(x,t) \equiv 0$.

Proof. For simplicity of notation we take $R = 1$. We consider the function

$$\varphi(x,t) = \varphi_0\left(\frac{t + |x|^2}{\rho^2}\right),$$

where

$$\varphi_0(s) = \begin{cases} 1, & \text{for} \quad s \le 1 \\ (2-s)^\beta, & \text{for} \quad 1 \le s \le 2 \\ 0, & \text{for} \quad s \ge 2 \end{cases}$$

and β is a rather large positive number. We multiply the first equation by $\varphi(x)\xi_1(x)$, the second one by $\varphi(x)\xi_2(x)$ and integrate with respect to Q_1'.

After integration by parts, we obtain the following relations:

$$\int\int_{Q_1'} |x|^{\sigma_1} v^{q_1} \xi_1 \varphi dxdt = -\int\int_{Q_1'} u\xi_1 \frac{\partial \varphi}{\partial t} dxdt -$$
$$-\alpha_1 \int\int_{Q_1'} u\Delta(\xi_1 \varphi)dxdt - \int\int_{Q_1'} \frac{C_1}{|x|^2} u\xi_1\varphi dxdt - \int_{B_1'} u_0(x)\xi_1(x)\varphi(0,x)dx =$$
$$= -\int\int_{Q_1'} u\xi_1 \frac{\partial \varphi}{\partial t} dxdt - \int\int_{Q_1'} u\varphi\left(\alpha_1 \Delta\xi_1 + \frac{C_1}{|x|^2}\xi_1\right) dxdt -$$
$$-\alpha_1 \int\int_{Q_1'} u(2(\nabla\xi_1, \nabla\varphi) + \xi_1 \nabla\varphi)dxdt - \int_{B_1'} u_0(x)\xi_1(x)\varphi(0,x)dx =$$

$$= -\int\int\limits_{Q_1'} u\xi_1 \frac{\partial \varphi}{\partial t} dxdt - \alpha_1 \int\int\limits_{Q_1'} u\left(2\left(\nabla\xi_1, \nabla\varphi\right) + \xi_1 \nabla\varphi\right) dxdt -$$

$$-\int\limits_{B_1'} u_0(x)\,\xi_1(x)\,\varphi(0,x)\,dx, \tag{2.1}$$

$$\int\int\limits_{Q_1'} |x|^{\sigma_2} u^{q_2} \xi_2 \varphi dxdt = -\int\int\limits_{Q_1'} v\xi_2 \frac{\partial \varphi}{\partial t} dxdt - \alpha_2 \int\int v\Delta\left(\xi_1\varphi\right) dxdt -$$

$$-\int\int\limits_{Q_1'} \frac{C_2}{|x|^2} v\xi_2\varphi dxdt - \int\limits_{B_1'} v_0(x)\,\xi_2(x)\,\varphi(0,x)\,dx =$$

$$= -\int\int\limits_{Q_1'} v\xi_2 \frac{\partial \varphi}{\partial t} dxdt - \alpha_2 \int\int\limits_{Q_1'} v\left(2\left(\nabla\xi_2, \nabla\varphi\right) + \xi_2 \Delta\varphi\right) dxdt -$$

$$-\int\limits_{B_1'} v_0(x)\,\xi_2(x)\,\varphi(0,x)\,dx. \tag{2.2}$$

As the last integrals in (2.1), (2.2) are non-negative, then using the Holder inequality, we get

$$\int\int\limits_{Q_1'} |x|^{\sigma_1} v^{q_1} \xi_1 \varphi dxdt \le \left(\int\int\limits_{\rho^2 \le t+|x|^2 \le 2\rho^2} |x|^{\sigma_2} |u|^{q_2} \xi_2 \varphi dxdt \right)^{\frac{1}{q_2}} \times$$

$$\times \left[\left(\int\int\limits_{\rho^2 \le t+|x|^2 \le 2\rho^2} \frac{\left|\frac{\partial\varphi}{\partial t}\right|^{q_2'} \xi_1^{q_2'}}{|x|^{\sigma_2\left(q_2'-1\right)} \varphi^{q_2'-1} \xi_2^{q_2'-1}} dxdt \right)^{\frac{1}{q_2'}} + \right.$$

$$\left. +\alpha_1 \left(\int\int\limits_{\rho^2 \le t+|x|^2 \le 2\rho^2} \frac{\left|2\left(\nabla\xi_1, \nabla\varphi\right) + \xi_1\Delta\varphi\right|^{q_2'}}{|x|^{\sigma_2\left(q_2'-1\right)} \xi_2^{q_2'-1} \varphi^{q_2'-1}} dxdt \right)^{\frac{1}{q_2'}} \right], \tag{2.3}$$

