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GLOBAL BIFURCATION FROM INFINITY IN SOME NONLINEARIZABLE EIGENVALUE PROBLEMS WITH INDEFINITE WEIGHT

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Abstract. We consider a nonlinear eigenvalue problem for a fourth order ordinary differential operator with indefinite weight. We study the structure of bifurcation set and investigate the behavior of the solution set bifurcating from infinity and contained in the classes of positive and negative functions near bifurcation points.

1. Introduction

We consider the following fourth order boundary value problem

$$(\ell u) \equiv (p(t)u'')'' - (q(t)u')' = \lambda r(t)u + h(t, u, u', u'', u''', \lambda),\ t \in (0,1), \tag{1.1}$$

$$\begin{aligned} & u'(0)\cos\alpha - (pu'')(0)\sin\alpha = 0, \\ & u(0)\cos\beta + Tu(0)\sin\beta = 0, \\ & u'(1)\cos\gamma + (pu'')(1)\sin\gamma = 0, \\ & u(1)\cos\delta - Tu(1)\sin\delta = 0, \end{aligned} \tag{1.2}$$

where $\lambda \in \mathbb{R}$ is a spectral parameter, $Ty \equiv (pu'')' - qu'$, the function $p(t)$ is strictly positive and continuous on $[0,1]$, $p(t)$ has an absolutely continuous derivative on $[0,1]$, $q(t)$ is nonnegative and absolutely continuous on $[0,1]$, the weight function $r(t)$ is sign-changing continuous on $[0,1]$ (i.e. meas$\{t \in (0,1) : \sigma r(t) > 0\} > 0$ for each $\sigma \in \{+, -\}$) and $\alpha, \beta, \gamma, \delta$ are real constants such that $0 \leq \alpha, \beta, \gamma, \delta \leq \pi/2$ except the cases $\alpha = \gamma = 0$, $\beta = \delta = \pi/2$ and $\alpha = \beta = \gamma = \delta = \pi/2$. The nonlinear term has the representation $h = f + g$, where $f, g \in C([0,1] \times \mathbb{R}^5)$ are real-valued functions satisfying the following conditions:

$$uf(t, u, s, v, w, \lambda) \leq 0,\ (t, u, s, v, w, \lambda) \in [0,1] \times \mathbb{R}^5, \tag{1.3}$$

there exists constants $M > 0$ such that

$$\left| \frac{f(t, u, s, v, w, \lambda)}{u} \right| \leq M,\ (t, u, s, v, w, \lambda) \in [0,1] \times \mathbb{R}^5, \tag{1.4}$$

and

$$g(t, u, s, v, w, \lambda) = o(|u| + |s| + |v| + |w|), \tag{1.5}$$

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in a neighborhood of $(u, s, v, w) = \infty$ uniformly in $t \in [0, 1]$ and in $\lambda \in \Lambda$, for every bounded interval $\Lambda \subset \mathbb{R}$.

Since condition (1.5) holds then we can consider bifurcation from infinity, i.e., the existence of solutions of the nonlinear problem (1.1)-(1.2) having arbitrarily large y. Global bifurcation from infinity for nonlinear Sturm-Liouville problems with definite weight functions have been considered in [5, 8, 10, 12-15]. These papers prove the existence of global continua of solutions in $\mathbb{R} \times C^1$ emanating from bifurcation points and intervals (in $\mathbb{R} \times \{\infty\}$, which we identify with $\mathbb{R}$) surrounding the eigenvalues of the corresponding linear problem. The approach used in these works is to transform the bifurcation from infinity problem to a problem involving bifurcation from zero at points and intervals surrounding the eigenvalues of the corresponding linear problems and then apply the global bifurcation theory from [12] and [15]. Global bifurcation from infinity of nonlinear eigenvalue problems for special class ordinary differential equations of fourth order with definite weight functions were investigated only in the works [9, 11]. This is due to the fact that before the recent work [1, 2] almost were not investigated the global bifurcation of solutions from zero of nonlinear eigenvalue problems ordinary differential equations of fourth order. But in the case of an indefinite weight functions, such problems have not actually been investigated since the oscillatory properties of all eigenfunctions, with the exception of the principal eigenfunctions, have not yet been studied. In addition, the global bifurcation of solutions of zero of problem (1.1)-(1.2) in classes of positive and negative functions has not been studied up to the recent paper [3].

The purpose of this work is to study the global bifurcation from infinity of solutions of problem (1.1)-(1.2) in the classes of positive and negative functions, bifurcating from the intervals at infinity. By extending the approximation technique from [4] and combining it with the global results in [2, 3, 13] we prove the existence of global sets of solutions bifurcating from infinity which are similar to those obtained in [13].

2. Preliminaries

The problem (1.1)-(1.2) for the case of $f \equiv 0$ is studied in [6]. In the case of $f \equiv 0$ the linearization of (1.1)-(1.2) at $u = 0$ is the linear eigenvalue problem

$$\begin{array}{l} (p(t)u''(t))'' - (q(t)u'(t))' = \lambda r(t)u(t), \ t \in (0,1), \\ u \in B.C., \end{array} \tag{2.1}$$

where by $B.C.$ we denote the set of boundary conditions (1.2). For the linear eigenvalue problem (2.1) we have the following result.

Theorem 2.1 [6, Theorem 2.1]. *The spectral problem* (2.1) *has two sequences of real eigenvalues*

$$0 < \lambda_1^+ < \lambda_2^+ \leq ... \leq \lambda_k^+ \mapsto +\infty,$$

and

$$0 > \lambda_1^- > \lambda_2^- \geq ... \geq \lambda_k^- \mapsto -\infty$$

and no other eigenvalues. Moreover, λ_1^+ *and* λ_1^- *are simple principal eigenvalues, i.e. the corresponding eigenfunctions* $u_1^+(t)$ *and* $u_1^-(t)$ *have no zeros in the interval* $(0,1)$*.*

Let $E = C^3[0,1] \cap B.C.$ be a Banach space with the norm $||u||_3 = ||u||_\infty + ||u'||_\infty + ||u''||_\infty + ||u'''||_\infty$, where $||\cdot||_\infty$ is the standard sup-norm in $C[0,1]$.

