Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 44, Number 2, 2018, Pages 171–186

NUMERICAL SOLUTION TO OPTIMAL CONTROL PROBLEMS WITH MULTIPOINT AND INTEGRAL CONDITIONS

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Abstract. Optimal control problems involving non-separated multipoint and integral conditions are investigated. For numerical solution to the problem, we propose to use first order optimization methods with application of the formulas for the gradient of the functional obtained in the work. To solve the adjoint boundary problems, we propose an approach. This approach makes it possible to reduce solving initial boundary problems to solving supplementary Cauchy problems and a linear algebraic system of equations. Results of numerical experiments are given.

1. Introduction

Much research activity in the past years has been directed at solving boundary problems involving non-local multipoint and integral conditions, and the corresponding optimal control problems. This is connected with non-local character of information provided by measurement equipment. Namely, the measurements are not taken instantly, but during some time interval and the measurements at a separate point actually characterize the state of the object in some domain which contains the measurement point. Problems of this kind arise when controlling an object if it is otherwise impossible to affect the object instantly at time and locally at its separate points.

The investigations of boundary problems involving non-local conditions commenced at the beginning of the 20^{th} century [21, 25, 26], and they became more active by the efforts of many authors whose works were dedicated to both ordinary and partial differential equations [8, 10, 11, 15, 18, 19, 22, 24]. In [4, 1, 5, 13, 27, 28], optimal control problems for boundary problems involving non-local multipoint and integral conditions are investigated and necessary optimality conditions are obtained.

For linear boundary problems involving multipoint conditions, there exist efficient numerical methods of sweep and shift of conditions [3, 2, 6, 9, 20]. For boundary problems involving integral conditions, the possibility of their reduction to problems involving multipoint conditions at the expense of introducing new variables and of increasing the number of differential equations has been implied.

2010 *Mathematics Subject Classification.* 49J15, 65L10.

Key words and phrases. Optimal control, integral conditions, multipoint conditions, transfer method for intermediate conditions, integral conditions convolution operation.

That is why any special numerical methods for problems of this kind have not been practically developed.

In the present work for optimal control problems, we obtain formulas for the gradient of the functional, as well as describe the numerical computation scheme based on first order optimization methods. Also, we propose an approach for numerical solution to the problem adjoint boundary problems.

2. Problem Statement

Consider the following optimal control problem for the process described by an ordinary differential equations system, linear in the phase variable:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + C(t), \quad t \in [t_0, T], \tag{2.1}$$

where $x(t) \in E^n$ is the phase variable; $u(t) \in U \subset E^r$ is the control function from the class of piecewise continuous functions, admissible values of $u(t)$ belong to a given compact set U; $A(t) \neq const$ is $(n \times n)$ matrix function, $B(t)$ is $(n \times r)$ matrix function, $C(t)$ is $n-$dimensional vector function, $A(t)$, $B(t)$, $C(t)$ are continuous with respect to t.

Non-separated multipoint and integral conditions are given in the following form:

$$\sum_{i=1}^{l_1} \int_{\bar{t}_{2i-1}}^{\bar{t}_{2i}} \bar{D}_i(\tau)x(\tau)d\tau + \sum_{j=1}^{l_2} \tilde{D}_j x(\tilde{t}_j) = C_0, \tag{2.2}$$

where $\bar{D}_i(\tau)$ is the continuously differentiable $(n \times n)$ matrix function; $\tilde{D}_j$ is the $(n \times n)$ scalar matrix; C_0 is the n-dimensional vector; $\bar{t}_i$,$\tilde{t}_j$ time instances from $[t_0, T]$; $\bar{t}_{i+1} > \bar{t}_i$, $\tilde{t}_{j+1} > \tilde{t}_j$, $\quad i = 1, 2, ..., 2l_1 - 1$, $j = 1, 2, ..., l_2 - 1$, l_1, l_2 are given.

