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A NEW APPROXIMATE METHOD FOR SOLVING HYPERSINGULAR INTEGRAL EQUATIONS WITH HILBERT KERNEL

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Abstract. In the present paper, the hypersingular integral operator with Hilbert kernel H is approximated by a sequence of operators of the special form. It is proved that the approximating operators H_n strongly converges to the operator H, for a trigonometric polynomial of degree not higher than n, the operators H_n and H coincide. A new method for the approximate solution of hypersingular integral equations of the first kind with Hilbert kernel is given and as an example its application to the 2D inner Neumann problem is shown.

1. Introduction

An active development of numerical methods for solving hypersingular integral equations is of considerable interest in modern numerical analysis. This is due to the fact that hypersingular integral equations have numerous applications in acoustics, aerodynamics, fluid mechanics, electrodynamics, elasticity, fracture mechanics, geophysics and etc. (see [4, 5, 15, 19, 28-30, 33, 34, 36]). Therefore the construction and justification of numerical schemes for approximate solutions of hypersingular integral equations is a topical issue and numerous works [3-14, 16-18, 21, 23, 25-27, 29-31, 34, 35] are devoted to their development. In the present paper, the hypersingular integral operator with Hilbert kernel H is approximated by a sequence of operators of the special form. It is proved that the approximating operators H_n strongly converges to the operator H, for a trigonometric polynomial of degree not higher than n, the operators H_n and H coincide. A new method for the approximate solution of hypersingular integral equations of the first kind with Hilbert kernel is given. The advantages of this method are as follows: using this method, we obtain an immediate estimate of the convergence rate for functions of the appropriate class, while other methods used earlier to approximate solutions of hypersingular integral equations require the study of the specific properties of the function classes under consideration; moreover, the estimate established in this paper yields more exact results in terms of the convergence rate than the methods used earlier (see [4-8, 10, 12-14, 16-18, 23, 26, 29, 34, 35]) ; approximate solutions of the system of linear algebraic equations can be

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found explicitly (not at isolated points) and, therefore, given a fixed convergence rate, this method requires smaller computational resources.

It is known (see [4, 10, 14, 29]) that, the solution of the inner Neumann problem for Laplace equation reduced to the hypersingular integral equation

$$(R\varphi)(t) = \frac{1}{4\pi}\int_0^{2\pi} \csc^2\frac{\tau - t}{2}\varphi(\tau)d\tau + \int_0^{2\pi} K(t,\tau)\varphi(\tau)d\tau = f(t),\ t \in T_0 = [0; 2\pi],$$

where $K(t,\tau) = \frac{\partial}{\partial t}F(t,\tau)$ is 2π-periodic for all arguments and continuous function.

In the present paper, the operator R is approximated by a sequence of operators of the form

$$(R_n\varphi)(t) = \sum_{k=0}^{2n-1} \alpha_k^{(n)}(t)\,\varphi\left(t + \frac{\pi k}{n}\right),\ t \in T_0, n \in N,$$

where the functions $\alpha_k^{(n)}(t)$ is continuous functions, $k = 0, 1, 2, ..., n \in N$, are expressed in terms of the given functions, and is given a new method for the approximate solution of hypersingular integral equations of the first kind. Note that, for the singular integral operators with Cauchy kernel and Hilbert kernel similar approximations and its application to singular integral equations are given in the papers [1], [2] and [20].

2. Hypersingular Hilbert integral

Consider the integral

$$\int_0^{2\pi} \csc^2\frac{\tau - t}{2}\varphi(\tau)\,d\tau, t \in T_0, \tag{2.1}$$

where φ is Lebesgue integrable on T_0 and 2π-periodic function. If we define this integral similar to the Cauchy integral, even if $\varphi \equiv 1$, we get the divergent integral.

Therefore, using the idea of Hadamard [22], we will define the integral (2.1) as follows:

Definition 2.1. If a finite limit

$$\lim_{\varepsilon \to 0+}\left(\int_{t-\pi}^{t-\varepsilon} \csc^2\frac{\tau - t}{2}\varphi(\tau)\,d\tau + \int_{t+\varepsilon}^{t+\pi} \csc^2\frac{\tau - t}{2}\varphi(\tau)\,d\tau - \frac{8\varphi(t)}{\varepsilon}\right)$$

exist, then the value of this limit is referred to as the hypersingular Hilbert integral of the function φ on T_0, and is denoted by $\int_0^{2\pi} \csc^2\frac{\tau-t}{2}\varphi(\tau)\,d\tau$.

From the definition 2.1 it follows that,

$$\int_0^{2\pi} \csc^2\frac{\tau - t}{2}d\tau = \lim_{\varepsilon \to 0+}\left(4\cot\frac{\varepsilon}{2} - \frac{8}{\varepsilon}\right) = 0,$$

and, therefore,

$$\int_0^{2\pi} \csc^2\frac{\tau - t}{2}\varphi(\tau)\,d\tau = \int_0^{2\pi} \csc^2\frac{\tau - t}{2}\left[\varphi(\tau) - \varphi(t)\right]d\tau,$$

where the integral standing in the right side is understood in the sense of the Cauchy principal value.

In the paper [4], it is proved that, if $\varphi(\tau) \in H_1(\alpha)$, where $H_1(\alpha)$ is the class of differentiable functions, whose derivatives are Hölder continuous with degree α, then there exist the hypersingular Hilbert integral (2.1) and the following equation holds:

$$\int_0^{2\pi} \csc^2 \frac{\tau - t}{2} \varphi(\tau)\, d\tau = 2 \int_0^{2\pi} \cot \frac{\tau - t}{2} \varphi'(\tau)\, d\tau, \tag{2.2}$$

where the integral standing in the right side is understood in the sense of the Cauchy principal value.

Now we show that the hypersingular integral (2.1) exists almost everywhere and the equation (2.2)satisfied under weak conditions, which we will need later.

