Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 44, Number 1, 2018, Pages 70–80

ON A SPACE OF μ-STATISTICAL CONTINUOUS FUNCTIONS

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Abstract. The concepts of μ-statistical discontinuities of the first and second kinds for functions in some measurable space with the measure μ are introduced in this work. The space of μ-statistical continuous functions on some interval is considered, some properties of functions in this space are studied. The relationship between this space and the spaces of continuous and Lebesgue-summable functions in the case where μ is a Lebesgue measure is also considered.

1. Introduction

Actually, the concept of statistical convergence of the sequences of complex numbers has long been known as "almost convergence" (see, e.g., the monograph of A.Zygmund [40]). It was introduced in the study of pointwise convergence of the Fourier series of summable functions. Equivalent definition for this concept was given by H. Fast in [11] (see also Steinhaus [37]), where it was (for the first time) referred to as "statistical convergence". In [33, 36, 12, 13], the basic properties of statistically convergent sequences were mainly generalized in two directions. The first direction included the generalizations of the concept of statistical convergence itself, so there arose I-convergence (ideal convergence), $\mathscr{F}$-convergence (filter convergence), lacunar convergence, etc. (see, e.g., [7, 8, 15, 32, 25, 9, 14, 28, 35, 16, 17]). The second direction treated these kinds of convergence in various mathematical structures (see [26, 1, 20, 19, 18, 3, 4, 5, 34, 38, 23, 2, 10, 24]). In [29, 31, 22], the statistical convergence was generalized for double sequences, and the properties of this convergence were studied. The number of all relevant works is too big, and it should be noted that it is impossible to name all of them here.

Quite naturally, there arises the question about the existence of a continuous analog of the concept of statistical convergence for number sequences (or for elements of other mathematical structure). The first step in this direction was made by F.Moricz [27] who introduced the concepts of statistical limit and statistical fundamentality for measurable functions at infinity and at a finite point generated by the Lebesgue measure. Moricz proved the equivalence of these concepts and studied some of their properties. He also studied the relationship between this kind of convergence and the one of Fourier series. But, this concept is not a

2010 *Mathematics Subject Classification.* 40A05, 26A15, 11B05.
Key words and phrases. μ-statistical discontinuities, μ-statistical continuity, the space of μ-statistical continuous functions.

generalization of the similar concept for sequences, because it does not imply, as a special case, the concept of statistical convergence for sequences.

The direct generalization of the concept of statistical convergence in continuous case was first carried out by B.T.Bilalov and S.R.Sadigova [6]. They introduced the concepts of *μ-statistical convergence* and *μ-statistical* fundamentality, proved their equivalence and studied some of their properties. They also introduced the concept of *μ-statistical* continuity. μ-stat convergence is a direct generalization of the statistical convergence in continuous case, as it turns out from this concept as a special case.

In this work, we give definitions of *μ-statistical* one-sided limits at a point and *μ-statistical* discontinuities of the first and second kind in some measurable space with a measure. We consider the space $C_{st}[a,b]$ of *μ-statistical* continuous functions on some interval $[a,b]$. We prove that the space of continuous functions is strictly embedded in $C_{st}[a,b]$. Moreover, we show that $C_{st}[a,b]$ is not embedded in the Lebesgue space $L_p[a,b]$ for $\forall p \in (0,+\infty)$, i.e. $C_{st}[a,b]\backslash L_p(a,b) \neq \emptyset$. We also make comparison between the concept of *μ-statistical* continuity at a point and the known concept of approximate continuity (see, e.g., [30]).

2. Needful Information

We will use the standard notation. N will be the set of all positive integers; R is the set of all real numbers; $\exists$ will mean "there exist(s)"; $\exists!$ will mean "there exists a unique"; $\Rightarrow$ will mean "it follows"; $\Leftrightarrow$ will mean equivalence. $I_a^{+\infty} \equiv [a\,;+\infty)$; $I_a^{-\infty} \equiv (-\infty; a]$.