To be specific, without loss of generality, let us assume that

$$\min\left(\bar{t}_1,\ \tilde{t}_1\right) = t_0, \quad \max\left(\bar{t}_{2l_1},\ \tilde{t}_{l_2}\right) = T, \tag{2.3}$$

and for all $i = 1, 2, ..., 2l_1$, $j = 1, 2, ..., l_2$ and that the following natural condition holds

$$\tilde{t}_j \bar{\in} \left[\bar{t}_{2i-1}, \bar{t}_{2i}\right]. \tag{2.4}$$

The target functional is as follows:

$$J(u) = \Phi(\hat{x}(\hat{t})) + \int_{t_0}^{T} f^0(x, u, t)dt \to \min_{u(t)\in U}, \tag{2.5}$$

where the function Φ and its partial derivatives are continuous with respect to its arguments, and $f^0(x, u, t)$ is continuously differentiable with respect to (x, u) and continuous with respect to t; $\hat{t} = (\hat{t}_1, \hat{t}_2, ..., \hat{t}_{2l_1+l_2})$ is the ordered union of the sets $\tilde{t} = (\tilde{t}_1, \tilde{t}_2, ..., \tilde{t}_{l_2})$ and $\bar{t} = (\bar{t}_1, \bar{t}_2, ..., \bar{t}_{2l_1})$, i.e. $\hat{t}_j < \hat{t}_{j+1}$, $j = 1, 2, ..., 2l_1 + l_2 - 1$, $\hat{x}(\hat{t}) = (x(\hat{t}_1), x(\hat{t}_2), ..., x(\hat{t}_{2l_1+l_2}))$.

$x(t,u)$ is absolutely continuous and satisfies equation (2.1) almost everywhere on $[t_0,T]$, and its derivative $\dot{x}(t,u)$ belongs to $L_2^n[t_0,T]$; equality (2.2) holds true, as well. Thus, (2.5) is defined for all $u=u(t)\in L_2^r[t_0,T]$.

The fundamental difference of problem statement (2.1)-(2.5) from the optimal control problems considered, for example, in [1], lies in non-separated non-local integral and multipoint conditions (2.2). By introducing some new phase variables, problem (2.1)-(2.5) can be reduced to a problem involving multipoint conditions. To demonstrate this, introduce new phase vector $X(t) = (x^1(t),...,x^{l_1+1}(t))$, $x^1(t)=x(t)$, which is the solution to the following differential equations system:

$$\begin{array}{l} \dot{x}^1(t) = A(t)x^1(t) + B(t)\,u(t) + C(t), \\ \dot{x}^{i+1}(t) = \bar{D}_i(t)x^1(t), \quad t\in(\bar{t}_{2i-1},\,\bar{t}_{2i}]\,, \quad i=1,2...,2l_1\,, \end{array} \tag{2.6}$$

involving the following initial conditions:

$$x^{i+1}(\bar{t}_{2i-1}) = 0, \;\; i=1,2,...,2l_1. \tag{2.7}$$

Then conditions (2.2) take the following form:

$$\sum_{i=1}^{l_1} x^{i+1}\left(\bar{t}_{2i}\right) + \sum_{j=1}^{l_2} \tilde{D}_j x^1(\tilde{t}_j) = C_0. \tag{2.8}$$

System (2.6)-(2.8) is obviously equivalent to (2.1) and (2.2). In system (2.6) and (2.7), there are $(l_1{+}1)n$ differential equations with respect to the phase vector $X(t)$ and there is the same number of conditions in (2.7) and (2.8). Obviously, the drawback of boundary problem (2.6) and (2.7) is its high dimension. This is an essential point for numerical methods of solution to boundary problems based, as a rule, on the methods of sweep or shift of boundary conditions [3, 9]. Also, the increase of the dimension of the phase variable complicates the solution to the optimal control problem itself due to the increase of the dimension of the adjoint problem.

Note that if we use the approach proposed in [20] then at the expense of the additional increase of the dimension of the phase variables vector up to $2(l_1+l_2+1)(l_1+1)n$ the problem (2.6)–(2.8) can be reduced to a two-point problem involving non-separated boundary conditions.

Using the technique of the works [14, 16, 17, 23], we can obtain existence and uniqueness conditions for the solution to problem (2.1), (2.2) under every admissible control $u\in U$, without reducing it to a problem involving multipoint conditions (2.8). But this kind of investigation is not the objective of the present work.

Assume that under every admissible control $u(t)\in U$, there is a unique solution to problem (2.1) and (2.2). For this purpose, we assume that the parameters of problem (2.1) and (2.2), after reducing it to (2.6)-(2.8), satisfy the conditions proposed in [14, 16, 17, 23, 25] dedicated to differential equations systems involving multipoint and two-point conditions.