Let

$$S = S_1 \cup S_2,$$

where

$$S_1 = \{u \in E : u^{(i)}(t) \neq 0, Tu(t) \neq 0,\ t \in [0,1],\ i = 0,\ 1,\ 2\}$$

and

$$S_2 = \{u \in E : \text{ there exists } i_0 \in \{0,\ 1,\ 2\} \text{ and } t_0 \in (0,1) \text{ such that } u^{(i_0)}(t_0) = 0, \text{ or } Tu(t_0) = 0 \text{ and if } u(t_0)u''(t_0) = 0, \text{ then } u'(t)Tu(t) < 0 \text{ in a neighborhood of } t_0, \text{ and if } u'(t_0)Tu(t_0) = 0, \text{ then } u(t)u''(t) < 0 \text{ in a neighborhood of } t_0\}.$$

Note that if $u \in S$ then the Jacobian $J = \rho^3 \cos\psi \sin\psi$ (see [1, 2]) of the Prüfer-type transformation

$$\begin{cases} u(t) = \rho(t)\sin\psi(t)\cos\theta(t), \\ u'(t) = \rho(t)\cos\psi(t)\sin\varphi(t), \\ (pu'')(t) = \rho(t)\cos\psi(t)\cos\varphi(t), \\ Tu(t) = \rho(t)\sin\psi(t)\sin\theta(t), \end{cases} \tag{2.2}$$

does not vanish on $(0,1)$.

For each $u \in S$ we define $\rho(u,t)$, $\theta(u,t)$, $\varphi(u,t)$ and $w(u,t)$ to be the continuous functions on $[0,1]$ satisfying

$$\rho(u,t) = u^2(t) + u'^2(t) + (p(t)u''(t))^2 + (Tu(t))^2,$$

$$\theta(u,t) = \operatorname{arctg}\frac{Tu(t)}{u(t)},\ \ \theta(u,0) = \beta - \pi/2\,,$$

$$\varphi(u,t) = \operatorname{arctg}\frac{u'(t)}{(pu'')(t)},\ \ \varphi(u,0) = \alpha\,,$$

$$w(u,t) = \operatorname{ctg}\psi(u,t) = \frac{u'(t)\cos\theta(u,t)}{u(t)\sin\varphi(u,t)},\ \ w(u,0) = \frac{u'(0)\sin\beta}{u(0)\sin\alpha}\,,$$

and $\psi(u,t) \in (0,\pi/2)$, $t \in (0,1)$, in the cases of $u(0)u'(0) > 0$; $u(0) = 0$; $u'(0) = 0$ and $u(0)u''(0) > 0$, $\psi(u,t) \in (\pi/2,\pi)$, $t \in (0,1)$, in the cases $u(0)u'(0) < 0$; $u'(0) = 0$ and $u(0)u''(0) < 0$; $u'(0) = u''(0) = 0$, $\beta = \pi/2$ in the case $\psi(u,0) = 0$ and $\alpha = 0$ in the case $\psi(u,0) = \pi/2$.

It is obvious that $\rho,\ \theta,\ \varphi, w : S \times [0,1] \to \mathbb{R}$ are continuous.

Remark 2.1. By (2.2) for each $u \in S$ the function $w(u,t)$ can be determined from one of the relations given by [3, Remark 3.1].

For each $\nu \in \{+\,,\,-\}$ let S_1^ν denote the subset of such $u \in S$ that:

1) $\theta(u,1) = \pi/2 - \delta$, where $\delta = \pi/2$ in the case $\psi(u,1) = 0$;

2) $\varphi(u,1) = 2\pi - \gamma$ or $\varphi(u,1) = \pi - \gamma$ in the case $\psi(u,0) \in [0,\pi/2)$; $\varphi(u,1) = \pi - \gamma$ in the case $\psi(y,0) \in [\pi/2,\pi)$, where $\gamma = 0$ in the case $\psi(y,l) = \pi/2$;

3) for fixed u, as t increases from 0 to 1, the function $\theta(u,t)$ ($\varphi(u,t)$) strictly increasingly takes values of $m\pi/2$, $m\in\{-1,\,0,\,1\}$ ($s\pi$, $s\in\{0,\,1,\,2\}$); as t decreases from 1 to 0, the function $\theta(u,t)$ ($\varphi(u,t)$), strictly decreasing takes values of $m\pi/2$, $m\in\{-1,\,0,\,1\}$ ($s\pi$, $s\in\{0,\,1,\,2\}$);

4) the function $\nu u(t)$ is positive in a neighborhood of $t=0$.

From the results [1, 2] it follows that the sets S_1^+ and S_1^- are nonempty. It immediately follows from the definition of these sets that they are disjoint and open in E.

3. Global bifurcation from infinity for problem (1.1)-(1.2) with $f\equiv 0$

This section is devoted to an analysis of the structure of the continuous branches of solutions of problem (1.1)-(1.2) for $f\equiv 0$ which bifurcate from the points (λ_1^+,∞) and (λ_1^-,∞).

Since zero is not an eigenvalue of the linear problem (2.1), the nonlinear problem (1.1)-(1.2) is equivalent to the following integral equation

$$u(t)=\lambda\int_0^1 H(t,\,s)\,r(s)u(s)ds+\int_0^1 H(t,\,s)f(s,u(s),u'(s),u''(s),u'''(s),\lambda)ds+$$
$$+\int_0^1 H(t,\,s)g(s,u(s),u'(s),u''(s),u'''(s),\lambda)ds, \tag{3.1}$$

where $H(x,t)$ is the Green's function for the differential expression $\ell(y)$ with boundary conditions (1.2).