Theorem 2.1. *If the 2π-periodic function φ is absolutely continuous on T_0, then the hypersingular Hilbert integral* (2.1) *exist for almost all $t \in T_0$, and the equation* (2.2) *holds.*

Proof. From the absolutely continuous of the function φ on T_0, it follows that, the function $\varphi'(\tau)$ is Lebesgue integrable on T_0. Then the integral standing of the right side of equation (2.2) exists for almost all $t \in T_0$ (see, for example, [37]).

For almost all $t \in T_0$ we have

$$\lim_{\varepsilon \to 0+} \left(\varphi(t - \varepsilon) \cot \frac{\varepsilon}{2} + \varphi(t + \varepsilon) \cot \frac{\varepsilon}{2} - \frac{4\varphi(t)}{\varepsilon} \right) =$$

$$= \lim_{\varepsilon \to 0+} \left\{ \left[\varphi(t) - \varepsilon\varphi'(t) + o(\varepsilon)\right] \cot \frac{\varepsilon}{2} + \left[\varphi(t) + \varepsilon\varphi'(t) + o(\varepsilon)\right] \cot \frac{\varepsilon}{2} - \frac{4\varphi(t)}{\varepsilon} \right\} =$$

$$= 2\varphi(t) \lim_{\varepsilon \to 0+} \left(\cot \frac{\varepsilon}{2} - \frac{2}{\varepsilon} \right) = 0.$$

Then, from the equation

$$\int_0^{2\pi} \csc^2 \frac{\tau - t}{2} \varphi(\tau) d\tau = \lim_{\varepsilon \to 0+} \left(- 2\varphi(t) \cot \frac{\tau - t}{2} \Big|_{t-\pi}^{t-\varepsilon} + 2 \int_{t-\pi}^{t-\varepsilon} \cot \frac{\tau - t}{2} \varphi'(\tau) d\tau - \right.$$

$$\left. - 2\varphi(t) \cot \frac{\tau - t}{2} \Big|_{t+\varepsilon}^{t+\pi} + 2 \int_{t-\pi}^{t-\varepsilon} \cot \frac{\tau - t}{2} \varphi'(\tau)\, d\tau - \frac{4\varphi(t)}{\varepsilon} \right) =$$

$$= \lim_{\varepsilon \to 0+} \left(\varphi(t - \varepsilon) \cot \frac{\varepsilon}{2} + \varphi(t + \varepsilon) \cot \frac{\varepsilon}{2} - \frac{4\varphi(t)}{\varepsilon} \right) + 2 \int_0^{2\pi} \cot \frac{\tau - t}{2} \varphi'(\tau)\, d\tau =$$

$$= 2 \int_0^{2\pi} \cot \frac{\tau - t}{2} \varphi'(\tau)\, d\tau$$

it follows that, the hypersingular Hilbert integral (2.1) exist for almost all $t \in T_0$ and the equation (2.2) holds. This completes the proof of the theorem. □

3. Approximation of the hypersingular integral operators with Hilbert kernel

Let $L_2 = L_2(T_0)$ be the space of the functions square-integrable on T_0 with the norm

$$\|\varphi\|_{L_2} = \left(\frac{1}{2\pi} \int_0^{2\pi} |\varphi(\tau)|^2 \, d\tau \right)^{\frac{1}{2}},$$

and let $W_2^1 = W_2^1(T_0)$ be the space of absolutely continuous on T_0 functions, which derivative belongs on the space L_2, with the norm

$$\|\varphi\|_{W_2^1} = \|\varphi\|_{L_2} + \|\varphi'\|_{L_2}.$$

Since the singular integral operator with Hilbert kernel

$$(S\varphi)(t) = \frac{1}{2\pi} \int_0^{2\pi} \cot \frac{\tau - t}{2} \varphi(\tau) \, d\tau$$

is bounded from the space L_2 into the space L_2, and $\|S\|_{L_2 \to L_2} = 1$ (see[37]), it follows from theorem 2.1 that the hypersingular integral operator with Hilbert kernel

$$(H\varphi)(t) = \frac{1}{4\pi} \int_0^{2\pi} \csc^2 \frac{\tau - t}{2} \varphi(\tau) \, d\tau, t \in T_0$$

is bounded from the space W_2^1 into the space L_2, and $\|H\|_{W_2^1 \to L_2} = 1$.

Consider the sequence of operators

$$(H_n\varphi)(t) = \frac{1}{2n} \sum_{k=0}^{n-1} \csc^2 \frac{\pi(2k+1)}{2n} \left(\varphi\left(t + \frac{\pi(2k+1)}{n} \right) - \varphi(t) \right), \quad t \in T_0, n \in N.$$

Let us calculate $H_n\left(e^{imt}\right)$ for any $m \in Z$ (Z is the set of integer real numbers):

$$H_n\left(e^{imt}\right) = \frac{1}{2n} \sum_{k=0}^{n-1} \csc^2 \frac{\pi(2k+1)}{2n} \left(\exp\left(imt + \frac{im\pi(2k+1)}{n} \right) - \exp(imt) \right) =$$

$$= \mu_m^{(n)} e^{imt}, \tag{3.1}$$

where $\mu_m^{(n)} = \frac{1}{2n} \sum_{k=0}^{n-1} \csc^2 \frac{\pi(2k+1)}{2n} \left(\exp \frac{im\pi(2k+1)}{n} - 1 \right)$.

It is clear that $\mu_0^{(n)} = 0$.