Let $(I_a^{\infty};\mathscr{B};\mu)$ be a measurable space with measure $\mu : \mathscr{B} \to I_a^{\infty}$, where $\mathscr{B}$ σ-algebra of Borel subsets in I_a. We will assume that the measure μ σ- finite measure and $\mu(I_a^{\infty}) = +\infty$. The measure of the set $M \in \mathscr{B}$ will be denoted by $|M|$, i.e. $|M| = \mu(M)$.

We will need some concepts and facts from the work [6].

Definition 2.1. We say that the infinitely remote point (∞) is a point *μ-stat* density for $M \in \mathscr{B}$, if

$$\lim_{x\to\infty} \frac{|M \cap I_a^x|}{|I_a^x|} = 1,$$

where $I_a^x = [a,x]\,,\ \forall x \in I_a^{\infty}$.

Let $f : I_a^{\infty} \to R$ be some $\mathscr{B}$-measurable function and $A \in R$ be some number. For a given $\varepsilon > 0$ assume

$$A_{\varepsilon}(f) \equiv \{x \in I_a^{\infty} : |f(x) - A| \geq \varepsilon\}\,.$$

Definition 2.2. We say that f has a *μ-stat* limit A at infinity if and only if

$$\lim_{x\to\infty} \frac{|A_{\varepsilon}(f) \cap I_a^x|}{|I_a^x|} = 0,\ \forall\varepsilon > 0,$$

and this limit will be denoted as *μ-st* $\lim\limits_{x\to\infty} f(x) = A$.

It is easy to see that the infinitely remote point (∞) is a point *μ-stat* density for M if and only if

$$\lim_{x\to\infty} \frac{|M^c \cap I_a^x|}{|I_a^x|} = 0,$$

where $M^c = I_a^\infty \backslash M$. It directly follows from the relation $I_a^x = (M \cap I_a^x) \cup (M^c \cap I_a^x)$. Consequently

$$\mu\text{-}st \lim_{x\to\infty} f(x) = A \Leftrightarrow \lim_{x\to\infty} \frac{|A_\varepsilon^c(f) \cap I_a^x|}{|I_a^x|} = 1, \ \forall \varepsilon > 0.$$

The set of all subsets of I_a^∞, or which the infinity (∞) is the point of μ-*stat* density, will be denoted by I_{st}^∞.

Definition 2.3. The sequence $\{a_n\}_{n\in N} \subset I_a^\infty$ is said to have a *st*-lim equal to (∞), if

$$\lim_{n\to\infty} \frac{card\,(a_\varepsilon \cap e_n)}{n} = 0, \ \forall \varepsilon > 0,$$

where $e_n = \{1; ....; n\}$ and $a_\varepsilon \equiv \{k \in N : |a_k| < \varepsilon\}$, this limit will be denoted as $a_n \overset{st}{\to} \infty$, $n \to \infty$, or $st\text{-}\lim\limits_{n\to\infty} a_n = \infty$.

Let $a_\varepsilon^c = \{k \in N : |a_k| \geq \varepsilon\}$. Then from the relation $e_n = (a_\varepsilon \cap e_n) \cup (a_\varepsilon^c \cap e_n)$ it directly follows that

$$st\text{-}\lim_{n\to\infty} a_n = \infty \Leftrightarrow \lim_{n\to\infty} \frac{card\,(a_\varepsilon^c \cap e_n)}{n} = 1, \ \forall \varepsilon > 0.$$

Definition 2.4. We say that the function $f : I_a^\infty \to R$ has a *st*-lim equal to $A \in R$, if $st\text{-}\lim\limits_{n\to\infty} f(a_n) = A$ with $\forall \{a_n\}_{n\in N} \subset I_a^\infty : st\text{-}\lim\limits_{n\to\infty} a_n = \infty$, and it will be denoted as $st\text{-}\lim\limits_{x\to\infty} f(x) = A$.

We will need the concept of μ-*stat*-fundamentality which introduced in [6].