Let

$$Lu(t)=\int_0^1 H(t,\,s)\,r(s)u(s)ds, \tag{3.2}$$

$$F(\lambda,u(t))=\int_0^1 H(t,\,s)\,f(s,u(s),u'(s),u''(s),u'''(s),\lambda)ds, \tag{3.3}$$

$$G(\lambda,u(t))=\int_0^1 H(t,\,s)\,g(s,u(s),u'(s),u''(s),u'''(s),\lambda)ds. \tag{3.4}$$

By the continuity of the function $r(t)$ it follows from the properties of the Green function $H(x,t)$ that $L:E\to E$ is a compact and linear operator. It is clear that λ_1^+ and λ_1^- are simple characteristic values of L. The operator F can be represented as the composition of the operator L with $r(t)\equiv 1$ and the continuous and bounded operator $\mathbf{f}(\lambda,u(t))=f(t,u(t),u'(t),u''(t),u'''(t),\lambda)$, and consequently, $F:\mathbb{R}\times E\to E$ is completely continuous. The operator G can be represented as the composition of the operator L with $r(t)\equiv 1$ and the continuous and bounded operator $\mathbf{g}(\lambda,u(t))=g(t,u(t),u'(t),u''(t),u'''(t),\lambda)$, and consequently, $G:\mathbb{R}\times E\to E$ is continuous.

By (3.1)-(3.4) we can write (1.1)-(1.2) in the following equivalent form:

$$u=\lambda Lu+F(\lambda,u)+G(\lambda,u). \tag{3.5}$$

Hence it is sufficient to investigate the structure of the set of solutions of problem (1.1)-(1.2) in the space $\mathbb{R}\times E$.

Now we consider the nonlinear eigenvalue problem

$$\begin{cases} \ell u = \lambda r(t)u + g(t, u, u', u'', u''', \lambda), \ t \in (0,1), \\ u \in B.C. \end{cases} \tag{3.6}$$

Then because of (3.3) and (3.5) problem (3.6) is equivalent to the problem

$$u = \lambda L u + G(\lambda, u). \tag{3.7}$$

Lemma 3.1. *The relation*

$$G(\lambda, u) = o(||u||_3) \ \text{ as } \ ||u||_3 \to \infty \tag{3.8}$$

holds uniformly in $\lambda \in \Lambda$ for any bounded interval $\Lambda \subset \mathbb{R}$.
Proof. Let $C_1 > 0$ be the constant such that

$$\left| \frac{\partial^i H(t,s)}{\partial t^i} \right| \leq C_1, \ i = 0,\ 1,\ 2,\ 3, \ \text{ for } \ (t,s) \in [0,1;0,1].$$

Let $\varepsilon > 0$ be fixed and Λ be the bounded interval in $\mathbb{R}$. It follows from (1.5) that there exists sufficiently large $\Delta_\varepsilon > 0$ such that for any $(u, s, v, w) \in \mathbb{R}^4$ satisfying the condition $|u| + |s| + |v| + |w| > \Delta_\varepsilon$ and for any $(t, \lambda) \in [0,1] \times \Lambda$ the following relation holds

$$|g(t, u, s, v, w, \lambda)| < \frac{\varepsilon}{2C_1} \left(|u| + |s| + |v| + |w| \right). \tag{3.9}$$

Moreover, by the continuity of $g(t, u, s, v, w, \lambda)$ on $[0,1] \times \mathbb{R}^5$ there exists $M_\varepsilon > 0$ such that for any $(u, s, v, w) \in \mathbb{R}^4$ satisfying the condition $|u|+|s|+|v|+|w| \leq \Delta_\varepsilon$ and for any $(t, \lambda) \in [0,1] \times \Lambda$ we have

$$|g(t, u, s, v, w, \lambda)| \leq M_\varepsilon. \tag{3.10}$$

Choosing $\bar{\Delta}_\varepsilon > \Delta_\varepsilon$ so large that

$$\frac{M_\varepsilon}{\bar{\Delta}_\varepsilon} < \frac{\varepsilon}{2C_1}, \tag{3.11}$$

and let $S = \{u \in E : ||u||_3 > \bar{\Delta}_\varepsilon\}$. Then by virtue of (3.9)-(3.11) for any $(\lambda, u) \in \Lambda \times S$ we obtain the following relation

$$\begin{aligned} ||G(\lambda,u)||_3 &= \sum_{i=0}^{3} \max_{t\in[0,1]} \left| \int_0^1 \frac{\partial^i H(t,s)}{\partial t^i}\, g(s, u(s), u'(s), u''(s), u'''(s), \lambda)\, ds \right| \leq \\ &\leq C_1 \left\{ \int\limits_{\sum_{i=0}^{3} |u^{(i)}(s)| \leq \Delta_\varepsilon} |g(s, u(s), u'(s), u''(s), u'''(s), \lambda)|\, ds + \right. \\ &+ \left. \int\limits_{\sum_{i=0}^{3} |u^{(i)}(s)| \geq \Delta_\varepsilon} |g(s, u(s), u'(s), u''(s), u'''(s), \lambda)|\, ds \right\} \leq C_1 \left\{ M_\varepsilon + \tfrac{\varepsilon}{2} ||u||_3 \right\} < \\ &< C_1 \left\{ \tfrac{\varepsilon}{2C_1} \bar{\Delta}_\varepsilon + \tfrac{\varepsilon}{2C_1} ||u||_3 \right\} < C_1 \left\{ \tfrac{\varepsilon}{2C_1} ||u||_3 + \tfrac{\varepsilon}{2C_1} ||u||_3 \right\} = \varepsilon ||u||_3, \end{aligned} \tag{3.12}$$

which implies that (3.8) holds. The proof of this lemma is complete.

