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VECTOR AND AFFINOR FIELDS ON CROSS-SECTIONS IN THE SEMI-COTANGENT BUNDLE

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Abstract. The main purpose of this paper is to study the behavior of complete lifts of vector and affinor (tensor of type (1,1)) fields on cross-sections for pull-back (semi-cotangent) bundle t*B.

1. Introduction

Let an $n-$dimensional differentiable manifold M_n of class C^∞ is a fiber bundle (M_n, π_1, B_m) with projection $\pi_1 : M_n \to B_m$. We use the notation $(x^i) = (x^a, x^\alpha)$, where the indices $i, j, ...$ run from 1 to n, the indices $a, b, ...$ from 1 to $n-m$ and the indices $\alpha, \beta, ...$ from $n-m+1$ to n, x^α are coordinates in B_m, x^a are fibre coordinates of the bundle $\pi_1 : M_n \to B_m$.

Let $(T^*(B_m), \widetilde{\pi}, B_m)$ be a cotangent bundle with base space B_m. Then the semi-cotangent [9], [10] bundle (induced or pull-back) of $(T^*(B_m), \widetilde{\pi}, B_m)$ is the bundle $(t^*(B_m), \pi_2, M_n)$ over M_n with a total space

$$\begin{aligned} t^*(B_m) &= \{((x^a, x^\alpha), x^{\overline{\alpha}}) \in M_n \times T^*_x(B_m) : \pi_1(x^a, x^\alpha) = \widetilde{\pi}(x^\alpha, x^{\overline{\alpha}}) = (x^\alpha)\} \\ &\subset M_n \times T^*_x(B_m) \end{aligned}$$

and with the projection map $\pi_2 : t^*(B_m) \to M_n$ defined by $\pi_2(x^a, x^\alpha, x^{\overline{\alpha}}) = (x^a, x^\alpha)$, where $T^*_x(B_m)\,(x = \pi_1(\widetilde{x}), \widetilde{x} = (x^a, x^\alpha) \in M_n)$ is the cotangent space at a point x of B_m (for definition of the pull-back bundle, see for example [1], [3], [5], [6]), where $x^{\overline{\alpha}} = p_\alpha\ (\overline{\alpha}, \overline{\beta}, ... = n+1, ..., m)$ are fiber coordinates of cotangent bundle $T^*(B_m)$. We denote by $\Im^p_q(M_n)$ and $\Im^p_q(B_m)$ the modules over $F(M_n)$ and $F(B_m)$ of all tensor fields of type (p, q) on M_n and B_m, respectively, where $F(M_n)$ and $F(B_m)$ denote the rings of real-valued $C^\infty-$functions on M_n and B_m, respectively.

If $\pi_1 : M_n \to B_m$ is a differentiable map between the manifolds M_n and B_m then the functions on B_m can be pulled back by π_1 to give functions on M_n. β_θ is differentiable as a mapping $M_n \to t^*(B_m)$ if and only if $\Phi \in C^\infty(B_m)$ implies $\beta_\theta(\Phi) \in C^\infty(M_n)$, where $(\beta_\theta(\Phi))(p) = \Phi(\pi_1(p))$ for all $p \in M_n$. Let θ be a covector field in an $n-$dimensional manifold M_n. Then the transformation $p \to \theta_p$, θ_p being the value of θ at $p \in M_n$, determines a cross-section β_θ of the semi-cotangent bundle. Thus if $\sigma : B_m \to T^*(B_m)$ is a cross-section of

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$(T^*(B_m), \widetilde{\pi}, B_m)$, such that $\widetilde{\pi} \circ \sigma = I_{(B_m)}$, an associated cross-section $\beta_\theta : M_n \to t^*(B_m)$ of semi-cotangent bundle $(t^*(B_m), \pi_2, M_n)$ defined by [[2], p. 217-218], [[8], p. 301]:

$$\beta_\theta\left(x^a, x^\alpha\right) = \left(x^a, x^\alpha, \sigma \circ \pi_1\left(x^a, x^\alpha\right)\right) = \left(x^a, x^\alpha, \sigma\left(x^\alpha\right)\right) = \left(x^a, x^\alpha, \theta_\alpha\left(x^\beta\right)\right).$$

2. Lifts of Vector Fields on a Cross-Section in the Semi-Cotangent Bundle

If the covector field θ has the local components $\theta_\alpha\left(x^\beta\right)$, the cross-section $\beta_\theta\left(M_n\right)$ of $t^*(B_m)$ is locally expressed by

$$\begin{cases} x^a = x^a, \\ x^\alpha = x^\alpha, \\ x^{\overline{\alpha}} = p_\alpha = \theta_\alpha\left(x^\beta\right), \end{cases} \tag{2.1}$$

with respect to the coordinates $x^A = (x^a, x^\alpha, x^{\overline{\alpha}})$ on $t^*(B_m)$. x^a being considered as parameters. Taking the derivative with respect to x^b, we have $k-$local vector fields $B_{(b)}$ $(k = 1, ..., n-m)$ with the components

$$B_{(b)} = \frac{\partial x^A}{\partial x^b} = \partial_b x^A = \begin{pmatrix} \partial_b x^a \\ \partial_b x^\alpha \\ \partial_b \theta_\alpha \end{pmatrix},$$