Definition 2.5. We say that the $\mathscr{B}$-measurable function $f : I_a^\infty \to R$ is μ-*stat* fundamental at infinity, if $\forall \varepsilon > 0$, $\exists x_\varepsilon \in I_a^\infty$

$$\lim_{x\to\infty} \frac{|X_f(\varepsilon) \cap I_a^x|}{|I_a^x|} = 0,$$

where $X_f(\varepsilon) \equiv \{x \in I_a^\infty : |f(x) - f(x_\varepsilon)| \geq \varepsilon\}$.

3. Main Results

Let $\mathscr{B}$ be a class of all Borel subsets of R and $(R; \mathscr{B}; \mu)$ be a measurable space with a σ-finite measure $\mu : \mathscr{B} \to I_0^\infty :\ \mu((-\infty, a)) = \mu((a, +\infty)) = +\infty, \forall a \in R$. Let $E \in \mathscr{B}$ be some set and x_0 be its limit point. Let

$$E(x_0) \equiv \left\{x : \left(x_0 + \frac{1}{x}\right) \in E\right\}$$

and

$$I_t(x_0) \equiv \begin{cases} I_a^{(t-x_0)^{-1}}, & (t-x_0)^{-1} \geq a, \\ I_{(t-x_0)^{-1}}^a, & (t-x_0)^{-1} < a. \end{cases}$$

Let $f : E \to R$ be some $(R; \mathscr{B})$ $-$measurable function. We give the following

Definition 3.1. We say that the function f has a μ-*stat* left-hand limit A, at the point x_0 if

$$\lim_{t\to x_0-0} \frac{\left|A_f^\varepsilon(x_0) \cap I_t(x_0)\right|}{|I_t(x_0)|} = 0, \ \forall \varepsilon > 0,$$

where

$$A_f^{\varepsilon}(x_0) \equiv \left\{x \in E(x_0) : \left|f\left(x_0 + x^{-1}\right) - A\right| \geq \varepsilon\right\}.$$

This fact will be denoted by $\mu\text{-}st \lim\limits_{x \to x_0 - 0} f(x) = \mu\text{-}st\, f(x_0 - 0) = A$.

Similarly, we define the concept of *μ-stat* right-hand limit at the point x_0 : $\mu\text{-}st \lim\limits_{x \to x_0 + 0} f(x) = A = \mu\text{-}st\, f(x_0 + 0)$.

Similar to the classical case, if

$$\mu\text{-}st \lim_{x \to x_0 - 0} f(x) = \mu\text{-}st \lim_{x \to x_0 + 0} f(x) \neq f(x_0),$$

then x_0 is called *μ-stat* removable discontinuity point, if $\exists\, \mu\text{-}st \lim\limits_{x \to x_0 \pm 0} f(x)$ and

$$\mu\text{-}st \lim_{x \to x_0 - 0} f(x) \neq \mu\text{-}st \lim_{x \to x_0 + 0} f(x),$$

then x_0 is called *μ-stat* discontinuity of the first kind and the quantity

$$\Delta_f^{st}(x_0) = \mu\text{-}st \lim_{x \to x_0 + 0} f(x) - \mu\text{-}st \lim_{x \to x_0 - 0} f(x),$$

is called a *μ-stat* jump of the function f at x_0.

In other cases, x_0 is called a *μ-stat* discontinuity point of the second kind.

Example 3.1. Let $(R; \mathscr{B}; \mu)$ be a measurable space with a Lebesgue measure. Consider the function

$$f(x) = \begin{cases} \sin x, & x \in Q, \\ sign\ x & x \in R \backslash Q, \end{cases}$$

where Q are rational numbers in R. The point $x_0 = 0$ is a *μ-statistical* discontinuity of the first kind and $\Delta_f^{st}(0) = 2$. All other points are *μ-statistical* continuity points.

If $\mu\text{-}st \lim\limits_{x \to x_0 - 0} f(x) = \mu\text{-}st \lim\limits_{x \to x_0 + 0} f(x) = f(x_0)$ holds, then $f(\cdot)$ is called a *μ-stat* continuous at the point x_0.