3. Formula for the gradient of the functional of problem (2.1)-(2.5)

To solve optimal control problem (2.1)-(2.5) numerically with the application of first order optimization methods (see [29]), we obtain formulas for the gradient of the functional.

With respect to an arbitrary admissible process $(u(t), x(t;u))$, we define problem (2.1), (2.2) in increments, corresponding to an admissible control $\tilde{u} = u + \Delta u$:

$$\Delta\dot{x}(t) = A(t)\Delta x(t) + B(t)\Delta u(t), \quad t \in [t_0, T] \ , \tag{3.1}$$

$$\sum_{i=1}^{l_1} \int_{\bar{t}_{2i-1}}^{\bar{t}_{2i}} \bar{D}_i(\tau)\Delta x(\tau)d\tau + \sum_{j=1}^{l_2} \tilde{D}_j \Delta x(\tilde{t}_j) = 0. \tag{3.2}$$

Here the following notations are used:

$$\Delta x(t) = x(t, \tilde{u}) - x(t, u), \quad \Delta u(t) = \tilde{u}(t) - u(t).$$

Let $\psi(t)$ be an almost everywhere continuously differentiable vector function and let $\lambda \in E^n$ be as yet arbitrary numerical vector. Taking into account that $x(t)$ and $x(t)+\Delta x(t)$ are the solutions to problem (2.1)-(2.2) under corresponding values of the controls, we can write:

$$J(u) = \Phi(\hat{x}(\hat{t})) + \int_{t_0}^{T} f^0(x,u,t)dt + \int_{t_0}^{T} \psi^*(t)\left[\dot{x}(t) - A(t)x(t) - B(t)u(t) - C(t)\right]dt +$$

$$+\lambda^* \left[\sum_{i=1}^{l_1} \int_{\bar{t}_{2i-1}}^{\bar{t}_{2i}} \bar{D}_i(\tau)x(\tau)d\tau + \sum_{j=1}^{l_2} \tilde{D}_j x(\tilde{t}_j) - C_0 \right],$$

$$J(u+\Delta u) = \Phi(\hat{x}(\hat{t}) + \Delta\hat{x}(\hat{t})) + \int_{t_0}^{T} f^0(x(t)+\Delta x(t), u(t)+\Delta u(t), t)dt +$$

$$+\int_{t_0}^{T} \psi^*(t)\left[(\dot{x}(t)+\Delta\dot{x}(t)) - A(t)(x(t)+\Delta x(t)) - B(t)(u(t)+\Delta u(t)) - C(t)\right]dt +$$

$$+\lambda^* \left[\sum_{i=1}^{l_1} \int_{\bar{t}_{2i-1}}^{\bar{t}_{2i}} \bar{D}_i(\tau)(x(\tau)+\Delta x(\tau))d\tau + \sum_{j=1}^{l_2} \tilde{D}_j (x(\tilde{t}_j)+\Delta x(\tilde{t}_j)) - C_0 \right],$$

where "*" is the transposition sign. Then for the increment of the functional, using the formula of partial integration, after grouping the corresponding terms, accurate within the terms of the first infinitesimal order, we obtain:

$$\Delta J(u) = \int_{t_0}^{T} \Big[-\dot{\psi}^*(t) - \psi^*(t)A(t) + \lambda^* \sum_{i=1}^{l_1} \left[\chi(\bar{t}_{2i}) - \chi(\bar{t}_{2i-1})\right] \bar{D}_i(t) +$$

$$+ f_x^0(x,u,t)\Big] \Delta x(t)dt + \int_{t_0}^{T} \big\{ f_u^0(x,u,t) - \psi^*(t)B(t) \big\} \Delta u(t)dt +$$

$$+ \sum_{k=2}^{2l_1+l_2-1} \left[\psi^{*-}(\hat{t}_k) - \psi^{*+}(\hat{t}_k) + \frac{\partial \Phi(\hat{x}(\hat{t}))}{\partial x(\hat{t}_k)} \right] \Delta x(\hat{t}_k) +$$

$$+ \sum_{j=1}^{l_2} \lambda^* \tilde{D}_j \Delta x(\tilde{t}_j) + \psi^*(T)\Delta x(T) - \psi^*(t_0)\Delta x(t_0) +$$