which are tangent to the cross-section $\beta_\theta\left(M_n\right)$. Thus $B_{(b)}$ has the components

$$B_{(b)} : \left(B_{(b)}^A\right) = \begin{pmatrix} \delta_b^a \\ 0 \\ 0 \end{pmatrix}$$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ on $t^*(B_m)$. Where

$$\delta_b^a = A_b^a = \frac{\partial x^a}{\partial x^b}.$$

Let $\widetilde{X} \in \Im_0^1\left(M_n\right)$ be a projectable vector field [7] with projection $X = X^\alpha(x^\alpha)\partial_\alpha$ i.e. $\widetilde{X} = \widetilde{X}^a(x^a, x^\alpha)\partial_a + X^\alpha(x^\alpha)\partial_\alpha$, we denote by BX the vector field with local components

$$BX : \left(B_{(b)}^A \widetilde{X}^b\right) = \begin{pmatrix} \delta_b^a \widetilde{X}^b \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} A_b^a \widetilde{X}^b \\ 0 \\ 0 \end{pmatrix} \tag{2.2}$$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ on $t^*(B_m)$, which is defined globally along $\beta_\theta\left(M_n\right)$. Then a mapping

$$B : \Im_0^1(M_n) \to \Im_0^1(\beta_\theta\left(M_n\right))$$

is defined by (2.2). The mapping B is the differential of $\beta_\theta : M_n \to t^*(B_m)$ and so an isomorphism of $\Im_0^1(M_n)$ onto $\Im_0^1(\beta_\theta\left(M_n\right))$.

Since a cross-section is locally expressed by

$$\begin{cases} x^a = const., \\ x^{\overline{\alpha}} = p_\alpha = const., \\ x^\alpha = x^\alpha, \end{cases}$$

x^α being considered as parameters. Taking the derivative with respect to x^β, we have $r-$local vector fields $C_{(\beta)}$ $(r = n - m + 1, ..., n)$ with the components

$$C_{(\beta)} = \frac{\partial x^A}{\partial x^\beta} = \partial_\beta x^A = \begin{pmatrix} \partial_\beta x^a \\ \partial_\beta x^\alpha \\ \partial_\beta \theta_\alpha \end{pmatrix},$$

which are tangent to the cross-section $\beta_\theta\,(M_n)$.

Thus $C_{(\beta)}$ has the components

$$C_{(\beta)} : \left(C^A_{(\beta)}\right) = \begin{pmatrix} A^a_\beta \\ \delta^\alpha_\beta \\ \partial_\beta \theta_\alpha \end{pmatrix}$$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ on $t^*(B_m)$. Where

$$A^a_\beta = \frac{\partial x^a}{\partial x^\beta}, \quad \delta^\alpha_\beta = A^\alpha_\beta = \frac{\partial x^\alpha}{\partial x^\beta}.$$

Let $\widetilde{X} \in \Im^1_0\,(M_n)$ be a projectable vector field [7] with projection $X = X^\alpha(x^\alpha)\partial_\alpha$ i.e. $\widetilde{X} = \widetilde{X}^a(x^a, x^\alpha)\partial_a + X^\alpha(x^\alpha)\partial_\alpha$. Then we denote by CX the vector field with local components

$$CX : \left(C^A_{(\beta)} X^\beta\right) = \begin{pmatrix} A^a_\beta X^\beta \\ X^\alpha \\ X^\beta \partial_\beta \theta_\alpha \end{pmatrix} \tag{2.3}$$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ on $t^*(B_m)$, which is defined globally along $\beta_\theta\,(M_n)$. Then a mapping

$$C : \Im^1_0(M_n) \to \Im^1_0(\beta_\theta\,(M_n))$$

is defined by (2.3). The mapping C is the differential of $\beta_\theta : M_n \to t^*(B_m)$ and so an isomorphism of $\Im^1_0(M_n)$ onto $\Im^1_0(\beta_\theta\,(M_n))$.

Now, consider $\omega \in \Im^0_1(B_m)$ and projectable vector field $\widetilde{X} \in \Im^1_0\,(M_n)$, then ${}^{vv}\omega$ (vertical lift) and ${}^{cc}\widetilde{X}$ (complete lift) have respectively, components on the semi-cotangent bundle $t^*(B_m)$ [10]:

$${}^{vv}\omega = \begin{pmatrix} 0 \\ 0 \\ \omega_\alpha \end{pmatrix}, \quad {}^{cc}\widetilde{X} = \begin{pmatrix} \widetilde{X}^a \\ X^\alpha \\ -p_\varepsilon(\partial_\alpha X^\varepsilon) \end{pmatrix} \tag{2.4}$$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$.