$$\int\int\limits_{Q_1'} |x|^{\sigma_2} u^{q_2} \xi_2 \varphi dxdt \le \left(\int\int\limits_{\rho^2 \le t+|x|^2 \le 2\rho^2} |x|^{\sigma_1} v^{q_1} \xi_1 \varphi dxdt \right)^{\frac{1}{q_1}} \times$$

$$\times \left[\left(\int\int\limits_{\rho^2 \le t+|x|^2 \le 2\rho^2} \frac{\left|\frac{\partial\varphi}{\partial t}\right|^{q_1'} \xi_2^{q_1'}}{|x|^{\sigma_1\left(q_1'-1\right)} \varphi^{q_1'-1} \xi_1^{q_1'-1}} dxdt \right)^{\frac{1}{q_1'}} + \right.$$

$$+\alpha_2\left(\int\int\limits_{\rho^2\le t+|x|^2\le 2\rho^2}\frac{\left|2\left(\nabla\xi_2,\nabla\varphi\right)+\xi_2\Delta\varphi\right|^{q_1'}}{|x|^{\sigma_1\left(q_1'-1\right)}\xi_1^{q_1'-1}\varphi^{q_1'-1}}dxdt\right)^{\frac{1}{q_1'}}\Bigg], \tag{2.4}$$

where $\frac{1}{q_1}+\frac{1}{q_1'}=1$, $\frac{1}{q_2}+\frac{1}{q_2'}=1$.

Denote the first integral in square bracket (2.3) by I_1, the second one by J_1 and the first integral in the square bracket (2.4) by I_2, the second one by J_2, respectively.

Then we can write (2.3) and (2.4) in the form:

$$\int\int\limits_{Q_1'}|x|^{\sigma_1}v^{q_1}\xi_1\varphi dxdt\le\left(\int\int\limits_{\rho^2\le t+|x|^2\le 2\rho^2}|x|^{\sigma_2}|u|^{q_2}\xi_2\varphi dxdt\right)^{\frac{1}{q_2}}\times$$

$$\times\left[I_1^{\frac{1}{q_2'}}+\alpha_1 J_1^{\frac{1}{q_2'}}\right],$$

$$\int\int\limits_{Q_1'}|x|^{\sigma_2}u^{q_2}\xi_2\varphi dxdt\le\left(\int\int\limits_{\rho^2\le t+|x|^2\le 2\rho^2}|x|^{\sigma_1}|v|^{q_1}\xi_1\varphi dxdt\right)^{\frac{1}{q_1}}\times$$

$$\times\left[I_2^{\frac{1}{q_1'}}+\alpha_2 J_2^{\frac{1}{q_1'}}\right].$$

Taking into account the second one of these inequalities in the first one, and the first one in the second one,

$$\int\int\limits_{Q_1'}|x|^{\sigma_1}v^{q_1}\xi_1\varphi dxdt\le\left(\int\int\limits_{\rho^2\le t+|x|^2\le 2\rho^2}|x|^{\sigma_1}|u|^{q_1}\xi_1\varphi dxdt\right)^{\frac{1}{q_1q_2}}\times$$

$$\times\left[I_1^{\frac{1}{q_2'}}+\alpha_1 J_1^{\frac{1}{q_2'}}\right]\cdot\left[I_2^{\frac{1}{q_1'}}+\alpha_2 J_2^{\frac{1}{q_1'}}\right]^{\frac{1}{q_2}}, \tag{2.5}$$

$$\int\int\limits_{Q_1'}|x|^{\sigma_2}u^{q_2}\xi_2\varphi dxdt\le\left(\int\int\limits_{\rho^2\le t+|x|^2\le 2\rho^2}|x|^{\sigma_2}|u|^{q_2}\xi_2\varphi dxdt\right)^{\frac{1}{q_1q_2}}\times$$

$$\times\left[I_2^{\frac{1}{q_1'}}+\alpha_2 J_2^{\frac{1}{q_1'}}\right]\cdot\left[I_1^{\frac{1}{q_1'}}+\alpha_1 J_2^{\frac{1}{q_2'}}\right]^{\frac{1}{q_1}}. \tag{2.6}$$