Let $f : [a, b] \to R$ be some function. It is clear that if $f \in C[a, b]$, then $f(\cdot)$ is a *μ-stat* continuous on $[a, b]$. The following question arises naturally.

Question 3.1. *Let $f : [a, b] \to R$ be a μ-stat continuous on $[a, b]$. Is it continuous on $[a, b]$?*

It is obvious that if $f(\cdot)$ has a discontinuity of the first kind at the point $x_0 \in (a, b)$, then x_0 is also a *μ-stat* discontinuity point of the first kind and moreover

$$\mu\text{-}st\, f(x_0 \pm 0) = f(x_0 \pm 0).$$

Therefore, if $f(\cdot)$ has a discontinuity of the first kind at the point x_0, then it can not be a *μ-stat* continuous at this point.

Denote the linear space of *μ-statistical* continuous functions on $[a, b]$ over the field K ($K \equiv C$ or R) by $C_{st}[a, b]$. It is absolutely clear that the pointwise limit of the sequence of *μ-statistical* continuous functions may not be *μ-statistical* continuous on $[a, b]$.

Let's give an example of a function on the interval $E = [-1, 1]$ which is not continuous on E, but, at the same time, is *μ-statistical* continuous on E.

The following lemma is true.

Lemma 3.1. *The strict embedding* $C[a,b] \subset C_{st}[a,b] : C_{st}[a,b] \backslash C[a,b] \neq \emptyset$ *holds true.*

Proof. The embedding $C[a,b] \subset C_{st}[a,b]$ is valid. We will prove the validity of $C_{st}[a,b] \backslash C[a,b] \neq \emptyset$.

Consider the following series

$$\sum_{k=1}^{\infty} \alpha_k, \tag{3.1}$$

such that the remainder terms satisfy the conditions

$$\sigma_n \leq \frac{1}{(n+1)^3}, \tag{3.2}$$

where $\sigma_n = \sum_{k=n}^{\infty} \alpha_k$, $\alpha_k > 0$, $\forall k \in N$.

Let $a = -1$, $b = 1$ and μ be a Lebesgue measure. Assume

$$O_\delta(x) \equiv (x-\delta, x+\delta) \cap [-1,1].$$

Denote by $i_n \subset \left(\frac{1}{n+1}, \frac{1}{n}\right)$ an arbitrary interval of length α_n, i.e. $|i_n| = \mu(i_n) = \alpha_n$, $n \in N$.

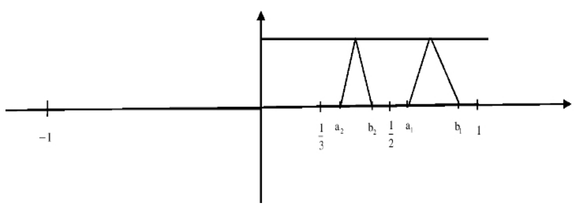

Figure 1

Let $x_n \in i_n$ be the middle of the interval $i_n = (a_n, b_n)$. Consider the points $(1;0)$, $(b_1;0)$, $(x_1;1)$, $(a_1;0)$, $(b_2;0)$,..., (see Fig. 1) and connect them with the intervals. Denote the function generated by this graph and the interval $[-1,0]$ by $f(x)$. It is easy to see that this function is defined by the formula

$$f(x) = \begin{cases} \frac{x-b_n}{x_n-b_n}, & \text{if } x \in [x_n, b_n], \\ \frac{x-a_n}{x_n-a_n}, & \text{if } x \in [a_n, x_n], \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $f \notin C[-1,1]$, because there exists no $f(+0)$. Let's show that $f \in C_{st}[-1,1]$. Obviously, $f(\cdot)$ is continuous at every point $x_0 \neq 0$, and therefore it is μ-*statistical* continuous at these points. Let's show that $f(\cdot)$ is μ-*statistical* continuous at the point $x = 0$, too. To do so, it suffices to show that there exist one-sided statistical limits at the point $x = 0$ and they are equal to each other. Let $t < 0$. Take $\forall \varepsilon > 0$. It is clear that