Define the operator $T : \mathbb{R} \times E \to E$ as follows:

$$T(\lambda, u) = ||u||_3^2 \, G\left(\lambda, \frac{u}{||u||_3^2||}\right).$$

Lemma 3.2. *The operator $T : \mathbb{R} \times E \to E$ is compact.*

Proof. It follows from the proof of Lemma 3.1 that for any $\varepsilon > 0$ there exists $\bar{\Delta}_\varepsilon > 0$ such that for any $(\lambda, y) \in \mathbb{R} \times E$ with $\lambda \in \Lambda$ and $||u||_3 > \bar{\Delta}_\varepsilon$ we have (see (3.12))

$$||G(\lambda, u)||_3 \le \varepsilon ||u||_3.$$

Let $\bar{\delta}_1 = \frac{1}{\bar{\Delta}_1}$ and $B_{\bar{\delta}_1} = \{u \in E : ||u||_3 \le \bar{\delta}_1\}$. Then it follows from the last inequality that

$$||T(\lambda, u)||_3 = ||y||_3^2 \left\| G\left(\lambda, \frac{u}{||u||_3^2||}\right) \right\|_3 \le ||u||_3 \le \bar{\delta}_1 \text{ for } (\lambda, u) \in \Lambda \times B_{\bar{\delta}_1},$$

which means that $T(\Lambda \times B_{\bar{\delta}_1})$ is bounded in E. Moreover, $v = T(\lambda, u)$ satisfies (3.6) which implies that

$$|v^4(t)| \le |\lambda||r(t)||u(t)| + |q(t)||v''(t)| + |q'(t)||v'(t)| +$$

$$+ |g(t, u(t), u'(t), u''(t), u'''(t), \lambda)| \le C_2,$$

where

$$C_2 = \max\{|\inf \Lambda| \max_{t\in[0,1]} |r(t)|, |\sup \Lambda| \max_{t\in[0,1]} |r(t)|, 2 \max_{t\in[0,1]} |q(t)|, 2 \max_{t\in[0,1]} |q'(t)|\} \, \delta_1$$

$$+ \frac{1}{2C_1 \delta_1^{-1}}.$$

Hence the set $T(\Lambda \times B_{\bar{\delta}_1})$ is precompact in E by the Arzelà-Ascoli theorem. Consequently, the operator T is compact. The proof of Lemma 3.2 is complete.

Let $\mathcal{F} \subset \mathbb{R} \times E$ denote the set of solutions of problem (1.1)-(1.2). We say (λ, ∞) is a bifurcation point (or asymptotic bifurcation point) for problem (1.1)-(1.2) if every neighborhood of (λ, ∞) contains solutions of this problem, i.e. there exists a sequence $\{(\lambda_n, u_n)\}_{n=1}^\infty \subset \mathcal{F}$ such that $\lambda_n \to \lambda$ and $||u_n||_3 \to +\infty$ as $n \to \infty$ (we add the points $\{(\lambda, \infty) : \lambda \in \mathbb{R}\}$ to space $\mathbb{R} \times E$). Next for any $\lambda \in \mathbb{R}$, we say that a subset $D \subset \mathcal{F}$ meets (λ, ∞) (respectively, $(\lambda, 0)$) if there exists a sequence $\{(\lambda_n, u_n)\}_{n=1}^\infty \subset D$ such that $\lambda_n \to \lambda$ and $||u_n||_3 \to +\infty$ (respectively, $||u_n||_3 \to 0$) as $n \to \infty$. Furthermore, we will say that $D \subset \mathcal{F}$ meets (λ, ∞) (respectively, $(\lambda, 0)$) through $\mathbb{R} \times S_1^\nu$, $\nu \in \{+, -\}$, if the sequence $\{(\lambda_n, u_n)\}_{n=1}^\infty \subset D$ can be chosen so that $u_n \in S_1^\nu$ for all $n \in \mathbb{N}$. If $I \in \mathbb{R}$ is a bounded interval we say that $D \subset \mathcal{F}$ meets $I \times \{\infty\}$ (respectively, $I \times \{0\}$) if D meets (λ, ∞) (respectively, $(\lambda, 0)$) for some $\lambda \in I$; we define $D \subset \mathcal{F}$ meets $I \times \{\infty\}$ (respectively, $I \times \{0\}$) through $\mathbb{R} \times S_1^\nu$, $\nu \in \{+, -\}$, similarly (see [13]).

Remark 3.1. It follows from the above consideration, Lemmas 3.1, 3.2 and [7, Ch. 4, § 3, Theorem 3.1] that (λ_1^+, ∞) and (λ_1^-, ∞) are asymptotic bifurcation points of problem (3.6) which correspond to continuous branches of solutions that meets these points.

By the direct application of Theorem 1.6 and Corollary 1.8 from [12] and using Lemmas 3.1, 3.2 and Remark 3.1, we obtain the following global results for problem (3.6) (or (1.1)-(1.2) with $f \equiv 0$).

Theorem 3.1. *For each $\sigma \in \{+, -\}$ the set $\mathcal{F}$ possesses an unbounded component C_1^σ which meets $(\lambda_1^\sigma, \infty)$. Moreover if $\Lambda \subset \mathbb{R}$ is an interval such that $\Lambda \cap X(L) = \{\lambda_1^\sigma\}$ (by X(L) denote the set of characteristic values of L) and $\mathcal{M}_1^\sigma$ is a neighborhood of $(\lambda_1^\sigma, \infty)$ whose projection on $\mathbb{R}$ lies in Λ and whose projection on E is bounded away from 0, then either*

1^o. $C_1^\sigma \backslash \mathcal{M}_1^\sigma$ is bounded in $\mathbb{R} \times E$ in which case $D_1^\sigma \backslash \mathcal{M}_\infty^\sigma$ meets $\mathcal{R} = \{(\lambda, 0) : \lambda \in \mathbb{R}\}$ or

2^o. $C_1^\sigma \backslash \mathcal{M}_1^\sigma$ is unbounded in $\mathbb{R} \times E$.

If 2^o occurs and $C_1^\sigma \backslash \mathcal{M}_1^\sigma$ has a bounded projection on $\mathbb{R}$, then $C_1^\sigma \backslash \mathcal{M}_1^\sigma$ meets $(\lambda_{k'}^{\sigma'}, \infty)$ where $\lambda_{k'}^{\sigma'} \in X(L)$, $(k', \sigma') \neq (1, \sigma)$.