Hence we have

$$\int\int\limits_{Q_1'}|x|^{\sigma_1}v^{q_1}\xi_1\varphi dxdt\le\left[I_1^{\frac{1}{q_2'}}+\alpha_1 J_1^{\frac{1}{q_2'}}\right]^{\frac{q_1q_2}{q_1q_2-1}}\cdot\left[I_2^{\frac{1}{q_1'}}+\alpha_2 J_2^{\frac{1}{q_1'}}\right]^{\frac{q_1}{q_1q_2-1}}, \tag{2.7}$$

$$\int\int_{Q_1'} |x|^{\sigma_2} u^{q_2} \xi_2 \varphi dxdt \leq \left[I_2^{\frac{1}{q_1'}} + \alpha_2 J_2^{\frac{1}{q_1'}} \right]^{\frac{q_1 q_2}{q_1 q_2 - 1}} \cdot \left[I_1^{\frac{1}{q_2'}} + \alpha_1 J_1^{\frac{1}{q_2'}} \right]^{\frac{q_2}{q_1 q_2 - 1}}. \quad (2.8)$$

Having made the substitution $t = \rho^2 \tau$, $x = \rho y$, $\xi_i(y) = \xi_i(\rho y)$, $\varphi(\tau, y) = \varphi(\rho^2 \tau, \rho y) = \varphi_0 \left(\tau + |y|^2 \right)$, $i = 1, 2$ estimate the integrals I_1, J_1, I_2, J_2

$$I_1 = \int\int_{\rho^2 \leq t + |x|^2 \leq 2\rho^2} \frac{\left| \frac{\partial \varphi}{\partial t} \right|^{q_2'} \xi_1^{q_2'}}{|x|^{\sigma_2 \left(q_2' - 1 \right)} \varphi^{q_2' - 1} \xi_2^{q_2' - 1}} dxdt =$$

$$= \rho^{-2q_2' + 2 + \lambda_1^+ q_2' - \sigma_2 \left(q_2' - 1 \right) - \lambda_2^+ \left(q_2' - 1 \right) + n} \times$$

$$\times \int\int_{1 \leq \tau + |y|^2 \leq 2} \frac{\left| \frac{\partial \varphi_0}{\partial \tau} \right|^{q_2'} \left(1 - \rho^{-2D_1} |y|^{-2D_1} \right)^{q_2'}}{|y|^{\sigma_2 \left(q_2' - 1 \right)} \varphi_0^{q_2' - 1} \left(1 - \rho^{-2D_2} |y|^{-2D_2} \right)^{q_2' - 1}} dyd\tau \leq$$

$$\leq \rho^{-\left(q_2' - 1 \right)(\sigma_2 + 2) + \lambda_1^+ q_2' - \lambda_2^+ \left(q_2' - 1 \right) + n} \times \widetilde{I}_1, \quad (2.9)$$

$$I_2 = \int\int_{\rho^2 \leq t + |x|^2 \leq 2\rho^2} \frac{\left| \frac{\partial \varphi}{\partial t} \right|^{q_1'} \xi_2^{q_1'}}{|x|^{\sigma_1 \left(q_1' - 1 \right)} \varphi^{q_1' - 1} \xi_1^{q_1' - 1}} dxdt =$$

$$= \rho^{-2q_1' + 2 + \lambda_2^+ q_1' - \sigma_1 \left(q_2' - 1 \right) - \lambda_1^+ \left(q_1' - 1 \right) + n} \times$$

$$\times \int\int_{1 \leq \tau + |y|^2 \leq 2} \frac{\left| \frac{\partial \varphi_0}{\partial \tau} \right|^{q_1'} \left(1 - \rho^{-2D_2} |y|^{-2D_2} \right)^{q_1'}}{|y|^{\sigma_1 \left(q_1' - 1 \right)} \varphi_0^{q_1' - 1} \left(1 - \rho^{-2D_1} |y|^{-2D_1} \right)^{q_1' - 1}} dyd\tau \leq$$

$$\leq \rho^{-\left(q_1' - 1 \right)(\sigma_1 + 2) + \lambda_2^+ q_1' - \lambda_1^+ \left(q_1' - 1 \right) + n} \cdot \widetilde{I}_2, \quad (2.10)$$

where by $\widetilde{I}_1$, $\widetilde{I}_2$ we denote the last integrals in (2.9), (2.10).