$$\forall x \in I_t(0) \Rightarrow x < 0 \Rightarrow f\left(x^{-1}\right) = 0 \Rightarrow \left|f\left(x^{-1}\right) - f(0)\right| < \varepsilon \Rightarrow A_f^\varepsilon(0) = \emptyset \Rightarrow$$

$$\Rightarrow \lim_{t \to -0} \frac{\left|A_f^\varepsilon(0) \cap I_t(0)\right|}{|I_t(0)|} = 0 \Rightarrow \mu\text{-}st\, f(-0) = 0.$$

Now, consider the case $t > 0$. We have

$$I_t(0) = \left(0, t^{-1}\right) \Rightarrow |I_t(0)| = t^{-1}.$$

Let $\varepsilon > 0$ be an arbitrary number. In this case $E = [-1, 1]$ and consequently

$$E(0) = \left\{x : x^{-1} \in E\right\} = (-\infty, -1) \cup (1, +\infty).$$

As a result ($x > 0$):

$$A_f^{\varepsilon}(0) = \left\{x \in E(0) : \left|f\left(x^{-1}\right) - f(0)\right| \geq \varepsilon\right\} =$$

$$= \left\{x \in E(0) : \left|f\left(x^{-1}\right)\right| \geq \varepsilon\right\} \subset \cup_k \left\{x : x^{-1} \in i_k\right\}.$$

It suffices to consider the case $t = n^{-1} \to 0$. We have

$$\left(A_f^{\varepsilon}(0) \cap I_{n^{-1}}(0)\right) \subset \cup_{k=1}^{n} \left\{x : x^{-1} \in i_k\right\} \Rightarrow \left|A_f^{\varepsilon}(0) \cap I_n(0)\right| \leq$$

$$\leq \sum_{k=1}^{n} \left|\left(\frac{1}{b_k}, \frac{1}{a_k}\right)\right| = \sum_{k=1}^{n} \left(\frac{1}{a_k} - \frac{1}{b_k}\right) = \sum_{k=1}^{n} \frac{b_k - a_k}{a_k b_k} = \sum_{k=1}^{n} \frac{\alpha_k}{a_k b_k}.$$

From $(a_k, b_k) \subset \left(\frac{1}{k+1}, \frac{1}{k}\right)$, it follows

$$a_k b_k \sim \frac{1}{k^2}.$$

Consequently

$$\left|A_f^{\varepsilon}(0) \cap I_{n^{-1}}(0)\right| \leq c \sum_{k=1}^{n} k^2 \alpha_k.$$

Let $\alpha_k = \frac{1}{k^4}$. Then it is clear that

$$\lim_{n\to\infty} \frac{\left|A_f^{\varepsilon}(0) \cap I_{n^{-1}}(0)\right|}{\left|I_{n^{-1}}(0)\right|} \leq \lim_{n\to\infty} \frac{c \sum_{k=1}^{n} \frac{1}{k^2}}{n} = 0.$$

Thus, it is proved that μ-*st* $f(+0) = 0$, and, as a result $f(\cdot)$ is a μ-*stat* continuous at $x = 0$, and, hence, $f \in C_{st}[-1, 1]$. ◄

Similarly, we can give an example of non-bounded function on the interval$[-1, 1]$ which is a μ-*statistical* continuous on $[-1, 1]$.

The following lemma is also true.

Lemma 3.2. *The relations* $C_{st}[a, b] \backslash L_p(a, b) \neq \emptyset \wedge L_p(a, b) \backslash C_{st}[a, b] \neq \emptyset$, $\forall p \in [1, +\infty)$ *hold true.*

Proof. The relation $L_p(a, b) \backslash C_{st}[a, b] \neq \emptyset$ is obvious, since the function having a removable discontinuity point does not belong to $C_{st}[a, b]$. Let us prove the relation $C_{st}[a, b] \backslash L_p(a, b) \neq \emptyset$.