Theorem 3.2. *For each $\sigma \in \{+, -\}$ the component D_1^σ can be decomposed into two subcontinua $C_1^{\sigma,+}$, $D_1^{\sigma,-}$ and there exists a neighborhood $\mathcal{Q}_1^\sigma \subset \mathcal{M}_1^\sigma$ of $(\lambda_1^\sigma, \infty)$ such that $(C_1^{\sigma,+} \cap \mathcal{Q}_1^\sigma) \subset ((\mathbb{R} \times S_1^+) \cup \{(\lambda_1^\sigma, \infty)\})$, $(C_1^{\sigma,-} \cap Q_1^\sigma) \subset ((\mathbb{R} \times S_1^-) \cup \{(\lambda_1^\sigma, \infty)\})$.*

4. Global bifurcation from intervals at infinity of solutions of problem (1.1)-(1.2)

Together with problems (2.1) we consider the following spectral problem

$$\begin{array}{l} \ell u(t) + \varphi(t) u(t) = \lambda r(t) u(t), \ t \in (0,1), \\ u \in B.C., \end{array} \tag{4.1}$$

where $\varphi(t) \in C[0,1]$ and $\varphi(t) \geq 0$, $t \in [0,1]$.

We need the following result which is basic in the sequel.

Lemma 4.1 [3, Lemma 3.5]. *For each $\sigma \in \{+, -\}$ the following relation is true:*

$$|\tilde{\lambda}_1^\sigma - \lambda_1^\sigma| \ \leq \ \frac{\sigma \tilde{K} \int\limits_0^1 (u_1^\sigma(t))^2 \, dt}{\int\limits_0^1 r(t) \, (u_1^\sigma(t))^2 \, dt}, \tag{4.2}$$

where $\tilde{\lambda}_1^\sigma$, $\sigma \in \{+, -\}$, is a principal eigenvalue of problem (3.1).

Remark 4.1. Since the class of continuous functions $C[0,1]$ is dense in $L_1[0,1]$ Lemma 3.2 also holds for $\varphi(t) \in L_1[0,1]$.

Let

$$J_1^+ = [\lambda_1^+, \lambda_1^+ + d_1^+], \ \ J_1^- = [\lambda_1^- - d_1^-, \lambda_1^-], \ \ I_1 = [\lambda_1^- - d_1^-, \lambda_1^+ + d_1^+],$$

where

$$d_1^\sigma = \frac{\sigma K \int\limits_0^1 (u_1^\sigma(t))^2 \, dt}{\int\limits_0^1 r(t) \, (u_1^\sigma(t))^2 \, dt}, \, \sigma \in \{+, -\}.$$

Along with (1.1)-(1.2), we look at the following approximation problem:

$$\begin{cases} \ell u = \lambda r(t)u + \frac{f(t,||u||_3^\varepsilon u,||u||_3^\varepsilon u',||u||_3^\varepsilon u'',||u||_3^\varepsilon u''',\lambda)}{||u||_3^{2\varepsilon}} + g(t,u,u',u'',u''',\lambda),\ t\in(0,1), \\ u \in B.C., \end{cases} \tag{4.3}$$

where $\varepsilon \in (0,1]$.

Lemma 4.2 *Let $\delta_1 > 0$ be the sufficiently small fixed number. There exists sufficiently large $R_1 > 0$ such that for given any $\varepsilon \in (0,1)$ problem (4.3) has no nontrivial solution (λ, y) such that $\delta_1 < \operatorname{dist}\{\lambda, J_1^\sigma\} < 2\delta_1,\ y \in S_1^\nu,\ \nu \in \{+,-\}$ and $||y||_3 > R_1$.*

Proof. Assume the contrary: there exists $\varepsilon_1 \in (0,1)$ such that (4.3) with $\varepsilon = \varepsilon_1$ for any $n \geq n_0$ (where n_0 is a sufficiently large natural number) has a nontrivial solution (λ_n, u_n) satisfying $\delta_1 < \operatorname{dist}\{\lambda_n, J_1^\sigma\} < 2\delta_1,\ y_n \in S_1^\nu,\ \nu \in \{+,-\}$ and $||y||_3 > n$.

For each $n \geq n_0$ and $t \in (0,1)$ we have

$$\begin{cases} \ell u_n = \lambda r(t)u_n + \frac{f(t,||u_n||_3^{\varepsilon_1} u_n,||u_n||_3^{\varepsilon_1} u_n',||u_n||_3^{\varepsilon_1} u_n'',||u_n||_3^{\varepsilon_1} u_n''',\lambda_n)}{||u_n||_3^{2\varepsilon_1}} + g(t,u_n,u_n',u_n'',u_n''',\lambda_n), \\ u_n \in B.C. \end{cases} \tag{4.4}$$

We define a function $\varphi_n(t)$, $n \geq n_0$, $t \in [0,l]$, as follows:

$$\varphi_n(t) = \begin{cases} -\frac{f\left(t,||u_n||_3^{\varepsilon_1} u_n(t),||u_n||_3^{\varepsilon_1} u_n'(t),||u_n||_3^{\varepsilon_1} u_n''(t),||u_n||_3^{\varepsilon_1} u_n'''(t),\lambda_n\right)}{||u_n||_3^{2\varepsilon_1} u_n(t)} & \text{if } u_n(t) \neq 0, \\ 0 & \text{if } u_n(t) = 0. \end{cases}$$

Then it follows from (4.4) that (λ_n, u_n), $n \geq n_0$, solves the nonlinear problem

$$\begin{cases} \ell u_n + \varphi_n(t)u_n = \lambda r(t)u_n + g(t,u_n,u_n',u_n'',u_n''',\lambda_n),\ t\in(0,1), \\ u_n \in B.C. \end{cases} \tag{4.5}$$

In view of (1.3) and (1.4) we have

$$\varphi_n(t) \geq 0 \text{ and } |\varphi_n(t)| \leq \frac{K}{||y_n(t)||_3^{\varepsilon_1}} \leq K \text{ for all } n \geq n_0,\ t\in[0,1].$$

Since $y_n(t)$, $n \geq n_0$, does not vanish in $(0,1)$ and is continuous on $[0,1]$, Remark 3.1 shows that the result of Lemma 3.2 also holds for the following linear problem

$$\begin{cases} \ell u + \varphi_n(t)u = \lambda r(t)u,\ t\in(0,1), \\ u \in B.C. \end{cases} \tag{4.6}$$