$$+ \int_{t_0}^{T} o_1(\|\Delta x(t)\|_{L_2^n[t_0,T]})dt + \int_{t_0}^{T} o_2(\|\Delta u(t)\|_{L_2^r[t_0,T]})dt + o_3(\left\|\Delta \hat{x}(\hat{t}_k)\right\|_{L_2^n[t_0,T]}), \quad (3.3)$$

where $\psi^+(\hat{t}_k) = \psi(\hat{t}_k + 0)$, $\psi^-(\hat{t}_k) = \psi(\hat{t}_k - 0)$, $k = 1, 2, ..., (2l_1 + l_2)$, $\chi(t)-$ is Heaviside function.

Let $\psi(t)$ be the solution to the following system of equations

$$\dot{\psi}(t) = -A^*(t)\psi(t) + \sum_{i=1}^{l_1} \left[\chi(\bar{t}_{2i}) - \chi(\bar{t}_{2i-1})\right] \bar{D}^*(t)\lambda + f_x^{0*}(x,u,t), \quad (3.4)$$

involving the following boundary conditions

$$\psi(t_0) = \begin{cases} \left(\frac{\partial \Phi(\hat{x}(\hat{t}))}{\partial x(\tilde{t}_1)}\right)^* + \tilde{D}_1^* \lambda, & if\ t_0 = \tilde{t}_1, \\ \left(\frac{\partial \Phi(\hat{x}(\hat{t}))}{\partial x(\bar{t}_1)}\right)^*, & if\ t_0 = \bar{t}_1, \end{cases} \quad (3.5)$$

$$\psi(T) = \begin{cases} -\left(\frac{\partial \Phi(\hat{x}(\hat{t}))}{\partial x(\tilde{t}_{l_2})}\right)^* - \tilde{D}_{l_2}^* \lambda, & if\ \tilde{t}_{l_2} = T, \\ -\left(\frac{\partial \Phi(\hat{x}(\hat{t}))}{\partial x(\bar{t}_{2l_1})}\right)^*, & if\ \bar{t}_{2l_1} = T, \end{cases} \quad (3.6)$$

as well as jump conditions at the intermediate $\tilde{t}_j$ such that $t_0 < \tilde{t}_j < T$,

$$\psi^+(\tilde{t}_j) - \psi^-(\tilde{t}_j) = \left(\frac{\partial \Phi(\hat{x}(\hat{t}))}{\partial x(\tilde{t}_j)}\right)^* + \tilde{D}_j^* \lambda, \qquad j = 1, 2, ... l_2, \quad (3.7)$$

and at the points $\bar{t}_i$, $i = 1, 2, ..., 2l_1$ such that $t_0 < \bar{t}_i < T$,

$$\psi^+(\bar{t}_i) - \psi^-(\bar{t}_i) = \left(\frac{\partial \Phi(\hat{x}(\hat{t}))}{\partial x(\bar{t}_i)}\right)^*, \qquad i = 1, 2, ... 2l_1. \quad (3.8)$$

The solutions to equation (3.4) are the functions that are absolutely continuous for $t < \bar{t}_i$, $t > \bar{t}_i$, $i = 1, 2, ..., 2l_1$, and $t < \tilde{t}_j$, $t > \tilde{t}_j$, $j = 1, 2, ..., l_2$, and satisfy equation (3.4) almost everywhere on $[t_0, T]$; moreover, they have a jump of the form (3.7) and (3.8) at $t = \bar{t}_i$ and $t < \tilde{t}_j$.

Instead of (3.4), (3.7)-(3.8), we can use a differential equations system involving impulse actions:

$$\dot{\psi}(t) = -A^*(t)\psi(t) + \sum_{i=1}^{l_1} \left[\chi(\bar{t}_{2i}) - \chi(\bar{t}_{2i-1})\right] \bar{D}^*(t)\lambda +$$
$$+\sum_{j=1}^{l_2} \left[\left(\frac{\partial \Phi(\hat{x}(\hat{t}))}{\partial x(\tilde{t}_j)} \right)^* + \tilde{D}_j^* \lambda \right] \delta(t - \tilde{t}_j) +$$
$$+\sum_{i=1}^{l_1} \left(\frac{\partial \Phi(\hat{x}(\hat{t}))}{\partial x(\bar{t}_i)} \right)^* \delta(t - \bar{t}_i) + f_x^{0*}(x, u, t)\,. \tag{3.9}$$