Consider the series (3.1), satisfying the condition (3.2). Similarly to the previous case, we consider the intervals

$$i_n = (a_n, b_n) \subset \left(\frac{1}{n+1}, \frac{1}{n}\right) : |i_n| = \alpha_n,$$

and let $x_n = \frac{a_n + b_n}{2}$. Consider the points $(1; 0)$, $(b_1; 0)$, $\left(x_1; \alpha_1^{-1}\right)$, $(a_1; 0)$, $(b_2; 0)$, $\left(x_2; \alpha_2^{-1}\right)$,....

Let us connect these points by segments. Denote by $f(x)$ the function obtained by these segments and the segment $[-1,0]$. From previous arguments it follows that $f \in C_{st}[-1,1]$. We have

$$\int_{-1}^{1} |f(x)|\, dx = \sum_{k=1}^{\infty} \int_{i_k} |f(x)|\, dx = \sum_{k=1}^{\infty} \frac{1}{2}\alpha_k f(x_k) = \frac{1}{2}\sum_{k=1}^{\infty} 1 = +\infty.$$

Thus, $f \notin L_p(0,1)$, $\forall p \in [1,+\infty)$. It is clear that

$$C[a,b] \subset (C_{st}[a,b] \cap L_p(a,b)) \,, \forall p \in [1,+\infty)\,,$$

is valid. ◄

The previous example shows that $C[a,b]$ is not dense in $C_{st}[a,b]$ with respect to the norm $\|\cdot\|_p$. The following question arises naturally.

Question 3.2. *Is there such a metric or such convergence, with respect to which the space $C_{st}[a,b]$ is complete?*

Assume

$$C_{st}^{J}[a,b] \equiv \{f \in C_{st}[a,b] : \|f\|_{\infty} < +\infty\}\,,$$

where

$$\|f\|_{\infty} = \sup_{[a,b]} |f(\cdot)|\,.$$

It is clear that the following strict embedding holds true

$$C[a,b] \subset C_{st}^{J}[a,b] \subset L_p(a,b) \ , \ \forall p \in (0,+\infty)\,.$$

Under $L_p(a,b)$ we understand the complete metric space of measurable (with respect to the Lebesgue measure) functions on (a,b), for $p \in (0,1)$, with finite integral

$$\int_a^b |f(t)|^p\, dt < +\infty,$$

with an ordinary metric. Thus, the following theorem is true.

Theorem 3.1. *Let $(R;\mathscr{B};\mu)$ be a measurable space with a σ-finite measure μ on the σ-algebra of Borel sets $\mathscr{B}$ and $\mu((-\infty,x_0)) = \mu((x_0,+\infty)) = +\infty$ for some $x_0 \in R$. Then the embeddings*

i) $C[a,b] \subset (C_{st}[a,b] \cap L_p(a,b))$, $\forall p \in (0,+\infty)$

and

ii) $C[a,b] \subset \left(C_{st}^{J}[a,b] \subset L_p(a,b)\right)$, $\forall p \in (0,+\infty)$

hold true, and they are strict.

Let us show that the space $C_{st}^{J}[a,b]$ is complete with respect to the norm $C[a,b] \subset (C_{st}[a,b] \cap L_p(a,b))$, $\forall p \in (0,+\infty)$. Let $\{f_n\}_{n\in N} \subset C_{st}^{J}[a,b]$ be some fundamental sequence, i.e.

$$\|f_n - f_m\|_{\infty} \to 0\,,\ n,m \to \infty.$$

Fixing $\forall x \in [a,b]$, hence we obtain that $\{f_n(x)\}_{n\in N}$ is a fundamental sequence and, as a result, it converges to a certain value $f(x)$. Let us show that $f \in C_{st}^{J}[a,b]$. Let $\varepsilon > 0$ be an arbitrary number and $x_0 \in [a,b]$ be an arbitrary point. For any arbitrary $n \in N$, we assume

$$E_n(\varepsilon) \equiv \left\{x : |f_n(x) - f_n(x_0)| \geq \frac{\varepsilon}{3}\right\},$$

$$E_n(f;\varepsilon) \equiv \left\{x : |f(x) - f_n(x)| \geq \frac{\varepsilon}{3}\right\}.$$