On the other hand, the fibre is locally represented by

$$\begin{cases} x^a = const., \\ x^\alpha = const., \\ x^{\overline{\alpha}} = p_\alpha = p_\alpha, \end{cases}$$

p_α being considered as parameters. Thus, by differentiating with respect to p_α, we easily see that the $l-$local vector fields $E_{(\overline{\beta})} =^{vv} \left(dx^\beta\right)$ $(l = n+1, ..., m)$ with components

$$E_{(\overline{\beta})} : \left(E^A_{(\overline{\beta})}\right) = \partial_{\overline{\beta}} x^A = \begin{pmatrix} \partial_{\overline{\beta}} x^a \\ \partial_{\overline{\beta}} x^\alpha \\ \partial_{\overline{\beta}} p_\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \delta^\beta_\alpha \end{pmatrix}$$

is tangent to the fibre, where

$$\delta^\beta_\alpha = A^\beta_\alpha = \frac{\partial x^\beta}{\partial x^\alpha}.$$

Let ω be an 1-form with local components ω_α on B_m, so that ω is a 1-form with local expression $\omega = \omega_\alpha dx^\alpha$. We denote by $E\omega$ the vector field with local components

$$E\omega : \left(E^A_{(\overline{\beta})}\omega_\beta\right) = \begin{pmatrix} 0 \\ 0 \\ \omega_\alpha \end{pmatrix}, \tag{2.5}$$

which is tangent to the fibre. Then a mapping

$$E : \Im^0_1(B_m) \to \Im^1_0(t^*(B_m))$$

is defined by (2.5) and so an isomorphism of $\Im^0_1(B_m)$ in to $\Im^1_0(t^*(B_m))$.

According to (2.2) and (2.3), we define new projectable vector field $H\widetilde{X}$ by

$$BX + CX = H\widetilde{X}$$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ in $t^*(B_m)$, where

$$H\widetilde{X} = \begin{pmatrix} A^a_b \widetilde{X}^b \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} A^a_\beta X^\beta \\ X^\alpha \\ X^\beta \partial_\beta \theta_\alpha \end{pmatrix} = \begin{pmatrix} A^a_b \widetilde{X}^b + A^a_\beta X^\beta \\ X^\alpha \\ X^\beta \partial_\beta \theta_\alpha \end{pmatrix} = \begin{pmatrix} \widetilde{X}^a \\ X^\alpha \\ X^\beta \partial_\beta \theta_\alpha \end{pmatrix}. \tag{2.6}$$

From (2.5) and (2.6), we obtain

Theorem 2.1. *Let $\widetilde{X}$ and $\widetilde{Y}$ be projectable vector fields on M_n with projections X and Y on B_m, respectively. For the Lie product, we have*

(i) $\left[H\widetilde{X}, H\widetilde{Y}\right] = H\widetilde{[X,Y]}$,

(ii) $[E\psi, E\omega] = 0$

for any $\psi, \omega \in \Im^0_1(B_m)$.

Proof. (i) If $\widetilde{X}$ and $\widetilde{Y}$ are projectable vector field on M_n and $\begin{pmatrix} \left[H\widetilde{X}, H\widetilde{Y}\right]^b \\ \left[H\widetilde{X}, H\widetilde{Y}\right]^\beta \\ \left[H\widetilde{X}, H\widetilde{Y}\right]^{\overline{\beta}} \end{pmatrix}$ are the components of $\left[H\widetilde{X}, H\widetilde{Y}\right]$ with respect to the coordinates $(x^b, x^\beta, x^{\overline{\beta}})$ on

$t^*(B_m)$, then we have

$$\left[H\widetilde{X},H\widetilde{Y}\right]^J=\left(H\widetilde{X}\right)^I\partial_I(H\widetilde{Y})^J-(H\widetilde{Y})^I\partial_I(H\widetilde{X})^J.$$

Firstly, if $J=b$, we have

$$\begin{aligned}\left[H\widetilde{X},H\widetilde{Y}\right]^b &= \left(H\widetilde{X}\right)^I\partial_I(H\widetilde{Y})^b-(H\widetilde{Y})^I\partial_I(H\widetilde{X})^b\\ &= \left(H\widetilde{X}\right)^a\partial_a(H\widetilde{Y})^b+\left(H\widetilde{X}\right)^\alpha\partial_\alpha(H\widetilde{Y})^b+\left(H\widetilde{X}\right)^{\overline{\alpha}}\partial_{\overline{\alpha}}(H\widetilde{Y})^b\\ &\quad -(H\widetilde{Y})^a\partial_a(H\widetilde{X})^b-(H\widetilde{Y})^\alpha\partial_\alpha(H\widetilde{X})^b-(H\widetilde{Y})^{\overline{\alpha}}\partial_{\overline{\alpha}}(H\widetilde{X})^b\\ &= \widetilde{X}^a\partial_a\widetilde{Y}^b+X^\alpha\partial_\alpha\widetilde{Y}^b+X^\beta\partial_\beta\theta_\alpha\partial_{\overline{\alpha}}\widetilde{Y}^b\\ &\quad -\widetilde{Y}^a\partial_a\widetilde{X}^b-Y^\alpha\partial_\alpha\widetilde{X}^b-Y^\beta\partial_\beta\theta_\alpha\partial_{\overline{\alpha}}\widetilde{X}^b\\ &= X^\alpha\partial_\alpha\widetilde{Y}^b-Y^\alpha\partial_\alpha\widetilde{X}^b\\ &= \widetilde{[X,Y]}^b\end{aligned}$$