Here $\delta(\cdot)$ is delta function. Problems (3.4)-(3.8) and (3.9), (3.5),(3.6) are equivalent. Numerical schemes of their approximation and the solution algorithms used are identical. Note the following about the estimate of the quantities $o_1(\|\Delta x(t)\|)$ and $o_3(\|\Delta x(\hat{t})\|)$. As it was noted above in the first paragraph, considered boundary problem (2.1)-(2.2) can be reduced to a non-local boundary problem involving both multipoint and two-point conditions. Problems of this kind are sufficiently well investigated in many works for the cases of different conditions imposed on the functions, on the parameters taking part in the statement, and for nonlinear problems as well. In these works, for different variants of the conditions, the following estimate is obtained:

$$\|\Delta x(t)\| \leq c\,\|\Delta u(t)\|\,, \tag{3.10}$$

where $c = const > 0$ does not depend on the choice of the admissible control ([13, 27, 28]).

It is clear that using the techniques of these works, we can obtain a similar estimate for boundary problem (2.1) and (2.2).

Thus the gradient of the target functional under the admissible control $u(t)$ in problem (2.1)-(2.3) is determined as follows:

$$\nabla J(u) = f_u^{0*}(x, u, t) - B^*(t)\psi(t), \tag{3.11}$$

where $x(t)$ and $\psi(t)$ are the solutions to direct system (2.1), (2.2) and to adjoint system (3.4)-(3.8), respectively, corresponding to this control.

On condition that there is a constructive algorithm for computing the value of the gradient of functional (3.11), it is not difficult to implement iterative techniques of first order minimization, particularly, of gradient projection method (see [29]):

$$u^{k+1}(t) = P_U(u^k(t) - \alpha_k\, \nabla J(u^k(t))), \quad k = 0, 1, ..., \tag{3.12}$$

$$\alpha_k = \arg\min_{\alpha \geq 0} J(P_U(u^k(t) - \alpha\, \nabla J(u^k(t))))\,,$$

where $P_U(v)$ is the projection operator of the element $v \in E^r$ on the admissible set U; α_k is the one-dimensional minimization step.

On every iteration (3.12), the calculation of the gradient of the functional under given control confronts with two the most essential difficulties associated with the specific character of the problem, namely, with the problem of solution to non-autonomous differential equations system involving non-separated multipoint and integral conditions (2.1), (2.2), and with the problem of solution to adjoint boundary problem (3.4)-(3.8), the non-local conditions of which contain an unknown n-dimensional vector of parameters λ. As a whole, system of relations (2.1), (2.2), (3.4)–(3.8) for determining the gradient of the functional under given control $u(t)$ is closed: to determine unknown $2n$ functions $x(t), \psi(t)$, their $2n$ initial conditions, and n-dimensional vector λ, we have $2n$-dimensional differential equations system, n conditions in (2.2) and $2n$ conditions in (3.5), (3.6).

Below, we give an algorithm for overcoming these difficulties. It is based on using the operation of shift of conditions developed in [3, 9] for solving systems of equations involving non-separated intermediate conditions and boundary conditions, including unknown parameters as well (see [3, 7, 12]). The concept of shift of intermediate conditions generalizes the known operation of shift of boundary conditions, and is based on developing the results of the [3, 9] applied to this class of problems.

4. Numerical scheme of solution to the problem

We shall give below an algorithm of computing the gradient of the target functional under given control. To solve the problem (2.1), (2.2) under given admissible function $u(t)$, we can make use of, for example, the numerical method proposed in [3, 9].

Solve the adjoint boundary problem (3.4)–(3.7) on the condition that the phase variable $x(t)$, the solution to problem (2.1) and (2.2), is already determined for given $u(t)$ by application of the procedure described above. To avoid cumbersome expressions in formulas and in description of numerical scheme of solution given below, we assume that $\tilde{t}_1 = t_0$ and $\tilde{t}_{l_2} = T$, and rewrite adjoint problem (3.4)–(3.7) in the following form:

$$\dot{\psi}(t) = A_1(t)\psi(t) + \sum_{i=1}^{l_1} \left[\chi(\bar{t}_{2i}) - \chi(\bar{t}_{2i-1})\right] \bar{D}_i(t)\lambda + C_1(t), \tag{4.1}$$