Then it follows from (4.2) that the principal eigenvalue $\lambda_{1,n}^\sigma$, $\sigma \in \{+,-\}$, of the linear problem (4.6) lies in J_1^σ. By Remark 3.1, Theorems 3.1 and 3.2 for each $\sigma \in \{+,-\}$ and $n \geq n_0$ the point $(\lambda_{1,n}^\sigma, \infty)$ is the unique asymptotic bifurcation point of (4.2) which correspond to continuous branches of solutions that meets this point through $\mathbb{R} \times S_1^\nu$. Hence for each sufficiently large $n > n_0$ we can assign a small $\delta_n > 0$ such that $\delta_n < \frac{\delta_1}{2}$ and $|\lambda_n - \lambda_{1,n}^\sigma| < \delta_n$. Then it follows that $\operatorname{dist}\{\lambda_n, J_1^\sigma\} < \frac{\delta_1}{2}$, contradicting $\operatorname{dist}\{\lambda, J_1^\sigma\} > \delta_1$. The proof of this lemma is complete.

We say that the (λ, ∞) is the asymptotic bifurcation point for problem (1.1)-(1.2) with respect to set $\mathbb{R} \times S_1^\nu$ if there exists a continuous branch of solutions of problem (1.1)-(1.2) that meets this point through $\mathbb{R} \times S_1^\nu$.

Lemma 4.3 *For each* $\sigma \in \{+, -\}$, $\nu \in \{+, -\}$ *and for each sufficiently large* $R > R_1$ *problem* (1.1)-(1.2) *has a solution* $(\lambda_R^{\sigma,\nu}, v_R^{\sigma,\nu})$ *such that* $v_R^{\sigma,\nu} \in S_1^\nu$ *and* $||v_R^{\sigma,\nu}||_3 = R$.

Proof. By (3.5) we can write (4.3) in an equivalent form as follows:

$$u = \lambda Lu + ||u||^{-2\varepsilon} F(\lambda, ||u||_3^\varepsilon y) + G(\lambda, u). \tag{4.7}$$

By (1.4) it follows from (3.3) that

$$||F(\lambda, u)||_3 \leq C_1 ||u||_3^{1+\varepsilon}. \tag{4.8}$$

In view of (4.8) we have

$$||u||^{-2\varepsilon} F(\lambda, ||u||_3^\varepsilon u) = o(||u||_3) \text{ as } ||u||_3 \to \infty, \tag{4.9}$$

uniformly in $\lambda \in \Lambda$ for any bounded interval $\Lambda \subset \mathbb{R}$. Then by virtue of Theorems 3.1 and 3.2 for each $\sigma \in \{+, -\}$ and each $\nu \in \{+, -\}$ there exists an unbounded component $C_{1,\varepsilon}^{\sigma,\nu}$ of solutions of (4.7) (or (4.3)) which meets $(\lambda_1^\sigma, \infty)$ and there exists a neighborhood $\mathcal{Q}_\varepsilon^\sigma$ of $(\lambda_1^\sigma, \infty)$ such that $\mathcal{Q}_\varepsilon^\sigma \cap (C_{1,\varepsilon}^{\sigma,\nu} \backslash (\lambda_1^\sigma, \infty)) \subset \mathbb{R} \times S_1^\nu$ and either $C_{1,\varepsilon}^{\sigma,\nu} \backslash \mathcal{Q}_\varepsilon^\sigma$ is bounded in $\mathbb{R} \times E$ in which case $C_{1,\varepsilon}^{\sigma,\nu} \backslash \mathcal{Q}_\varepsilon^\sigma$ meets $\mathcal{R}$ or $C_{1,\varepsilon}^{\sigma,\nu} \backslash \mathcal{Q}_\varepsilon^\sigma$ is unbounded in $\mathbb{R} \times E$. Moreover, if $C_{1,\varepsilon}^{\sigma,\nu} \backslash \mathcal{Q}_\varepsilon^\sigma$ is unbounded and has a bounded projection on $\mathbb{R}$, then this set meets $(\lambda_{k'}^{\sigma'}, \infty)$ where $\lambda_{k'}^{\sigma'} \in X(L)$, $(k', \sigma') \neq (1, \sigma)$. Then by Lemma 4.1 it follows that for any $\varepsilon \in (0, 1]$ and each sufficiently large $R > R_1$ there exists a solution $(\lambda_{R,\varepsilon}^{\sigma,\nu}, v_{R,\varepsilon}^{\sigma,\nu}) \in \mathbb{R} \times E$ of (4.3) such that $\operatorname{dist}\{\lambda_{R,\varepsilon}^{\sigma,\nu}, J_1^\sigma\} \leq \delta_1$, $v_{R,\varepsilon}^{\sigma,\nu} \in S_1^\nu$ and $||v_{R,\varepsilon}^{\sigma,\nu}||_3 = R$.

Since $\{v_{R,\varepsilon}^{\sigma,\nu} \in E \ : \ 0 < \varepsilon \leq 1\}$ is a bounded subset of $C^3[0,1]$, the functions f and g are continuous in $[0,1] \times \mathbb{R}^5$, and the set $\{\lambda_{R,\varepsilon}^\sigma \in \mathbb{R} \ : \ 0 < \varepsilon \leq 1\}$ is bounded in $\mathbb{R}$, it follows from (4.3) that $\{v_{R,\varepsilon}^\sigma \in E \ : \ 0 < \varepsilon \leq 1\}$ is also bounded in $C^4[0,1]$. Hence it is precompact in E by the Arzelà-Ascoli theorem.

Let $\{\varepsilon_n\}_{n=2}^\infty \subset (0, 1)$ be a sequence such that $\varepsilon_n \to 0$ and $(\lambda_{R,\varepsilon_n}^{\sigma,\nu}, v_{R,\varepsilon_n}^{\sigma,\nu}) \to (\lambda_R^{\sigma,\nu}, v_R^{\sigma,\nu})$ as $n \to \infty$. Taking the limit (as $n \to \infty$) in (4.3) we see that $(\lambda_R^{\sigma,\nu}, v_R^{\sigma,\nu})$ is a solutions of (1.1)-(1.2).