by virtue of (2.6). Secondly, if $J=\beta$, we have

$$\begin{aligned}\left[H\widetilde{X},H\widetilde{Y}\right]^\beta &= \left(H\widetilde{X}\right)^I\partial_I(H\widetilde{Y})^\beta-(H\widetilde{Y})^I\partial_I(H\widetilde{X})^\beta\\ &= \left(H\widetilde{X}\right)^a\partial_a(H\widetilde{Y})^\beta+\left(H\widetilde{X}\right)^\alpha\partial_\alpha(H\widetilde{Y})^\beta+\left(H\widetilde{X}\right)^{\overline{\alpha}}\partial_{\overline{\alpha}}(H\widetilde{Y})^\beta\\ &\quad -(H\widetilde{Y})^a\partial_a(H\widetilde{X})^\beta-(H\widetilde{Y})^\alpha\partial_\alpha(H\widetilde{X})^\beta-(H\widetilde{Y})^{\overline{\alpha}}\partial_{\overline{\alpha}}(H\widetilde{X})^\beta\\ &= \widetilde{X}^a\partial_aY^\beta+X^\alpha\partial_\alpha Y^\beta+X^\beta\partial_\beta\theta_\alpha\partial_{\overline{\alpha}}Y^\beta\\ &\quad -\widetilde{Y}^a\partial_aX^\beta-Y^\alpha\partial_\alpha X^\beta-Y^\beta\partial_\beta\theta_\alpha\partial_{\overline{\alpha}}X^\beta\\ &= X^\alpha\partial_\alpha Y^\beta-Y^\alpha\partial_\alpha X^\beta\\ &= [X,Y]^\beta\end{aligned}$$

by virtue of (2.6). Thirdly, if $J=\overline{\beta}$ then we have

$$\begin{aligned}\left[H\widetilde{X},H\widetilde{Y}\right]^{\overline{\beta}} &= \left(H\widetilde{X}\right)^I\partial_I(H\widetilde{Y})^{\overline{\beta}}-(H\widetilde{Y})^I\partial_I(H\widetilde{X})^{\overline{\beta}}\\ &= \left(H\widetilde{X}\right)^a\partial_a(H\widetilde{Y})^{\overline{\beta}}+\left(H\widetilde{X}\right)^\alpha\partial_\alpha(H\widetilde{Y})^{\overline{\beta}}+\left(H\widetilde{X}\right)^{\overline{\alpha}}\partial_{\overline{\alpha}}(H\widetilde{Y})^{\overline{\beta}}\\ &\quad -(H\widetilde{Y})^a\partial_a(H\widetilde{X})^{\overline{\beta}}-(H\widetilde{Y})^\alpha\partial_\alpha(H\widetilde{X})^{\overline{\beta}}-(H\widetilde{Y})^{\overline{\alpha}}\partial_{\overline{\alpha}}(H\widetilde{X})^{\overline{\beta}}\\ &= \widetilde{X}^a\partial_aY^\gamma\partial_\gamma\theta_\beta+X^\alpha\partial_\alpha Y^\gamma\partial_\gamma\theta_\beta+X^\beta\partial_\beta\theta_\alpha\partial_{\overline{\alpha}}Y^\gamma\partial_\gamma\theta_\beta\\ &\quad -\widetilde{Y}^a\partial_aX^\gamma\partial_\gamma\theta_\beta-Y^\alpha\partial_\alpha X^\gamma\partial_\gamma\theta_\beta-Y^\beta\partial_\beta\theta_\alpha\partial_{\overline{\alpha}}X^\gamma\partial_\gamma\theta_\beta\\ &= X^\alpha\partial_\alpha Y^\gamma\partial_\gamma\theta_\beta-Y^\alpha\partial_\alpha X^\gamma\partial_\gamma\theta_\beta\\ &= \left(X^\alpha\partial_\alpha Y^\gamma-Y^\alpha\partial_\alpha X^\gamma\right)\partial_\gamma\theta_\beta\\ &= [X,Y]^\gamma\,\partial_\gamma\theta_\beta\end{aligned}$$

by virtue of (2.6). On the other hand, we know that $H\widetilde{[X,Y]}$ has the components

$$H\widetilde{[X,Y]}=\begin{pmatrix}\widetilde{[X,Y]}^b\\ [X,Y]^\beta\\ [X,Y]^\gamma\,\partial_\gamma\theta_\beta\end{pmatrix}$$

with respect to the coordinates $(x^b, x^\beta, x^{\overline{\beta}})$ on $t^*(B_m)$. Thus, we have $\left[H\widetilde{X}, H\widetilde{Y}\right] = H\widetilde{[X,Y]}$.

(*ii*) If $\psi, \omega \in \Im_1^0(B_m)$ and $\begin{pmatrix} [E\psi, E\omega]^b \\ [E\psi, E\omega]^\beta \\ [E\psi, E\omega]^{\overline{\beta}} \end{pmatrix}$ are the components of $[E\psi, E\omega]$ with respect to the coordinates $(x^b, x^\beta, x^{\overline{\beta}})$ on $t^*(B_m)$, then we have

$$\begin{aligned} [\psi, \omega]^J &= \psi^I \partial_I \omega^J - \omega^I \partial_I \psi^J \\ &= \psi^a \partial_a \omega^J + \psi^\alpha \partial_\alpha \omega^J + \psi^{\overline{\alpha}} \partial_{\overline{\alpha}} \omega^J - \omega^a \partial_a \psi^J - \omega^\alpha \partial_\alpha \psi^J - \omega^{\overline{\alpha}} \partial_{\overline{\alpha}} \psi^J \\ &= \psi_\alpha \partial_{\overline{\alpha}} \omega^J - \omega_\alpha \partial_{\overline{\alpha}} \psi^J. \end{aligned}$$