with the following boundary conditions:

$$\tilde{G}_1\psi(t_0) = \tilde{K}_1 + \tilde{D}_1\lambda, \tag{4.2}$$

$$\psi(T) = -\tilde{K}_{l_2} - \tilde{D}_{l_2}\lambda, \tag{4.3}$$

and jump conditions at the intermediate points $\tilde{t}_j$, for which $t_0 < \tilde{t}_j < T$:

$$\psi^+(\tilde{t}_j) - \psi^-(\tilde{t}_j) = \tilde{K}_j + \tilde{D}_j\lambda, \quad j = 2, 3, ..., l_2 - 1, \tag{4.4}$$

and at the points $\bar{t}_i$, for which $t_0 < \bar{t}_i < T$, $i = 1, 2, ... 2l_1$:

$$\psi^+(\bar{t}_i) - \psi^-(\bar{t}_i) = \bar{K}_i, \quad i = 1, 2, ...2\, l_1. \tag{4.5}$$

Here $\bar{G}_1 = I_n$ is the n-dimensional identity matrix, and the following notations are introduced for the matrices and vectors:

$$A_1(t) = -A^*(t),\ C_1(t) = \partial f^0(x,u,t)/\partial x,$$

$$\tilde{K}_j = \partial\Phi(x(\hat{t}))/\partial x(\tilde{t}_j), \tilde{D}_j^* = \tilde{D}_j,\ j = 1,2,...,l_2,$$

$$\bar{D}_i^*(t) = \bar{D}_i(t),\ i = 1,2,....,l_1, \quad \bar{K}_i = \partial\Phi(x(\hat{t}))/\partial x(\bar{t}_i), \quad i = 1,2...,2\,l_1,$$

that were already calculated when solving the direct problem.

In problem (4.1)–(4.5), defined by system of n differential equations (4.1), in general case, we have $2n$ boundary conditions that include the unknown n-dimensional vector λ. Thus the conditions of problem (4.1)–(4.4) are closed, but there is a specific character which lies in the presence of discontinuities of the function $\psi(t)$ defined by jumps (4.4).

Condition (4.2) is called a condition shifted to the right in the semi-interval $t \in [\tilde{t}_1, \tilde{t}_2)$ by the matrix and vector functions $G_1(t),\ D_1(t) \in E^{n\times n}$, $K_1(t) \in E^n$ such that

$$G_1(t_0) = G_1(\tilde{t}_1) = \tilde{G}_1, \quad K_1(t_0) = K_1(\tilde{t}_1) = \tilde{K}_1, \quad \tilde{D}_1(t_0) = \tilde{D}_1(\tilde{t}_1) = \tilde{D}_1, \tag{4.6}$$

if for the solution $\psi(t)$ to (4.1), the following relation holds:

$$G_1(t)\psi(t) = K_1(t) + D_1(t)\lambda\,, \quad t \in [\tilde{t}_1, \tilde{t}_2). \tag{4.7}$$

Next, using the results of [3, 7, 12], we give the techniques to find the shifting functions $G_1(t),\ D_1(t), K_1(t)$. By using formula (4.7), we shift initial conditions (4.5) to the point $t = \tilde{t}_2 - 0$ and, taking shift condition (4.4) into account at the point $t = \tilde{t}_2 + 0$, we obtain

$$G_1(\tilde{t}_2)\psi(\tilde{t}_2+0) = \left[\ K_1(\tilde{t}_2) + G(\tilde{t}_2)\tilde{K}_2\right] + \left[\ D_1(\tilde{t}_2) + G_1(\tilde{t}_2)\tilde{D}_2\right]\ \lambda.$$

Introducing the notations

$$\tilde{t}_2 = \tilde{t}_2 + 0, \quad \tilde{G}_1^1 = G_1(\tilde{t}_2), \quad \tilde{K}_1^1 = K_1(\tilde{t}_2) + G(\tilde{t}_2)\tilde{K}_2, \quad \tilde{D}_1^1 = D_1(\tilde{t}_2) + G_1(\tilde{t}_2)\tilde{D}_2,$$

we obtain the conditions similar to (3.27) and defined at the point $\tilde{t}_2$:

$$\tilde{G}_1^1\psi(\tilde{t}_2) = \tilde{K}_1^1 + \tilde{D}_1^1\lambda.$$

By shifting condition (4.2) $l_2 - 1$ times and taking (4.6) into account, we obtain a linear system of $2n$ algebraic equations with respect to $\psi(\tilde{t}_{l_2}) = \psi(T)$ and λ. After solving this system, we determine the vector function $\psi(t)$ from right to left from Cauchy problem with respect to (4.1).