We have

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|. \quad (3.3)$$

It is clear that $\|f_n - f\|_\infty \to 0$, $n \to \infty$. Therefore, it is clear that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3}, \forall x \in [a,b],$$

holds. Then from (3.3) it follows

$$\{x : |f(x) - f(x_0)| \geq \varepsilon\} \subset E_n(\varepsilon), \ \forall n \geq n_\varepsilon.$$

Since, otherwise

$$|f(x) - f(x_0)| \leq \frac{2}{3}\varepsilon + |f_n(x) - f_n(x_0)| < \frac{2}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon.$$

Consequently

$$(\{x : |f(x) - f(x_0)| \geq \varepsilon\} \cap O_\delta(x_0)) \subset (E_n(\varepsilon) \cap O_\delta(x_0)),$$

and, as a result

$$|\{x : |f(x) - f(x_0)| \geq \varepsilon\} \cap O_\delta(x_0)| \leq |E_n(\varepsilon) \cap O_\delta(x_0)|, \forall n \geq n_\varepsilon. \quad (3.4)$$

Take $\forall n \geq n_\varepsilon$ and fix. So, $f_{n_0} \in C_{st}^J[a,b]$, then from (3.4) we obtain

$$\lim_{\delta\to 0} \frac{|\{x : |f(x) - f(x_0)| \geq \varepsilon\} \cap O_\delta(x_0)|}{|O_\delta(x_0)|} \leq \lim_{\delta\to 0} \frac{|E_{n_0}(\varepsilon) \cap O_\delta(x_0)|}{O_\delta(x_0)} = 0.$$

Since x_0 is arbitrary, we obtain $f \in C_{st}^J[a,b]$. So, the following theorem is true.

Theorem 3.2. *The space $C_{st}^J[a,b]$ is a Banach space with respect to the norm $\|\cdot\|_\infty$.*

Compare the concept *μ-stat* continuity with the concept of approximate continuity. Let us recall the definition of approximate continuity.

Let $E \subset R$ be some measurable (with respect to the Lebesgue measure) set and assume

$$E(x_0;h) = E \cap [x_0 - h, x_0 + h].$$

Definition 3.2. The limit

$$D_{x_0}E = \lim_{h\to 0} \frac{mE(x_0;h)}{2h}$$

(in case it exists) is called a density of the set E at the point x_0.

If $D_{x_0}E = 1$, then x_0 is a point of density for the set E, and if $D_{x_0}E = 0$, then x_0- is a rarefaction point E.

In our case, x_0 is a point of *m-stat* density for the set E, where m is a Lebesgue measure.

The following theorem is known.

Theorem 3.3. *Almost all points of measurable set E are its density point.*

More details about the following concept can be found in [30].

Definition 3.3. Let the function $f(x)$ be given on the segment $[a,b]$ and $x_0 \in [a,b]$. If there exists a measurable set $E \subset [a,b]$ with a density point x_0 such that $f(x)$ is continuous along E at the point x_0, then $f(x)$ is said to be approximate continuous at the point x_0.

In our case, the concept of approximate continuity coincides with the one of *m-statistical* continuity at the point x_0. Let's recall the following Denjoy theorem.

Theorem 3.4. *(Denjoy) If $f(x)$ is a measurable and almost everywhere finite function in $[a,b]$, then it is approximate continuous at almost every point in $[a,b]$.*

Consequently, if $f(\cdot)$ is measurable and almost everywhere finite in $[a,b]$, then it is *m-statistical* continuous almost everywhere in $[a,b]$.

Acknowledgements

The authors would like to express their profound gratitude to Prof. Bilal Bilalov, for his valuable guidance to this article.

This research was supported by the Azerbaijan National Academy of Sciences under the program "Approximation by neural networks and some problems of frames".

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Received: October 12, 2017; Revised: March 19, 2018; Accepted: March 29, 2018