For all $\alpha, \beta \in R$ we have

$$e^{i\alpha} - e^{i\beta} = exp\frac{i(\alpha+\beta)}{2} \left[exp\frac{i\alpha - i\beta}{2} - exp\frac{i\beta - i\alpha}{2} \right] =$$

$$= 2i \exp \frac{i(\alpha+\beta)}{2} \sin \frac{(\alpha-\beta)}{2}.$$

So,

$$\lambda_m^{(n)} = \mu_{m+1}^{(n)} - \mu_m^{(n)} =$$

$$= \frac{1}{2n} \sum_{k=0}^{n-1} \csc^2 \frac{\pi(2k+1)}{2n} \left(\exp \frac{i(m+1)\pi(2k+1)}{n} - \exp \frac{im\pi(2k+1)}{n} \right) =$$

$$= \frac{i}{n}\sum_{k=0}^{n-1}\csc\frac{\pi\left(2k+1\right)}{2n}\exp\frac{i\,\left(m+1/2\right)\pi\left(2k+1\right)}{n}.$$

As

$$\lambda_0^{(n)} = \frac{i}{n}\sum_{k=0}^{n-1}\csc\frac{\pi\left(2k+1\right)}{2n}\exp\frac{i\,\pi\left(2k+1\right)}{2n} =$$

$$= \frac{i}{n}\sum_{k=0}^{n-1}\csc\frac{\pi\left(2k+1\right)}{2n}\left[\cos\frac{\pi\left(2k+1\right)}{2n}+i\sin\frac{\pi\left(2k+1\right)}{2n}\right] =$$

$$= \frac{i}{n}\sum_{k=0}^{n-1} ctg\frac{\pi\left(2k+1\right)}{2n} - 1 = -1,$$

$$\delta_m^{(n)} = \lambda_{m+1}^{(n)} - \lambda_m^{(n)} = \frac{i}{n}\sum_{k=0}^{n-1}\csc\frac{\pi\left(2k+1\right)}{2n}\left(\exp\frac{i\,\left(m+3/2\right)\pi\left(2k+1\right)}{n} -\right.$$

$$\left. - \exp\frac{i\,\left(m+1/2\right)\pi\left(2k+1\right)}{n}\right) = -\frac{2}{n}\sum_{k=0}^{n-1}\exp\frac{i\,\left(m+1\right)\pi\left(2k+1\right)}{n},$$

and, therefore, $\delta_m^{(n)} = -2$ for $m = -1\ (mod\,2n)$, $\delta_m^{(n)} = 2$ for $m = n-1\ (mod\,2n)$, $\delta_m^{(n)} = 0$ for $m \neq -1\ (mod\,n)$, it follows that, $\lambda_m^{(n)} = -1$ for $m = \overline{0, n-1}$, $\lambda_m^{(n)} = 1$ for $m = \overline{n, 2n-1}$, $\lambda_{m\pm 2n}^{(n)} = \lambda_m^{(n)}$ for all $m \in Z$, and, therefore, $\mu_m^{(n)} = -m$ for $m = \overline{0, n}$, $\mu_m^{(n)} = m - 2n$ for $m = \overline{n+1, 2n}$, $\mu_{m\pm 2n}^{(n)} = \mu_m^{(n)}$ for all $m \in Z$.

Suppose that $\varphi\left(t\right) = \sum\limits_{k=-\infty}^{+\infty} c_k e^{ikt} \in W_2^1$. Then $\varphi'\left(t\right) = \sum\limits_{k=-\infty}^{+\infty} ikc_k e^{ikt} \in L_2$, and taking (3.1) into account, we obtain

$$\left(H_n\varphi\right)\left(t\right) = \sum_{k=-\infty}^{+\infty} c_k \mu_k^{(n)} e^{ikt}.$$

Since $\mu_k^{(n)} = -k$ for $k = \overline{0, n}$, $\mu_k^{(n)} = k$ for $k = \overline{-0, n}$, and $|\mu_k^{(n)}| \leq |n|$ holds for all $k \in Z$, then the the inequality $\left|\mu_k^{(n)}\right| \leq |k|$ holds for all $k \in Z$. It follows that

$$\left\|H_n\varphi\right\|_{L_2} = \left[\sum_{k=-\infty}^{+\infty}\left|c_k\right|^2\left|\mu_k^{(n)}\right|^2\right]^{1/2} \leq \left[\sum_{k=-\infty}^{+\infty} k^2\left|c_k\right|^2\right]^{1/2} = \left\|\varphi'\right\|_{L_2} \leq \left\|\varphi\right\|_{W_2^1}.$$

Moreover, from the equations (2.2) and (3.1) it follows that for any $k = -\overline{n, n}$

$$H(e^{ikt}) = S(ike^{ikt}) = -|k|e^{ikt} = \mu_k^n e^{ikt} = H_n(e^{ikt}).$$

This implies the following properties of the operators H_n:

The operators H_n, $n = 1, 2, \ldots$ is bounded from the space W_2^1 into the space L_2, $\|H_n\|_{W_2^1 \to L_2} \leq 1$, and for any trigonometric polynomial $q\left(t\right) = \sum_{k=-n}^{n} q_k e^{ikt}$ the following relation holds

$$\left(H_n q\right)\left(t\right) = \left(Hq\right)\left(t\right). \tag{3.2}$$

Suppose that $E_n\left(\varphi;\, W_2^1\right) = \inf\limits_{q\in T_n} \|\varphi(\cdot) - q_n(\cdot)\|_{W_2^1}$ is the best approximation of the function $\varphi \in W_2^1$ by polynomials T_n, where T_n is the set of trigonometric polynomials of the form $\sum_{k=-n}^{n} \alpha_k e^{ikt}$, $\alpha_k \in C$.

Theorem 3.1. *The sequence of operators $\{H_n\}$ strongly converges to the operator H and, for any $\varphi \in W_2^1$, the following estimate holds:*

$$\|H\varphi - H_n\varphi\|_{L_2} \leq 2E_n\left(\varphi;\, W_2^1\right).$$

Proof. Suppose that $q_n(t) = \sum_{k=-n}^{n} q_k^{(n)} e^{ikt}$ is the best approximation polynomial for the function $\varphi \in W_2^1$ from T_n. Then, it follows from equation (3.2) that

$$(H\varphi - H_n\varphi)(t) = H(\varphi - q_n)(t) - H_n(\varphi - q_n)(t).$$

This yields

$$\|H\varphi - H_n\varphi\|_{L_2} \leq \left(\|H\|_{W_2^1\to L_2} + \|H_n\|_{W_2^1\to L_2}\right) \cdot \|\varphi - q_n\|_{W_2^1} \leq 2E_n\left(\varphi;\, W_2^1\right).$$