Firstly, if $J = b$, we have

$$\begin{aligned} [\psi, \omega]^b &= \psi_\alpha \partial_{\overline{\alpha}} \omega^b - \omega_\alpha \partial_{\overline{\alpha}} \psi^b \\ &= 0 \end{aligned}$$

by virtue of (2.5). Secondly, if $J = \beta$, we have

$$\begin{aligned} [\psi, \omega]^\beta &= \psi_\alpha \partial_{\overline{\alpha}} \omega^\beta - \omega_\alpha \partial_{\overline{\alpha}} \psi^\beta \\ &= 0 \end{aligned}$$

by virtue of (2.5). Thirdly, if $J = \overline{\beta}$. Then we have

$$\begin{aligned} [\psi, \omega]^{\overline{\beta}} &= \psi_\alpha \partial_{\overline{\alpha}} \omega^{\overline{\beta}} - \omega_\alpha \partial_{\overline{\alpha}} \psi^{\overline{\beta}} \\ &= \psi_\alpha \partial_{\overline{\alpha}} \omega_\beta - \omega_\alpha \partial_{\overline{\alpha}} \psi_\beta \\ &= 0 \end{aligned}$$

by virtue of (2.5). Thus, we have $[E\psi, E\omega] = 0$. □

We consider in $\pi^{-1}(U)$ $n + m$ local vector fields $B_{(b)}, C_{(\beta)}$ and $E_{(\overline{\beta})}$ along $\beta_\theta(M_n)$, which are respectively represented by

$$B_{(b)} = B\frac{\partial}{\partial x^b}, \quad C_{(\beta)} = C\frac{\partial}{\partial x^\beta}, \quad E_{(\overline{\beta})} = E dx^\beta.$$

Theorem 2.2. *Let $\widetilde{X}$ be a projectable vector field on M_n with projection X on B_m. We have along $\beta_\theta(M_n)$ the formulas*

$$\begin{aligned} (i) \quad {}^{cc}\widetilde{X} &= H\widetilde{X} + E(-L_X\theta), \\ (ii) \quad {}^{vv}\omega &= E\omega \end{aligned} \tag{2.7}$$

for any $\omega \in \Im_1^0(B_m)$, where $L_X\theta$ denotes the Lie derivative of θ with respect to X.

Proof. (i) Using (2.4), (2.5) and (2.6), we have

$$\begin{aligned} H\widetilde{X} + E(\,-L_X\theta) &= \begin{pmatrix} \widetilde{X}^a \\ X^\alpha \\ X^\beta \partial_\beta \theta_\alpha \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -X^\beta \partial_\beta \theta_\alpha - \theta_\beta \partial_\alpha X^\beta \end{pmatrix} \\ &= \begin{pmatrix} \widetilde{X}^a \\ X^\alpha \\ -\theta_\beta \partial_\alpha X^\beta \end{pmatrix} \\ &= {}^{cc}\widetilde{X}. \end{aligned}$$

Thus, we have Theorem 2.2.

(ii) This immediately follows from (2.4). □

On the other hand, on putting $C_{(\overline{\beta})} = E_{(\overline{\beta})}$, we write the adapted frame of $\beta_\theta(M_n)$ as $\left\{B_{(b)}, C_{(\beta)}, C_{(\overline{\beta})}\right\}$. The adapted frame $\left\{B_{(b)}, C_{(\beta)}, C_{(\overline{\beta})}\right\}$ of $\beta_\theta(M_n)$ is given by the matrix

$$\widetilde{A} = \left(\widetilde{A}_B^A\right) = \begin{pmatrix} \delta_b^a & A_\beta^a & 0 \\ 0 & \delta_\beta^\alpha & 0 \\ 0 & \partial_\beta \theta_\alpha & \delta_\alpha^\beta \end{pmatrix}. \tag{2.8}$$

Where

$$\delta_b^a = A_b^a = \frac{\partial x^a}{\partial x^b}, \quad \delta_\beta^\alpha = A_\beta^\alpha = \frac{\partial x^\alpha}{\partial x^\beta}, \quad \delta_\alpha^\beta = A_\alpha^\beta = \frac{\partial x^\beta}{\partial x^\alpha}, \quad A_\beta^a = \frac{\partial x^a}{\partial x^\beta}.$$

Since the matrix $\widetilde{A}$ in (2.8) is non-singular, it has the inverse. Denoting this inverse by $\left(\widetilde{A}\right)^{-1}$, we have

$$\left(\widetilde{A}\right)^{-1} = \left(\widetilde{A}_C^B\right)^{-1} = \begin{pmatrix} \delta_c^b & -A_\theta^b & 0 \\ 0 & \delta_\theta^\beta & 0 \\ 0 & -\partial_\theta \theta_\beta & \delta_\beta^\theta \end{pmatrix}, \tag{2.9}$$

since $\widetilde{A}\left(\widetilde{A}\right)^{-1} = \widetilde{A}_B^A\left(\widetilde{A}_C^B\right)^{-1} = \delta_C^A = \widetilde{I}$. Where $A = (a, \alpha, \overline{\alpha})$, $B = \left(b, \beta, \overline{\beta}\right)$, $C = \left(c, \theta, \overline{\theta}\right)$.