$$J_1 = \int\int_{\rho^2 \leq t + |x|^2 \leq 2\rho^2} \frac{\left| 2 \left(\nabla \xi_1, \nabla \varphi \right) + \xi_1 \Delta \varphi \right|^{q_2'}}{|x|^{\sigma_2 \left(q_2' - 1 \right)} \xi_2^{q_2' - 1} \varphi^{q_2' - 1}} dxdt \leq$$

$$\leq \rho^{\left(\lambda_1^+ - 2 \right) q_2' - \lambda_2^+ \left(q_2' - 1 \right) - \sigma_2 \left(q_2' - 1 \right) + n + 2} \times$$

$$\times \int\int_{1 \leq \tau + |y|^2 \leq 2} \frac{\left| C_3 |y|^{\lambda_1^+ - 2} \left(y, \nabla_y \varphi_0 \right) + C_4 |y|^{\lambda_1^+} \Delta_y \varphi_0 \right|^{q_2'}}{|y|^{\sigma_2 \left(q_2' - 1 \right)} \left(1 - \rho^{-2D_2} |y|^{-2D_2} \right)^{q_2' - 1} \varphi_0^{q_2' - 1}} dyd\tau \leq$$

$$\leq \rho^{-\left(q_2' - 1 \right)(\sigma_2 + 2) + \lambda_1^+ q_2' - \lambda_2^+ \left(q_2' - 1 \right) + n} \times \widetilde{J}_1, \quad (2.11)$$

$$J_2 \leq \rho^{-\left(q_1' - 1 \right)(\sigma_1 + 2) + \lambda_2^+ q_1' - \lambda_2^+ \left(q_1' - 1 \right) + n} \times$$

$$\times \iint\limits_{1\le\tau+|y|^2\le 2} \frac{\left|C_5\,|y|^{\lambda_2^+-2}\,(y,\nabla_y\varphi_0)+C_6\,|y|^{\lambda_2^+}\,\Delta_y\varphi_0\right|^{q_1'}}{|y|^{\sigma_2\left(q_2'-1\right)}\left(1-\rho^{-2D_2}\,|y|^{-2D_2}\right)^{q_2'-1}\varphi_0^{q_2'-1}}dyd\tau \le$$

$$\le \rho^{-\left(q_1'-1\right)(\sigma_1+2)+\lambda_2^+q_1'-\lambda_1^+\left(q_1'-1\right)+n}\times \widetilde{J}_2, \tag{2.12}$$

where by $\widetilde{J}_1$, $\widetilde{J}_2$ we denote the last integrals in (2.11), (2.12). It is known that for large β , $\widetilde{I}_1$, $\widetilde{I}_2$,$\widetilde{J}_1$,$\widetilde{J}_2 < \infty$ (see [12]). Using (2.9), (2.10), (2.11), (2.12),from (2.7),(2.8) we get

$$\iint\limits_{Q_1'} |x|^{\sigma_1} v^{q_1}\xi_1\varphi dxdt \le \rho^{\left(-\left(q_2'-1\right)(\sigma_2+2)+\lambda_1^+q_2'-\lambda_2^+\left(q_2'-1\right)+n\right)\frac{q_1q_2}{q_2'(q_1q_2-1)}}\times$$

$$\times\rho^{\left(-\left(q_1'-1\right)(\sigma_1+2)+\lambda_2^+q_1'-\lambda_1^+\left(q_1'-1\right)+n\right)\frac{q_1}{q_1'(q_1q_2-1)}}\times$$

$$\times\left[\widetilde{I}_1^{\frac{1}{q_2'}}+\alpha_1\widetilde{J}_1^{\frac{1}{q_2'}}\right]^{\frac{q_1q_2}{q_1q_2-1}}\cdot\left[\widetilde{I}_2^{\frac{1}{q_1'}}+\alpha_2\widetilde{J}_2^{\frac{1}{q_1'}}\right]^{\frac{q_1q_2}{q_1q_2-1}}=$$

$$=\rho^{\left[\left(-(\sigma_2+2)+\lambda_1^+q_2-\lambda_2^++n(q_2-1)\right)q_1+\left(-(\sigma_1+2)+\lambda_2^+q_1-\lambda_1^++n(q_1-1)\right)\right]\frac{1}{q_1q_2-1}}\times$$

$$\times\left[\widetilde{I}_1^{\frac{1}{q_2'}}+\alpha_1\widetilde{J}_1^{\frac{1}{q_2'}}\right]^{\frac{q_1q_2}{q_1q_2-1}}\cdot\left[\widetilde{I}_2^{\frac{1}{q_1'}}+\alpha_2\widetilde{J}_2^{\frac{1}{q_1'}}\right]^{\frac{q_1}{q_1q_2-1}}=$$