This completes the proof of the theorem. □

4. The approximate solution of the hypersingular integral equations of the first kind

At first consider the simple hypersingular integral equation of the first kind

$$(H\varphi)(t) = f(t),\, t \in T_0, \tag{4.1}$$

where $f \in L_2$. Since for any function $\varphi \in W_2^1$ the equation

$$\int_0^{2\pi} (H\varphi)(\tau)\, d\tau = \int_0^{2\pi} \left(S\varphi'\right)(\tau)\, d\tau = \int_0^{2\pi} \varphi'(\tau)\, d\tau = 0,$$

holds, then the equation (4.1) is solvable only if

$$\int_0^{2\pi} f(\tau)\, d\tau = 0, \tag{4.2}$$

where $(S\varphi)(t) \equiv \frac{1}{2\pi}\int_0^{2\pi} \cot\frac{\tau-t}{2}\varphi(\tau)\, d\tau$, $t \in T_0$ is singular integral operator with Hilbert kernel. If the condition (4.2) is satisfies, then the equation (4.1) has infinitely many solutions in the general form

$$\varphi^*(t) = d_0 - \sum_{k\in Z\setminus\{0\}} \frac{c_k(f)}{|k|} e^{ikt} \in W_2^1, \tag{4.3}$$

where $c_k(f) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\tau} f(\tau)\, d\tau$ is Fourier's coefficients of function $f \in L_2$, and d_0 is a constant. Therefore if we consider the equation

$$(H\varphi)(t) = f(t) - \frac{1}{2\pi}\int_0^{2\pi} f(\tau)\, d\tau, t \in I, \tag{4.4}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi(\tau)\, d\tau = d_0, \tag{4.5}$$

then, from the equation (3.1) we will receive that, the equation (4.4)-(4.5) is unique solvable for any $(f;\, d_0) \in L_2 \times C$, and the solution of (4.4)-(4.5) is the function, defined by (4.3).

Now we consider the equation

$$(H_n\varphi)\,(t) = f\,(t)\,, t \in T_0, \tag{4.6}$$

where $f \in L_2$. Considering equation (4.6) at the points t, $t+\frac{\pi}{n}$, ..., $t+\frac{\pi(2n-1)}{n}$ we obtain the following system of linear algebraic equations:

$$(H_n\varphi)\left(t+\frac{\pi k}{n}\right) = f\left(t+\frac{\pi k}{n}\right), k=\overline{0, 2n-1}, t \in T_0 \tag{4.7}$$

with respect to $\left(\varphi\,(t)\;,\;\varphi\left(t+\frac{\pi}{n}\right),...,\varphi\left(t+\frac{\pi(2n-1)}{n}\right)\right)$. Since for any $t \in T_0$ the equation

$$\sum_{k=0}^{2n-1}(H_n\varphi)\left(t+\frac{\pi k}{n}\right) = 0,$$

holds, then we obtain that, the equation (4.6) is solvable only if

$$\sum_{k=0}^{2n-1} f\left(t+\frac{\pi k}{n}\right) = 0. \tag{4.8}$$

If the condition (4.8) is satisfies, then the equation (4.6) has infinitely many solutions. Denote $L_2 \times C = \{(f;d) : f \in L_2 \,\text{and}\, d \in C\}$. If we consider the equation

$$(H_n\varphi)\,(t) = f\,(t) - \frac{1}{2n}\sum_{k=0}^{2n-1} f\left(t+\frac{\pi k}{n}\right), t \in T_0, \tag{4.9}$$

$$\frac{1}{2n}\sum_{k=0}^{2n-1}\varphi\left(t+\frac{\pi k}{n}\right) = d_0, \tag{4.10}$$

we will receive that, the equations (4.9)-(4.10) is unique solvable for any $(f;\, d_0) \in L_2 \times C$, and the solutions of (4.9)-(4.10) is the function

$$\varphi_n^*\,(t) = d_0 + \sum_{\substack{k=-\infty \\ k\neq 0\ (mod\, 2n)}}^{+\infty} \frac{c_k\,(f)}{\mu_k^{(n)}} e^{ikt} \in L_2. \tag{4.11}$$

Theorem 4.1. *For any $(f;\, d_0) \in L_2 \times C$ the system of linear algebraic equations* (4.9)-(4.10) *unique solvable with respect to* $\left(\varphi\,(t)\;,\;\varphi\left(t+\frac{\pi}{n}\right),...,\;\varphi\left(t+\frac{\pi(2n-1)}{n}\right)\right)$; *the solutions $\varphi_n^*\,(t)$ of the equations* (4.9)-(4.10) *converge in the norm of the space L_2 to the solution $\varphi^*\,(t)$ of the equation* (4.4)-(4.5), *and the following estimate is holds:*

$$\|\varphi_n^* - \varphi^*\|_{L_2} \leq E_n\,(f;\, L_2)\,. \tag{4.12}$$

Proof. We already proved that for any $(f;d_0) \in L_2 \times C$ equations (4.4)-(4.5) and (4.9)-(4.10) have unique solutions (4.3) and (4.11). Since $\mu_k^{(n)} = -|k|$ for

$k=-\overline{n,n}$ and $-n \le \mu_k^{(n)} \le -1$ for $k \ne 0$ (mod 2n), it follows that

$$\|\varphi_n^* - \varphi^*\|_{L_2} = \left\| \sum_{\substack{k=-\infty \\ k\ne 0 (mod\, 2n)}}^{+\infty} \frac{c_k(f)}{\mu_k^{(n)}} e^{ikt} + \sum_{|k|>n} \frac{c_k(f)}{|k|} e^{ikt} \right\|_{L_2} =$$

$$= \left\| \sum_{\substack{|k|>n \\ k\ne 0\ (mod\, 2n)}} \frac{c_k(f)}{\mu_k^{(n)}} e^{ikt} + \sum_{|k|>n} \frac{c_k(f)}{|k|} e^{ikt} \right\|_{L_2} =$$