Illustrate the stages of the shift process applied to condition (4.2). To be specific we assume that $[\bar{t}_1, \bar{t}_2]\ \subset [\tilde{t}_1, \tilde{t}_2)$ and $\tilde{t}_1 = t_0$. Shift of condition is carried out successively in the intervals $[\tilde{t}_1, \bar{t}_1)$, $[\bar{t}_1, \bar{t}_2)$, $[\bar{t}_2, \tilde{t}_2)$, by using formulas (4.7).

1) For $t \in [\tilde{t}_1, \bar{t}_1)$, we shift initial conditions (4.2) to the point $t = \bar{t}_1 - 0$ and, taking shift condition (4.5) into account at the point $t = \bar{t}_1$, we obtain

$$G_1(\bar{t}_1)\psi(\bar{t}_1+0) = \left[\ K_1(\bar{t}_1) + G_1(\bar{t}_1)\bar{K}_1\right] + D_1(\bar{t}_1)\,\lambda.$$

Assuming $\bar{t}_1 = \bar{t}_1 + 0$, introduce the notations

$$\tilde{G}_1^1 = G_1(\bar{t}_1), \quad \tilde{K}_1^1 = K_1(\bar{t}_1) + G_1(\bar{t}_1)\bar{K}_1, \quad \tilde{D}_1^1 = D_1(\bar{t}_1),$$

following which, we obtain the initial conditions similar to (4.2) and defined at the point $\bar{t}_1$,

$$\tilde{G}_1^1\psi(\bar{t}_1) = \tilde{K}_1^1 + \tilde{D}_1^1\lambda. \tag{4.8}$$

2) For $t \in [\bar{t}_1, \bar{t}_2)$, we shift conditions (4.8) to the point $t = \bar{t}_2 - 0$ and, taking shift condition (4.5) into account at the point $t = \bar{t}_2$, we obtain

$$G_1(\bar{t}_2)\psi(\bar{t}_2 + 0) = \left[\ K_1(\bar{t}_2) + G_1(\bar{t}_2)\bar{K}_2\ \right] + D_1(\bar{t}_2)\,\lambda.$$

Assuming $\bar{t}_2 = \bar{t}_2 + 0$, introduce the notations

$$\tilde{G}_1^2 = G_1(\bar{t}_2), \quad \tilde{K}_1^2 = K_1(\bar{t}_2) + G_1(\bar{t}_2)\bar{K}_2\,, \quad \tilde{D}_1^2 = D_1(\bar{t}_2)\,,$$

and obtain initial conditions equivalent to (4.8) and defined at the point $\bar{t}_2$

$$\tilde{G}_1^2\psi(\bar{t}_2) = \tilde{K}_1^2 + \tilde{D}_1^2\lambda\,. \tag{4.9}$$

3) For $t \in [\bar{t}_2, \tilde{t}_2)$ we shift conditions (4.9) to the point $t = \tilde{t}_2 - 0$ and, taking shift condition (4.4) into account at the point $t = \tilde{t}_2$, we obtain

$$G_1(\tilde{t}_2)\psi(\tilde{t}_2 + 0) = \left[\ K_1(\tilde{t}_2) + G_1(\tilde{t}_2)\bar{K}_2\ \right] + \left[D_1(\tilde{t}_2) + G_1(\tilde{t}_2)\tilde{D}_2\right]\ \lambda\,.$$

Assuming $\tilde{t}_2 = \tilde{t}_2 + 0$, introduce the notations

$$\tilde{G}_1^3 = G_1(\tilde{t}_2)\,, \quad \tilde{K}_1^3 = K_1(\tilde{t}_2) + G_1(\tilde{t}_2)\bar{K}_2\,, \quad \tilde{D}_1^3 = D_1(\tilde{t}_2) + G_1(\tilde{t}_2)\tilde{D}_2\,,$$

and obtain the conditions equivalent to (3.34) and defined at the point $\tilde{t}_2$

$$\tilde{G}_1^3\psi(\tilde{t}_2) = \tilde{K}_1^3 + \tilde{D}_1^3\lambda\,. \tag{4.10}$$

The functions $G_j(t),\ K_j(t),\ D_j(t),\ j = 1, 2, ..., l_2$ that shift conditions (4.2) successively to the right (i.e. the functions $G_j(t),\ K_j(t),\ D_j(t),\ j = 1, 2, ..., l_2$ must satisfy (4.6), (4.7)), are not determined uniquely. For example, it is possible to use functions proposed in the following theorem.