Proof. In fact, from (2.8) and (2.9), we easily see that

$$\begin{aligned} \widetilde{A}\left(\widetilde{A}\right)^{-1} &= \widetilde{A}_B^A\left(\widetilde{A}_C^B\right)^{-1} \\ &= \begin{pmatrix} \delta_b^a & A_\beta^a & 0 \\ 0 & \delta_\beta^\alpha & 0 \\ 0 & \partial_\beta \theta_\alpha & \delta_\alpha^\beta \end{pmatrix} \begin{pmatrix} \delta_c^b & -A_\theta^b & 0 \\ 0 & \delta_\theta^\beta & 0 \\ 0 & -\partial_\theta \theta_\beta & \delta_\beta^\theta \end{pmatrix} \\ &= \begin{pmatrix} \delta_c^a & -A_\theta^a + A_\theta^a & 0 \\ 0 & \delta_\theta^\alpha & 0 \\ 0 & \partial_\theta \theta_\alpha - \partial_\theta \theta_\alpha & \delta_\alpha^\theta \end{pmatrix} = \begin{pmatrix} \delta_c^a & 0 & 0 \\ 0 & \delta_\theta^\alpha & 0 \\ 0 & 0 & \delta_\alpha^\theta \end{pmatrix} = \delta_C^A = \widetilde{I}. \end{aligned}$$

□

Then we see from (2.7) that the complete lift ${}^{cc}\widetilde{X}$ of a projectable vector field [7] with projection $X = X^{\alpha}(x^{\alpha})\partial_{\alpha}$ on M_n has along $\beta_{\theta}(M_n)$ components of the form

$$ {}^{cc}\widetilde{X} : \begin{pmatrix} \widetilde{X}^{a} \\ X^{\alpha} \\ -L_X\theta_{\alpha} \end{pmatrix} \tag{2.10} $$

with respect to the adapted frame $\left\{B_{(b)}, C_{(\beta)}, C_{(\overline{\beta})}\right\}$.

BX, CX and $E\omega$ also have the components:

$$ BX = \begin{pmatrix} \widetilde{X}^{a} \\ 0 \\ 0 \end{pmatrix}, \quad CX = \begin{pmatrix} 0 \\ X^{\alpha} \\ 0 \end{pmatrix}, \quad E\omega = \begin{pmatrix} 0 \\ 0 \\ \omega_{\alpha} \end{pmatrix} \tag{2.11} $$

respectively, with respect to the adapted frame $\left\{B_{(b)}, C_{(\beta)}, C_{(\overline{\beta})}\right\}$ of the cross-section $\beta_{\theta}(M_n)$ determined by a 1-form θ in M_n.

A vector field ${}^{cc}\widetilde{X}$ on a differentiable map $\Omega : M_n \to t^*(B_m)$ is a mapping ${}^{cc}\widetilde{X} : M_n \to T(t^*(B_m))$ such that $\pi_4 \circ \left({}^{cc}\widetilde{X}\right) = \Omega$, where π_4 is the projection $T(t^*(B_m)) \to t^*(B_m)$. Thus ${}^{cc}\widetilde{X}$ assigns to each point $(x^a, x^{\alpha}) = p \in M_n$ a tangent vector to $t^*(B_m)$ at $\Omega(p)$. ${}^{cc}\widetilde{X}$ is differentiable as a mapping $M_n \to T(t^*(B_m))$ if and only if $f \in \Im(t^*(B_m))$ implies, ${}^{cc}\widetilde{X}f \in \Im(M_n)$, where $\left({}^{cc}\widetilde{X}f\right)(p) ={}^{cc} \widetilde{X}(p)f$ for all $p \in M_n$ [4].

Thus, from (2.10), we have

Theorem 2.3. *The complete lift ${}^{cc}\widetilde{X}$ of a projectable vector field $\widetilde{X}$ on M_n to $t^*(B_m)$ is tangent to the cross-section $\beta_{\theta}(M_n)$ determined by a $1-$form θ in M_n if and only if the Lie derivative of θ with respect to X vanishes in M_n, i.e., if and only if $L_X\theta = 0$.*

3. Complete Lift of Tensor Fields of Type (1,1) on a Cross-Section in Semi-Cotangent Bundle

Let $\widetilde{F} \in \Im_1^1(M_n)$ be a projectable affinor field [7] with projection $F = F_{\beta}^{\alpha}(x^{\alpha})\partial_{\alpha}\otimes dx^{\beta}$, i.e. $\widetilde{F}$ has the components

$$ \widetilde{F} = (\widetilde{F}_j^i) = \begin{pmatrix} \widetilde{F}_b^a(x^a, x^{\alpha}) & \widetilde{F}_{\beta}^a(x^a, x^{\alpha}) \\ 0 & F_{\beta}^{\alpha}(x^{\alpha}) \end{pmatrix} $$

with respect to the coordinates (x^a, x^{α}). Then the semi-cotangent bundle $t^*(B_m)$ admits the complete lift ${}^{cc}\widetilde{F}$ of $\widetilde{F}$ with components [10]:

$$ {}^{cc}\widetilde{F} = ({}^{cc}\widetilde{F}_J^I) = \begin{pmatrix} \widetilde{F}_b^a & \widetilde{F}_{\beta}^a & 0 \\ 0 & F_{\beta}^{\alpha} & 0 \\ 0 & p_{\sigma}(\partial_{\beta}F_{\alpha}^{\sigma} - \partial_{\alpha}F_{\beta}^{\sigma}) & F_{\alpha}^{\beta} \end{pmatrix} \tag{3.1} $$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ on $t^*(B_m)$. Then ${}^{cc}\widetilde{F}$ has the components $\widetilde{F}_B^A$ given by

$$ {}^{cc}\widetilde{F} = ({}^{cc}\widetilde{F}_B^A) = \begin{pmatrix} \widetilde{F}_b^a & \widetilde{F}_\beta^a & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & \phi_F\theta & F_\alpha^\beta \end{pmatrix} \tag{3.2} $$

with respect to the adapted frame $\left\{B_{(b)}, C_{(\beta)}, C_{(\overline{\beta})}\right\}$ of the cross-section $\beta_\theta(M_n)$ determined by a 1-form θ in M_n. Where $A = (a, \alpha, \overline{\alpha})$, $B = \left(b, \beta, \overline{\beta}\right)$. Also, the component ${}^{cc}\widetilde{F}_\beta^{\overline{\alpha}}$ of ${}^{cc}\widetilde{F}_B^A$ is defined as Tachibana operator $\phi_F\theta$ of F, i.e.,

$$ {}^{cc}\widetilde{F}_\beta^{\overline{\alpha}} = \phi_F\theta = (\partial_\beta F_\alpha^\sigma - \partial_\alpha F_\beta^\sigma)\theta_\sigma - F_\beta^\gamma \partial_\gamma \theta_\alpha + F_\alpha^\gamma \partial_\beta \theta_\gamma. $$

Proof. Let $F \in \Im_1^1(M_n)$. Then we have by (2.8), (2.9) and (3.1):

$$ \begin{aligned} {}^{cc}\widetilde{F} &= \left(\widetilde{A}_A^B\right)^{-1} ({}^{cc}\widetilde{F}_C^A) \left(\widetilde{A}_D^C\right) \\ &= \begin{pmatrix} \delta_a^b & -A_\alpha^b & 0 \\ 0 & \delta_\alpha^\beta & 0 \\ 0 & -\partial_\alpha\theta_\beta & \delta_\beta^\alpha \end{pmatrix} \begin{pmatrix} \widetilde{F}_c^a & \widetilde{F}_\gamma^a & 0 \\ 0 & F_\gamma^\alpha & 0 \\ 0 & \theta_\sigma(\partial_\gamma F_\alpha^\sigma - \partial_\alpha F_\gamma^\sigma) & F_\alpha^\gamma \end{pmatrix} \begin{pmatrix} \delta_d^c & A_\psi^c & 0 \\ 0 & \delta_\psi^\gamma & 0 \\ 0 & \partial_\psi\theta_\gamma & \delta_\gamma^\psi \end{pmatrix} \\ &= \begin{pmatrix} \widetilde{F}_c^b & 0 & 0 \\ 0 & F_\gamma^\beta & 0 \\ 0 & -F_\gamma^\alpha\partial_\alpha\theta_\beta + \theta_\sigma\partial_\gamma F_\beta^\sigma - \theta_\sigma\partial_\beta F_\gamma^\sigma & F_\beta^\gamma \end{pmatrix} \begin{pmatrix} \delta_d^c & A_\psi^c & 0 \\ 0 & \delta_\psi^\gamma & 0 \\ 0 & \partial_\psi\theta_\gamma & \delta_\gamma^\psi \end{pmatrix} \\ &= \begin{pmatrix} \widetilde{F}_d^b & A_\psi^c \widetilde{F}_c^b & 0 \\ 0 & F_\psi^\beta & 0 \\ 0 & -F_\psi^\alpha\partial_\alpha\theta_\beta + \theta_\sigma\partial_\psi F_\beta^\sigma - \theta_\sigma\partial_\beta F_\psi^\sigma + F_\beta^\gamma\partial_\psi\theta_\gamma & F_\beta^\psi \end{pmatrix} \\ &= \begin{pmatrix} \widetilde{F}_d^b & \widetilde{F}_\psi^b & 0 \\ 0 & F_\psi^\beta & 0 \\ 0 & \phi_F\theta & F_\beta^\psi \end{pmatrix} \\ &= ({}^{cc}\widetilde{F}_D^B), \end{aligned} $$

where $A = (a, \alpha, \overline{\alpha})$, $B = \left(b, \beta, \overline{\beta}\right)$, $C = (c, \gamma, \overline{\gamma})$, $D = \left(d, \psi, \overline{\psi}\right)$. □

Theorem 3.1. *Let $\widetilde{F}$ and $\widetilde{X}$ be projectable affinor and vector fields on M_n with projections F and X on B_m, respectively, and $\omega \in \Im_1^0(B_m)$. Then we have along $\beta_\theta(M_n)$*

(i) ${}^{cc}\widetilde{F}(BX + CX) = B(FX) + C(FX) + E(P_X)$,

(ii) ${}^{cc}\widetilde{F}(E\omega) = E(\omega \circ F)$,

where $P \in \Im_2^0(B_m)$ with local components

$$ P_{\beta\alpha} = \phi_F\theta = (\partial_\beta F_\alpha^\sigma - \partial_\alpha F_\beta^\sigma)\theta_\sigma - F_\beta^\gamma\partial_\gamma\theta_\alpha + F_\alpha^\gamma\partial_\beta\theta_\gamma, $$

θ_β being the local components of θ, and $P_X \in \Im_1^0(B_m)$ defined by $P_X(Y) = P(X, Y)$, for $Y \in \Im_0^1(M_n)$.