$$= \left(\sum_{\substack{|k|>n \\ k\ne 0 (mod 2n)}} \left(\frac{1}{\mu_k^{(n)}} + \frac{1}{|k|} \right)^2 |c_k(f)|^2 + \sum_{\substack{|k|>n \\ k=0 (mod 2n)}} \frac{|c_k(f)|^2}{|k|^2} \right)^{1/2} \le$$

$$\le \left(\sum_{|k|>n} |c_k(f)|^2 \right)^{1/2} \le E_n(f;\, L_2).$$

This completes the proof of the theorem. □

Now we consider the hypersingular integral equation of the first kind

$$(H\varphi)(t) + (\mathrm{K}\varphi)(t) = f(t), t \in T_0, \tag{4.13}$$

where $(\mathrm{K}\varphi)(t) = \frac{1}{2\pi}\int_0^{2\pi} K(t,\, \tau)\, \varphi(\tau)\, d\tau$, $K(t,\, \tau) = \frac{\partial}{\partial t} F(t,\, \tau)$ is 2π-periodic for all arguments and continuous function. The equation (4.13) is also solvable only if the condition (4.2) is satisfied. Therefore, we consider the equation

$$(H\varphi)(t) + (\mathrm{K}\varphi)(t) = f(t) - \frac{1}{2\pi}\int_0^{2\pi} f(\tau)\, d\tau, t \in T_0, \tag{4.14}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi(\tau)\, d\tau = d_0. \tag{4.15}$$

Since the solution of the equation (4.4)-(4.5) is a function (see [4]) $(\varphi)(t) = d_0 + (Bf)(t)$, where $(Bf)(t) = \frac{1}{\pi}\int_0^{2\pi} \ln 2 \left|\sin \frac{\tau - t}{2}\right| f(\tau)\, d\tau$, then equation (4.14)-(4.15) is equivalent to the equation

$$(\varphi)(t) + (B\mathrm{K})\, \varphi(t) = d_0 + (Bf)(t), t \in T_0. \tag{4.16}$$

Consider the equation

$$(H_n\varphi)(t) + (\mathrm{K}_n\varphi)(t) - \frac{1}{4n^2}\sum_{k=0}^{2n-1}\sum_{p=0}^{2n-1} K\left(t + \frac{\pi p}{n}, t + \frac{\pi k}{n}\right) \varphi\left(t + \frac{\pi k}{n}\right) =$$

$$= f(t) - \frac{1}{2n}\sum_{k=0}^{2n-1} f\left(t + \frac{\pi k}{n}\right), t \in T_0, \tag{4.17}$$

$$\frac{1}{2n}\sum_{k=0}^{2n-1} \varphi\left(t + \frac{\pi k}{n}\right) = d_0, \tag{4.18}$$

where $(\mathrm{K}_n\varphi)(t) = \frac{1}{2n}\sum_{k=0}^{2n-1} K\left(t, t+\frac{\pi k}{n}\right)\varphi\left(t+\frac{\pi k}{n}\right)$. By theorem 4.1 for any $(f;\, d_0) \in L_2 \times C$ the equation (4.9)-(4.10) has the unique solution

$$(\varphi)(t) = d_0 + (B_n f)(t);$$

the operators B_n are the form $(B_n f)(t) = \sum_{k=0}^{2n-1} \beta_k^{(n)} f\left(t+\frac{\pi k}{n}\right)$, and the sequence of operators $\{B_n\}$ strongly converges to the operator B in L_2. Then equation (4.17)-(4.18) is equivalent to the equation

$$(\varphi)(t) + (B_n\mathrm{K}_n)\varphi(t) = d_0 + (B_n f)(t). \tag{4.19}$$

We need the following theorem and lemma proved in [2].

Theorem 4.2. *The sequence of operators* $\{\mathrm{K}_n\}$ *strongly converges to the operator* K *in* L_2 *and, for any* $\varphi \in L_2$*, the following estimate holds:*

$$\|\mathrm{K}\varphi - \mathrm{K}_n\varphi\|_{L_2} \le 2\|K\|_\infty E_{n-1}(\varphi;\, L_2) + 2E_{n-1}(K)\left\{E_{n-1}(\varphi;\, L_2) + \|\varphi\|_{L_2}\right\},$$

where $\|K\|_\infty = \max\limits_{t,\tau\in[0,\,2\pi]} |K(t,\tau)|$, $E_{n-1}(K) = \inf\left\|K(t,\tau) - \sum\limits_{k=-n+1}^{n-1} \alpha_k(t)e^{ik\tau}\right\|_\infty$ *and infimum is taken on all trigonometric polynomials* $\alpha_k(t)$, $k = \overline{-n+1,\, n-1}$ *is degree not higher than* $n-1$.

Lemma 4.1. *Suppose that* B, B_n, $n = 1, 2, ...,$ *are bounded linear operators acting in* L_2*; the operators* B_n *are of the form* $(B_n f)(t) = \sum_{k=0}^{2n-1} \beta_k^{(n)} f\left(t+\frac{\pi k}{n}\right)$ *and the sequence of operators* $\{B_n\}$ *strongly converges to the operator* B *in* L_2*. If the inverse operator* $(I + B\mathrm{K})^{-1}$ *exist, then, for large values of* n*, the operators* $(I + B_n\mathrm{K}_n)$ *are also invertible and the sequence of operators* $\left\{(I + B_n\mathrm{K}_n)^{-1}\right\}$ *strongly converges to the operator* $(I + B\mathrm{K})^{-1}$ *in* L_2*.*

Theorem 4.3. *If for any* $(f;\, d_0) \in L_2 \times C$ *the equation* (4.14)-(4.15) *unique solvable in* L_2*, then for large values of* n*, the systems of linear algebraic equations* (4.17)-(4.18) *are also unique solvable for any* $(f;\, d_0) \in L_2 \times C$ *with respect to* $\left(\varphi(t),\ \varphi\left(t+\frac{\pi}{n}\right), ..., \varphi\left(t+\frac{\pi(2n-1)}{n}\right)\right)$*; the solutions* $\varphi_n^*(t)$ *of the equations* (4.17)-(4.18) *converge in the norm of the space* L_2 *to the solution* $\varphi^*(t)$ *of the equation* (4.14)-(4.15)*, and the following estimate is holds:*

$$\|\varphi_n^* - \varphi^*\|_{L_2} \le const\cdot\{E_n(f;\, L_2) + E_n(\mathrm{K}\varphi^*;\, L_2) +$$

$$+2\|K\|_\infty E_{n-1}(\varphi^*;\, L_2) + 2E_{n-1}(K)\left[E_{n-1}(\varphi^*;\, L_2) + \|\varphi^*\|_{L_2}\right]\}. \tag{4.20}$$