We define a function $\varphi_R^{\sigma,\nu}(t)$, $t \in [0, l]$, as follows:

$$\varphi_R^{\sigma,\nu}(t) = \begin{cases} -\frac{f(t, v_R^{\sigma,\nu}(t), (v_R^{\sigma,\nu})'(t), (v_R^{\sigma,\nu})''(t), (v_R^{\sigma,\nu})'''(t), \lambda_R^{\sigma,\nu})}{v_R^{\sigma,\nu}(t)} & \text{if } v_R^{\sigma,\nu}(t) \neq 0, \\ 0, & \text{if } v_{R,\sigma,\nu}(t) = 0. \end{cases}$$

It is clear that $(\lambda_R^{\sigma,\nu}, y_R^{\sigma,\nu})$ solves the nonlinear problem

$$\begin{cases} \ell u + \varphi_R^{\sigma,\nu}(t) u = \lambda r(t) u + g(t, u, u', u'', u''', \lambda), \ t \in (0,1), \\ u \in B.C. \end{cases}$$

Since $R > R_1$ is sufficiently large it follows from [12, Corollary 1.8] that $y_R^{\sigma,\nu} \in S_1^\nu$. The proof of this lemma is complete.

Corollary 4.1. *The set of asymptotic bifurcation points for problem* (1.1)-(1.2) *with respect to set* $\mathbb{R} \times S_1^\nu$ *is nonempty. Moreover, if* (λ, ∞) *is a bifurcation point for* (1.1)-(1.2) *with respect to set* $\mathbb{R} \times S_1^\nu$ *then* $\lambda \in J_1^+ \cup J_1^-$.

For each $\sigma \in \{+,-\}$ and $\nu \in \{+,-\}$ we define the set $\bar{D}_1^{\sigma,\nu} \subset \mathcal{F}$ to be the union of all the components of $\mathcal{F}$ which meet $J_1^\sigma \times \{\infty\}$ through $\mathbb{R} \times S_1^\nu$. It follows from Corollary 4.1 that this set is nonempty. The set $\bar{D}_1^{\sigma,\nu}$ may not be connected in $\mathbb{R}\times E$, but the set $D_1^{\sigma,\nu} = \bar{D}_1^{\sigma,\nu} \cup (J_1^\sigma \times \{\infty\})$ is connected in $\mathbb{R}\times E$.

Theorem 4.1. *Let* $\mathcal{P}_1^\sigma = \{(\lambda,u) \in \mathbb{R}\times E : \operatorname{dist}\{\lambda, J_1^\sigma\} < \delta_1, ||u||_3 > R_1\}$ *for* $\sigma \in \{+,-\}$. *Then either*

1^o. $D_1^{\sigma,\nu} \backslash \mathcal{P}_1^\sigma$ *is bounded in* $\mathbb{R}\times E$ *in which case* $D_1^{\sigma,\nu} \backslash \mathcal{P}_1^\sigma$ *meets* $\mathcal{R}$ *or*

2^o. $D_1^{\sigma,\nu} \backslash \mathcal{P}_1^\sigma$ *is unbounded in* $\mathbb{R}\times E$. *If additionally* $D_1^{\sigma,\nu} \backslash \mathcal{P}_1^\sigma$ *has a bounded projection on* $\mathbb{R}$, *then* $D_1^{\sigma,\nu} \backslash \mathcal{P}_1^\sigma$ *meets* $J_{k'}^{\sigma'} \times \{\infty\}$ *for some* $(k',\sigma') \neq (1,\sigma)$ *where* $J_{k'}^{\sigma'} \times \{\infty\}$ *is a bifurcation interval for* (1.1)-(1.2) *which surrounds an eigenvalue of the linear problem* (2.1)

Proof. We choose some fixed σ_0 and ν_0, we will prove the theorem for σ_0 and ν_0.

Let (λ,u) be the nontrivial solution of problem (1.1)-(1.2). Dividing (1.1)-(1.2) by $||u||_3^2$ and setting $v = \frac{u}{||u||_3^2}$ we get

$$\begin{cases} \ell v = \lambda r(t) v + \frac{h(t,u,u',u'',u''',\lambda)}{||y||_3^2}, \ t \in (0,1), \\ v \in B.C. \end{cases} \tag{4.10}$$

It is obvious that $||v||_3 = \frac{1}{||u||_3}$ and $u = \frac{v}{||v||_3^2}$. We define the function $\tilde{h}(t,v,v',v'',v''',\lambda)$ as follows:

$$\tilde{h}(t,v,v',v'',v''',\lambda) = \begin{cases} ||v||_3^2\, h\left(t, \frac{v}{||v||_3^2}, \frac{v'}{||v||_3^2}, \frac{v''}{||v||_3^2}, \frac{v'''}{||v||_3^2}, \lambda\right) & \text{if } v \neq 0, \\ 0, & \text{if } v = 0. \end{cases}$$

Then (4.10) is rewritten in the following equivalent form

$$\begin{cases} \ell v = \lambda r(t) v + \tilde{h}(t,v,v',v'',v''',\lambda), \ t \in (0,1), \\ v \in B.C. \end{cases} \tag{4.11}$$