Proof. (*i*) *If* $\widetilde{F}$ *and* $\widetilde{X}$ *are projectable affinor and vector fields on* M_n *with projections* F *and* X *on* B_m*, respectively,* then by (2.11) and (3.2), we have

$$\begin{aligned}
{}^{cc}\widetilde{F}\left(BX+CX\right) &= \begin{pmatrix} \widetilde{F}_b^a & \widetilde{F}_\beta^a & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & \phi_F\theta & F_\alpha^\beta \end{pmatrix}\begin{pmatrix} \widetilde{X}^b \\ X^\beta \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \widetilde{F}_b^a\widetilde{X}^b + \widetilde{F}_\beta^a X^\beta \\ F_\beta^\alpha X^\beta \\ X^\beta\partial_\beta F_\alpha^\sigma\theta_\sigma - X^\beta\partial_\alpha F_\beta^\sigma\theta_\sigma - F_\beta^\gamma X^\beta\partial_\gamma\theta_\alpha + F_\alpha^\gamma X^\beta\partial_\beta\theta_\gamma \end{pmatrix} \\
&= \begin{pmatrix} \widetilde{(FX)}^a \\ (FX)^\alpha \\ X^\beta\partial_\beta F_\alpha^\sigma\theta_\sigma - X^\beta\partial_\alpha F_\beta^\sigma\theta_\sigma - F_\beta^\gamma X^\beta\partial_\gamma\theta_\alpha + F_\alpha^\gamma X^\beta\partial_\beta\theta_\gamma \end{pmatrix} \\
&= \begin{pmatrix} \widetilde{(FX)}^b \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ (FX)^\beta \\ 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 \\ 0 \\ X^\beta\partial_\beta F_\alpha^\sigma\theta_\sigma - X^\beta\partial_\alpha F_\beta^\sigma\theta_\sigma - F_\beta^\gamma X^\beta\partial_\gamma\theta_\alpha + F_\alpha^\gamma X^\beta\partial_\beta\theta_\gamma \end{pmatrix} \\
&= B\left(FX\right) + C\left(FX\right) + E\left(P_X\right).
\end{aligned}$$

Thus, we have

$${}^{cc}\widetilde{F}\left(BX+CX\right) = B\left(FX\right) + C\left(FX\right) + E\left(P_X\right).$$

(*ii*) If $\omega \in \Im_1^0(B_m)$, $\widetilde{F}$ is a projectable affinor fields on M_n with projection $F \in \Im_1^1(B_m)$, then by (2.11) and (3.2), we have

$$\begin{aligned}
{}^{cc}\widetilde{F}\left(E\omega\right) &= \begin{pmatrix} \widetilde{F}_b^a & \widetilde{F}_\beta^a & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & (\partial_\beta F_\alpha^\sigma - \partial_\alpha F_\beta^\sigma)\theta_\sigma - F_\beta^\gamma\partial_\gamma\theta_\alpha + F_\alpha^\gamma\partial_\beta\theta_\gamma & F_\alpha^\beta \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \omega_\beta \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ \omega_\beta F_\alpha^\beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ (\omega\circ F)_\alpha \end{pmatrix} = E\left(\omega\circ F\right).
\end{aligned}$$

Thus, we have (*ii*) of Theorem 3.1. □

On the other hand, for an arbitrary symmetric affine connection ∇ in B_m, we have

$$P_{\beta\alpha} = (\nabla_\beta F_\alpha^\sigma - \nabla_\alpha F_\beta^\sigma)\theta_\sigma - F_\beta^\gamma\nabla_\gamma\theta_\alpha + F_\alpha^\gamma\nabla_\beta\theta_\gamma.$$

When ${}^{cc}\widetilde{F}\left(BX+CX\right)$ is always tangent to $\beta_\theta\left(M_n\right)$ for any projectable vector field $\widetilde{X} \in \Im_0^1\left(M_n\right)$, ${}^{cc}\widetilde{F}$ is said to leave the cross-section $\beta_\theta\left(M_n\right)$ invariant.

Thus we have

Theorem 3.2. *The complete lift* ${}^{cc}\widetilde{F}$ *of an element of* $\widetilde{F} \in \Im_1^1(M_n)$ *leaves the cross-section* $\beta_\theta\,(M_n)$ *invariant if and only if*

$$(\partial_\beta F_\alpha^\sigma - \partial_\alpha F_\beta^\sigma)\theta_\sigma - F_\beta^\gamma \partial_\gamma \theta_\alpha + F_\alpha^\gamma \partial_\beta \theta_\gamma = 0 \ (i.e. \phi_F \theta = 0),$$

where F_β^α *and* θ_β *are the local components of* F *and* θ *respectively.*

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