Proof. Since the equation (4.14)-(4.15) is equivalent to the equation (4.16), we will get that, the operator $I+B\mathrm{K}$ invertible in L_2. Then from lemma 4.1 it follows that, the operators $(I + B_n\mathrm{K}_n)$ are also invertible and the sequence of operators $\left\{(I + B_n\mathrm{K}_n)^{-1}\right\}$ strongly converges to the operator $(I + B\mathrm{K})^{-1}$ in L_2. Since the equation (4.17)-(4.18) is equivalent to the equation (4.19), we will get that, the equations (4.17)-(4.18) are also unique solvable for any $(f;\, d_0) \in L_2 \times C$ with respect to $\left(\varphi(t),\ \varphi\left(t+\frac{\pi}{n}\right), ..., \varphi\left(t+\frac{\pi(2n-1)}{n}\right)\right)$ and the solutions $\varphi_n^*(t)$ of the equations (4.17)-(4.18) converge in the norm of the space L_2 to the solution $\varphi^*(t)$ of the equation (4.14)-(4.15).

Estimate (4.20) follows from the inequality

$$\|\varphi_n^* - \varphi^*\|_{L_2} = \left\|(I + B_n \mathrm{K}_n)^{-1}(d_o + B_n f) - (I + B\mathrm{K})^{-1}(d_o + Bf)\right\|_{L_2} \leq$$

$$\leq \left\|(I + B_n \mathrm{K}_n)^{-1}\right\|_{L_2 \to L_2} \cdot \|B_n f - Bf\|_{L_2} +$$

$$+ \left\|(I + B_n \mathrm{K}_n)^{-1}\right\|_{L_2 \to L_2} \cdot \|(B\mathrm{K} - B_n \mathrm{K}_n)\varphi^*\|_{L_2},$$

and from the theorems 4.1 and 4.2. This completes the proof of the theorem. □

5. Numerical examples

Example 1. In numerical examples most of the authors consider the continuous and bounded functions. To show the efficiency and accuracy of our methods, we will consider as an example the unbounded function. Consider the following hypersingular integral equation

$$(H\varphi)(t) = \frac{2}{\pi} \ln \left|\tan \frac{t}{2}\right|, \tag{5.1}$$

with a condition

$$\frac{1}{2\pi} \int_0^{2\pi} \varphi(\tau)\, d\tau = \frac{\pi}{2}. \tag{5.2}$$

It is easy to verify that the function $\varphi^*(t) = |t - \pi|$ is a solution of (5.1) - (5.2) on the segment $[0\,,\, 2\pi]$.

Now we consider the equation

$$(H_n\varphi)(t) = \frac{2}{\pi} \ln \left|\tan \frac{t}{2}\right|, \tag{5.3}$$

$$\frac{1}{2n} \sum_{k=0}^{2n-1} \varphi\left(t + \frac{\pi k}{n}\right) = \frac{\pi}{2}. \tag{5.4}$$

If we study the equation (5.3) at the points $t + \frac{\pi k}{n}$, $k = \overline{0\,,\, 2n-2}$, and add the equation (5.4) we will receive the system of linear algebraic equations with respect to $\left(\varphi(t)\,,\ \varphi\left(t + \frac{\pi}{n}\right), ..., \varphi\left(t + \frac{\pi(2n-1)}{n}\right)\right)$.

Fix a point $t_0 = \frac{\pi}{2n}$ and solving the system of linear algebraic equations

$$\begin{cases} \frac{1}{2n}\sum_{k=0}^{n-1} \csc^2 \frac{\pi(2k+1)}{2n}\left(\varphi\left(\frac{\pi(4k+3)}{2n}\right) - \varphi\left(\frac{\pi}{2n}\right)\right) = \frac{2}{\pi}\ln\left|\tan\frac{\pi}{4n}\right|, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{1}{2n}\sum_{k=0}^{n-1} \csc^2 \frac{\pi(2k+1)}{2n}\left(\varphi\left(\frac{\pi(4k-1)}{2n}\right) - \varphi\left(\frac{\pi(4n-3)}{2n}\right)\right) = \frac{2}{\pi}\ln\left|\tan\frac{\pi(4n-3)}{4n}\right|, \\ \frac{1}{2n}\sum_{k=0}^{2n-1} \varphi\left(\frac{\pi(2k+1)}{2n}\right) = \frac{\pi}{2}, \end{cases} \tag{5.5}$$

with respect to $\left(\varphi\left(\frac{\pi}{2n}\right)\,,\ \varphi\left(\frac{3\pi}{2n}\right), ..., \varphi\left(\frac{\pi(4n-1)}{2n}\right)\right)$ we obtain the value of the approximate solution $\varphi_n^*(t)$ of the equation (5.1)-(5.2) at the points $t_k = \frac{\pi(2k+1)}{2n}$, $k = \overline{0\,,\, 2n-1}$.

In table 1 we show the supremum-error $\Delta_n = \sup\limits_{k=\overline{0\,,\,2n-1}} |\varphi^*(t_k) - \varphi_n^*(t_k)|$ and the mean-square-error $\Delta_n^{(2)} = \left(\frac{1}{2n}\sum_{k=0}^{2n-1}|\varphi^*(t_k) - \varphi_n^*(t_k)|^2\right)^{\frac{1}{2}}$ for $n = 256$, $n = 512$ and $n = 1024$.