Note that the function $\tilde{h} = \tilde{f} + \tilde{g}$, where the terms $\tilde{f}$ and $\tilde{g}$ are defined analogously to $\tilde{h}$, satisfies the conditions (1.3)-(1.5):

$$v(t)\tilde{f}(t,v(t),v'(t),v''(t),v'''(t),\lambda) = \frac{1}{||u||_3^4}\, u(t)\, f(t,u(t),u'(t),u''(t),u'''(t),\lambda) \leq 0,$$

and

$$\left|\frac{\tilde{f}(t,v(t),v'(t),v''(t),v'''(t),\lambda)}{v(t)}\right| = ||v||_3^2\,||u||_3^2 \left|\frac{f\ (t,u(t),u'(t),u''(t),u'''(t),\lambda)}{u(t)}\right| \leq M;$$

$$\frac{|\tilde{g}(t,v(t),v'(t),v''(t),v'''(t),\lambda)|}{|v(t)|+|v'(t)|+|v''(t)|+|v'''(t)|} = ||v||_3^2\,||u||_3^2 \frac{|g(t,u(t),u'(t),u''(t),u'''(t),\lambda)|}{|u(t)|+|u'(t)|+|u''(t)|+|u'''(t)|}$$

and it follows from condition (1.5) that

$$\tilde{g}(t,u,s,v,w,\lambda) = o(|y|+|s|+|v|+|w|) \text{ near } (u,s,v,w) = (0,0,0,0),$$

uniformly in $t \in [0,1]$ and in $\lambda \in \Lambda$. In this way the transformation $(\lambda,u) \to A(\lambda,u) = (\lambda,v)$ allows us to convert problem (1.1)-(1.2) into an equivalent problem (4.11) (with a nonlinear term $\tilde{g}$ satisfying the condition (1.5) in a neighborhood of $(u,s,v,w) = (0,0,0,0)$ uniformly in $t \in [0,1]$ and in $\lambda \in \Lambda$) which was studied in [3]. In other words the transformation $(\lambda,u) \to A(\lambda,u) = (\lambda,v)$ turns a "bifurcation at infinity" problem into a "bifurcation at zero" problem (heuristically, interchanges points at $u = \infty$ (respectively, $u = 0$) with points at $v = 0$ (respectively, $v = \infty$)).

Let $\tilde{F} \subset \mathbb{R} \times E$ be the set of nontrivial solutions of problem (4.11). It is obvious that the transformation $(\lambda, u) \to A(\lambda, u)$ maps F into $\tilde{F}$ and $J_1^\sigma \times \{\infty\}$ into $J_1^\sigma \times \{0\}$. Let $\tilde{D}_1^{\sigma_0, \nu_0}$ be the union of all the components of $\tilde{F}$ which contains $J_1^\sigma \times \{0\}$ and emanating from bifurcation points $(\lambda, 0) \in J_1^{\sigma_0} \times \{0\}$ with respect to $S_1^{\nu_0}$. Then $D_1^{\sigma_0, \nu_0}$ is the inverse image $A^{-1}(\tilde{D}_1^{\sigma_0, \nu_0})$ of $\tilde{D}_1^{\sigma_0, \nu_0}$ and $\tilde{P}_1^{\sigma_0}$ is the inverse image $A^{-1}(\tilde{P}_1^{\sigma_0})$ of $P_1^{\sigma_0}$ under the transformation A, where $\tilde{\mathcal{P}}_1^{\sigma_0} = \{(\lambda, v) \in \mathbb{R} \times E : \text{dist}\,\{\lambda, J_1^\sigma\} < \delta_1,\ ||v||_3 > R_1^{-1}\}$. Now following the proofs of [2, Theorem 1.3], [3, Theorem 4.8] and [13, Theorem 3.1] we can prove that either $\tilde{D}_1^{\sigma_0, \nu_0} \backslash \tilde{\mathcal{P}}_1^{\sigma_0}$ meets some bifurcation interval $J_k^\sigma \times \{0\}$ with $(k, \sigma) \neq (k, \sigma_0)$ or is unbounded in $\mathbb{R} \times E$. Clearly, $\tilde{D}_1^{\sigma_0, \nu_0}$ satisfies the conclusions of this theorem. The proof of Theorem 4.1 is complete.

References

[1] Z.S. Aliyev, Some global results for nonlinear fourth order eigenvalue problems, *Cent. Eur. J. Math.* **12**(12) (2014), 1811-1828.

[2] Z.S. Aliyev, Global bifurcation of solutions of certain nonlinear eigenvalue problems for ordinary differential equations of fourth order, *Sb. Math.* **207**(12) (2016), 1625-1649.

[3] Z.S. Aliyev, R.A. Huseynova, Bifurcation in nonlinearizable eigenvalue problems for ordinary differential equations of fourth order with indefinite weight, *Electron. J. Qual. Theory Differ. Equ.* **2016**(46) (2016), 1-14.

[4] H. Berestycki, On some nonlinear SturmLiouville problems, *J. Differential Equations* **26**(3) (1977), 375390.

[5] G. Dai, Global bifurcation from intervals for Sturm-Liouville problems which are not linearizable, *Electron. J. Qual. Theory Differ. Equ.* **65** (2013), 17.

[6] R.A. Huseynova, Global bifurcation from principal eigenvalues for nonlinear fourth order eigenvalue problem with indefinite weight, *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb* **42**(2) (2016), 202-211.

[7] M. A. Krasnoselskii, Topological methods in the theory of nonlinear integral equations, Gostekhizdat, Pergamon Press, Oxford 1964.

[8] R. Ma, G. Dai, Global bifurcation and nodal solutions for a Sturm-Liouville problem with a nonsmooth nonlinearity, *J. Functional Analysis* **265** (2013), 1443-1459.

[9] N.A. Mustafayeva, On the structure of global continua of solutions bifurcating from infinity of some nonlinear fourth order eigenvalue problems, *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.* **43**(2) (2017), 270-277.

[10] J. Przybycin, Bifurcation from infinity for the special class of nonlinear differential equations, *J. Differential Equations* **65** (1986), 235-239.

[11] J. Przybycin, Some applications of bifurcation theory to ordinary differential equations of the fourth order, *Ann. Polon. Math.* **53**(2) (1991), 153-160.

[12] P.H. Rabinowitz, On bifurcation from infinity, *J. Differential Equations* **14** (1973), 462-475.

[13] B.P. Rynne, Bifurcation from zero or infinity in Sturm-Liouville problems which are not linearizable, *J. Math.Anal. Appl.* **228** (1998), 141-156.

[14] C.A. Stuart, Solutions of large norm for non-linear Sturm-Liouville problems, *Quart. J. of Math. (Oxford)* **24**(2) (1973), 129-139.

[15] J.F. Toland, Asymptotic linearity and non-linear eigenvalue problems, *Quart. J. Math. (Oxford)* **24**(2) (1973), 241-250.

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