$$=\rho^{-\theta_1}\left[\widetilde{I}_1^{\frac{1}{q_2'}}+\alpha_1\widetilde{J}_1^{\frac{1}{q_2'}}\right]^{\frac{q_1q_2}{q_1q_2-1}}\cdot\left[\widetilde{I}_2^{\frac{1}{q_1'}}+\alpha_2\widetilde{J}_2^{\frac{1}{q_1'}}\right]^{\frac{q_1}{q_1q_2-1}}, \tag{2.13}$$

$$\iint\limits_{Q_1'} |x|^{\sigma_2} u^{q_2}\xi_2\varphi dxdt \le \rho^{\left(-\left(q_1'-1\right)(\sigma_1+2)+\lambda_2^+q_1'-\lambda_1^+\left(q_1'-1\right)+n\right)\frac{q_1q_2}{q_2'(q_1q_2-1)}}\times$$

$$\times\rho^{\left(-\left(q_1'-1\right)(\sigma_2+2)+\lambda_1^+q_2'-\lambda_2^+\left(q_2'-1\right)+n\right)\frac{q_2}{q_2'(q_1q_2-1)}}\times$$

$$\times\left[\widetilde{I}_2^{\frac{1}{q_1'}}+\alpha_2\widetilde{J}_2^{\frac{1}{q_1'}}\right]^{\frac{q_1q_2}{q_1q_2-1}}\cdot\left[\widetilde{I}_1^{\frac{1}{q_2'}}+\alpha_1\widetilde{J}_1^{\frac{1}{q_2'}}\right]^{\frac{q_1q_2}{q_1q_2-1}}=$$

$$=\rho^{-\theta_2}\left[\widetilde{I}_2^{\frac{1}{q_1'}}+\alpha_2\widetilde{J}_2^{\frac{1}{q_1'}}\right]^{\frac{q_1q_2}{q_1q_2-1}}\cdot\left[\widetilde{I}_1^{\frac{1}{q_2'}}+\alpha_1\widetilde{J}_1^{\frac{1}{q_2'}}\right]^{\frac{q_1}{q_1q_2-1}}. \tag{2.14}$$

Let now $\max(\theta_1,\theta_2)>0$.

If for example $\theta_1>0$, then passing to limit in (2.13), as $\rho\to+\infty$ we get

$$\iint\limits_{Q_1'} |x|^{\sigma_1} v^{q_1}\xi_2 dxdt \le 0.$$

This means that $v\equiv 0$. Then from the second equation of (1.1) we get $u\equiv 0$. And if $\theta_2>0$, then from (2.14) we get $u\equiv 0$ and $v\equiv 0$, respectively.

Let now $(\theta_1, \theta_2) = 0$. For example, if $\theta_1 = 0$, then from (2.13) we get

$$\int\int_{Q_1'} |x|^{\sigma_1} v^{q_1} \xi_2 dxdt \le C_7.$$

Consequently, from the property of integral it follows that

$$\int\int_{\rho^2 \le t+|x|^2 \le 2\rho^2} |x|^{\sigma_1} v^{q_1} \xi_2 dxdt \to 0 \quad \text{as} \quad \rho \to +\infty. \tag{2.15}$$

Then from (2.5), considering (2.15), we obtain that

$$\int\int_{Q_1'} |x|^{\sigma_1} v^{q_1} \xi_1 \varphi dxdt \le \left(\int\int_{\rho^2 \le t+|x|^2 \le 2\rho^2} |x|^{\sigma_1} |v|^{q_1} \xi_1 \varphi dxdt \right)^{\frac{1}{q_1 q_2}} \times$$

$$\times \left[I_1^{\frac{1}{q_1'}} + \alpha_1 J_1^{\frac{1}{q_2'}} \right] \cdot \left[\widetilde{I}_2^{\frac{1}{q_1'}} + \alpha_2 J_2^{\frac{1}{q_2'}} \right]^{\frac{1}{q_2}} \le$$

$$\le \left(\int\int_{\rho^2 \le t+|x|^2 \le 2\rho^2} |x|^{\sigma_1} |v|^{q_1} \xi_1 dxdt \right)^{\frac{1}{q_1 q_2}} \times$$

$$\times \left[\widetilde{I}_1^{\frac{1}{q_1'}} + \alpha_1 J_1^{\frac{1}{q_2'}} \right] \cdot \left[\widetilde{I}_2^{\frac{1}{q_1'}} + \alpha_2 J_2^{\frac{1}{q_2'}} \right]^{\frac{1}{q_2}} \to 0$$

This means that in this case $v \equiv 0$ and $u \equiv 0$ as well. Furthermore, if $\theta_2 = 0$, then similarly, from (2.6) it follows that $u \equiv 0$ and $v \equiv 0$, respectively. This completely proves the theorem. □

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