Table 1.

n	Δ_n	$\Delta_n^{(2)}$
256	1.0149E-002	2.6891E-003
512	5.672E-003	1.3487E-003
1024	3.135E-003	6.755E-004

Example 2. To show the efficiency and accuracy of our methods, we will consider as an second example the 2D inner Neumann problem. Let D be a bounded open simply-connected domain on the R^2 and ∂D be its boundary. Consider the inner Neumann problem

$$\Delta U(M) = 0, M \in D, \tag{5.6}$$

$$\frac{\partial U(P)}{\partial \vec{n}_P} = f(P), P \in \partial D, \tag{5.7}$$

where $\vec{n}_P$ is the unit normal vector at the point P on the boundary ∂D to be directed into D and $f(P)$ is given function satisfying the condition

$$\int_{\partial D} f(P)\, ds_P = 0.$$

This condition is necessary and sufficient for the existence of solutions of the equation (5.6)-(5.7) and all solutions of the equation (5.6)-(5.7) is different constant (see [24]). If we add the condition

$$U(M_0) = \alpha_0, \tag{5.8}$$

where $M_0 = (x_0,\, y_0) \in D$, $\alpha_0 \in R$, then the equation (5.6)-(5.8) has a unique solution.

If the solution $U(M)$ of (5.6)-(5.7) can be expressed by a double-layer potential

$$U(M) = \int_{\partial D} g(Q)\frac{\partial}{\partial \vec{n}_Q}\ln|M-Q|\, ds_Q, M \in D,$$

then the function $U(M)$ is satisfy the equation (5.6) and the equation (5.7) is equivalent to the hypersingular integral equation (see [14])

$$\frac{\partial}{\partial \vec{n}_P}\int_{\partial D} g(Q)\frac{\partial}{\partial \vec{n}_Q}\ln|P-Q|\, ds_Q = f(P), P \in \partial D. \tag{5.9}$$

Now we give one numerical example for the interior Neumann problem (5.6)-(5.8).

Let the domain D is an ellipse and its boundary ∂D is $\beta(t) = (\cos t,\, 2\sin t)$, $t \in T_0$. Introducing

$$b(t,\tau) = -\frac{1}{\pi}\frac{\partial}{\partial \tau}\left[\frac{\beta'(t)(\beta(\tau)-\beta(t))}{|\beta(\tau)-\beta(t)|^2} - \frac{\sin(\tau - t)}{2(1-\cos(\tau - t))}\right] =$$

$$= -\frac{1}{2\pi} \cdot \frac{9 + 15\cos(t+\tau)}{[5 + 3\cos(t+\tau)]^2}, \ t, \tau \in T_0$$

and $\varphi(t) = g(\beta(t))$, $h(t) = \frac{1}{\pi} f(\beta(t)) \cdot |\beta'(t)|$, $t \in T_0$, equation (5.9) is rewritten as (see [10]):

$$(H\varphi)(t) + \int_0^{2\pi} b(t, \tau)\varphi(\tau)\,d\tau = h(t), t \in T_0, \tag{5.10}$$

where $\int_0^{2\pi} h(\tau)\,d\tau = \frac{1}{\pi}\int_{\partial D} f(P)\,ds_P = 0$. The solution $U(M)$, $M = (x, y) \in D$ of the equation (5.6)-(5.8) is a function

$$U(x, y) = U(x, y) + \alpha_0 - U(x_0, y_0) = \alpha_0 + \int_0^{2\pi} [G(x, y, t) - G(x_0, y_0, t)]\,\varphi(t)\,dt,$$

where $G(x, y, t) = \frac{-2\cos t \cdot (x - \cos t) - \sin t \cdot (y - 2\sin t)}{(x - \cos t)^2 + (y - 2\sin t)^2}$, $(x, y) \in D$, $t \in T_0$ and $\varphi(t)$, $t \in T_0$ is a solution of the equation (5.10).

We suppose that the solution of the problem is

$$U(x, y) = e^x \sin y, (x, y) \in D.$$

We reformulate the Neumann problem as a hypersingular integral equation (5.10) with the right-hand function

$$h(t) = -\frac{1}{\pi} e^{\cos t} \cdot [2\sin(2\sin t) \cdot \cos t + \cos(2\sin t) \cdot \sin t], t \in T_0.$$

Table 2 show the errors $\delta_n^{(k)} = |U(M_k) - U_n(M_k)|$ of the approximate solution for $U(x, y) = e^x \sin y$, where $M_k = 0.2k \cdot \left(\cos\frac{\pi}{4}, 2\sin\frac{\pi}{4}\right)$, $k = 1, 2, 3, 4$.

Table 2.

n	$\delta_n^{(1)}$	$\delta_n^{(2)}$	$\delta_n^{(3)}$	$\delta_n^{(4)}$
10	2.49002E-004	2.03515E-003	1.62083E-002	1.95521E-001
16	2.21581E-006	4.80044E-005	6.95798E-004	1.10161E-002
32	3.35232E-013	1.51967E-010	1.04752E-007	2.1614E-004
64	0E+000	1.66533E-015	1.11022E-014	3.74569E-009
128	0E+000	0E+000	0E+000	2.6645E-015

Table 2 show that if we take only $n = 128$, we get the order accuracy more than E-015.

References

[1] R.A.Aliev, A new constructive method for solving singular integral equations, *Mathematical Notes* 79:(5-6) (2006), 749-770.

[2] R.A.Aliev, A.F.Amrakhova, A constructive method for the solution of integral equations with Hilbert kernel, *Proceedings of the Institute of Mathematics and Mechanics: Ural branch of the Russian Academy of Sciences*, 18:4 (2012), 14-25 (in Russian).

[3] R.A.Aliev, Ch.A.Gadjieva, Approximation of hypersingular integral operators with Cauchy kernel, *Numerical Functional Analysis and Optimization*, 37:9 (2016), 1055-1065.

[4] A.Yu.Anfinogenov, I.K.Lifanov, P.I.Lifanov, On certain one and two dimensional hypersingular integral equations, *Sbornik:Mathematics*, 192:8 (2001), 1089-1131.

[5] W.T.Ang, *Hypersingular Integral Equations in Fracture Analysis*, Woodhead Publishing, Cambridge, 2013.

[6] Avram Sidi, Analysis of errors in some recent numerical quadrature formulas for periodic singular and hypersingular integrals via regularization, *Applied Numerical Mathematics*, 81 (2014), 30-39.

[7] Avram Sidi, Compact Numerical Quadrature Formulas for Hypersingular Integrals and Integral Equations, *Journal of Scientific Computing*, 54:1 (2013), 145-176.

[8] Avram Sidi, Richardson Extrapolation on Some Recent Numerical Quadrature Formulas for Singular and Hypersingular Integrals and Its Study of Stability, *Journal of Scientific Computing*, 60:1 (2014), 141-159.

[9] I.V.Boykov, E.S.Ventsel, A.I.Boykova, An approximate solution of hypersingular integral equations, *Applied Numerical Mathematics*, 60:6 (2010), 607-628.

[10] H.T.Cai, A fast solver for a hypersingular boundary integral equation, *Applied Numerical Mathematics*, 59:8 (2009), 1960-1969.

[11] T.Cao, Adaptive-additive multilevel methods for hypersingular integral equation, *Applicable Analysis*, 81:3 (2002), 539-564.

[12] Chaolang Hu, Xiaoming He, Tao Lu, Euler-Maclaurin expansions and approximations of hypersingular integrals, *Discret and continuous dynamical systems Series B*, 20:5 (2015), 1355-1375.

[13] Z.Chen, Y.F.Zhou, A new method for solving hypersingular integral equations of the first kind, *Applied mathematics letters*, 24 (2011), 636-641.

[14] D.D.Chien, K.Atkinson, A discrete Galerkin method for a hypersingular boundary integral equation, *IMA Journal of Numerical Analysis*, 17:3 (1997), 463-478.

[15] A.G.Davydov, E.V.Zakharov, Y.V.Pimenov, Hypersingular integral equations for the diffraction of electromagnetic waves on homogeneous magneto-dielectric bodies, *Computational Mathematics and Modeling*, 17:2 (2006), 97-104.

[16] V.J.Ervin, E.P.Stephan, Collocation with Chebyshev polynomials for a hypersingular integral equation on an interval, *Journal of Computational and Applied Mathematics*, 43 (1992), 221-229.

[17] L.Farina, P.A.Martin, V.Peron, Hypersingular integral equations over a disc: Convergence of a spectral method and connection with Tranter's method, *Journal of Computational and Applied Mathematics*, 269 (2014), 118-131.

[18] H.Feng, X.Zhang, J.Li, Numerical solution of a certain hypersingular integral equation of the first kind, *BIT Numerical Mathematics*, 51:3 (2011), 609-630.

[19] R.Gayen, Arpita Mondal, A hypersingular integral equation approach to the porous plate problem, *Applied Ocean Research*, 46 (2014), 70-78.

[20] Ch.A.Gadjieva, A new constructive method for solving of bisingular integral equations with Cauchy kernel, *Proc. of the IMM of NAS of Azerbaijan*, 42:1 (2016), 50-66.

[21] M.Gülsu, Y.Öztürk, Numerical approach for the solution of hypersingular integro-differential equations, *Applied Mathematics and Computation*, 230 (2014), 701-710.

[22] J.Hadamard, *Lectures on Cauchy's problem in linear partial differential equations*, Dover publication, New-York, 2003.

[23] J.Huang, Z.Wang, R.Zhu, Asymptotic error expansions for hypersingular integrals, *Advances in Computational Mathematics*, 38:2 (2013), 257-279.

[24] R.Kress, *Linear integral equations*, Springer, New York, 2014.

[25] W.McLean, O.Steinbach, Boundary element preconditioners for a hypersingular integral equation on an interval, *Advances in Computational Mathematics*, 11:4 (1999), 271-286.

[26] J. Li, X.P.Zhang, D.H.Yu, *Extrapolation methods to compute hypersingular integral in boundary element methods*, Science China Mathematics, 56:8 (2013), 1647-1660.

[27] S.Li, J.Xian, A multiscale Galerkin method for the hypersingular integral equation reduced by the harmonic equation, *Applied Mathematics-A Journal of Chinese Universities*, 28:1 (2013), 75-89.

[28] I.K.Lifanov, *Singular integral equations and Discrete Vortices*, VSP, the Netherlands, 1996.

[29] I.K.Lifanov, L.N.Poltavskii, G.M.Vainikko, *Hypersingular integral equations and their applications,* CRC Press, 2004.

[30] B.N.Mandal, G.H.Bera, Approximate solution for a class of hypersingular integral equations, *Applied mathematics letters*, 19 (2006), 1286-1290.

[31] P.A.Martin, Exact solution of a simple hypersingular integral equation, *Journal of integral equations and applications*, 4:2 (1992), 197-204.

[32] N.M.A. Nik Long, Z.K. Eshkuvatov, Hypersingular integral equation for multiplen curved cracks problem in plane elasticity, *International Journal of Solids and Structures*, 46 (2009), 2611–2617.

[33] G.Rus, R.Gallego, Hypersingular shape sensitivity boundary integral equation for crack identification under harmonic elastodynamic excitation, *Computer Methods in Applied Mechanics and Engineering*, 196 (2007), 2596-2618.

[34] J.Saranen, G.Vainikko, *Periodic integral and Pseudodifferential Equations with Numerical Approximation*, Springer-Verlag, Berlin, 2002.

[35] Ch. Yang, A unified approach with spectral convergence for the evaluation of hypersingular and supersingular integrals with a periodic kernel, *Journal of Computational and Applied Mathematics*, 239 (2013), 322-332.

[36] C.Zheng, T.Matsumoto, T.Takahashi, H.Chen, Explicit evaluation of hypersingular boundary integral equations for acoustic sensitivity analysis based on direct differentiation method, *Engineering Analysis with Boundary Elements*, 35:11 (2011), 1225-1235.

[37] A. Zygmund, *Trigonometric Series*, vols. 1 and 2, Cambridge Univ. Press, Cambridge, 2003.

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