Theorem 4.1. *Let the functions $G_1(t),\ K_1(t),\ D_1(t)$ be the solution to the following Cauchy problems for $t \in (\tilde{t}_1, \tilde{t}_2]$:*

$$\dot{G}_1(t) = Q^0(t)G_1(t) - G_1(t)A_1(t), \qquad G_1(\tilde{t}_1) = \tilde{G}_1,$$

$$\dot{D}_1(t) = Q^0(t)D_1(t) + G_1(t)\sum_{i=1}^{l_1}\left[\chi(\bar{t}_{2i}) - \chi(\bar{t}_{2i-1})\right]\bar{D}_i(t), \quad D_1(\tilde{t}_1) = \tilde{D}_1,$$

$$\dot{K}_1(t) = Q^0(t)K_1(t) + G_1(t)C_1(t), \qquad K_1(\tilde{t}_1) = \tilde{K}_1,$$

$$\dot{Q}(t) = Q^0(t)Q(t), \qquad Q(\tilde{t}_1) = I_{n\times n}, \tag{4.11}$$

$$Q^0(t) = \left[G_1(t)A_1(t)G_1^*(t) - G_1(t)\sum_{i=1}^{l_1}\left[\chi(\bar{t}_{2i}) - \chi(\bar{t}_{2i-1})\right]\bar{D}_i(t)D_1^*(t) - \right.$$

$$-G_1(t)C_1(t)K_1^*(t)] \times [G_1(t)G_1^*(t) + D_1(t)D_1^*(t) + K_1(t)K_1^*(t)]^{-1}.$$

Then these functions shift condition (4.2) to the right on the semi-interval $t \in [\tilde{t}_1, \tilde{t}_2)$*, and relation (4.7) holds true for them. The following condition*

$$\|G_1(t)\|^2_{R^{n\times n}} + \|D_1(t)\|^2_{R^{n\times n}} + \|K_1(t)\|^2_{R^n} =$$

$$= \left\|\tilde{G}_1\right\|^2_{R^{n\times n}} + \left\|\tilde{D}_1\right\|^2_{R^{n\times n}} + \left\|\tilde{K}_1\right\|^2_{R^n} = const, \quad t \in \left(\tilde{t}_1, \tilde{t}_2\right), \tag{4.12}$$

also holds. Condition (4.12) provides stability for the solution to Cauchy problem (4.11).

Proof. Differentiating expression (4.7)

$$\dot{G}_1(t)\psi(t) + G_1(t)\,\dot{\psi}(t) = \dot{K}_1(t) + \dot{D}_1(t)\lambda,$$

and taking (4.1) into account, we come to the equality

$$\dot{G}_1(t)\psi(t) + G_1(t)\left[A_1(t)\psi(t) + \sum_{i=1}^{l_1}\left[\chi(\bar{t}_{2i}) - \chi(\bar{t}_{2i-1})\right]\bar{D}_i(t)\lambda + C_1(t)\right] =$$

$$= \dot{K}_1(t) + \dot{D}_1(t)\lambda$$

After grouping, we obtain the following equation:

$$\left[\dot{G}_1(t) + G_1(t)A_1(t)\right]\psi(t) + \left[-\dot{D}_1(t) + G_1(t)\sum_{i=1}^{l_1}\left[\chi(\bar{t}_{2i}) - \chi(\bar{t}_{2i-1})\right]\bar{D}_i(t)\right]\lambda +$$

$$+\left[-\dot{K}_1(t) + G_1(t)C_1(t)\right] = 0.$$

Setting the expressions in brackets equal to 0, we obtain

$$\dot{G}_1(t) = -G_1(t)A_1(t), \quad \dot{K}_1(t) = G_1(t)C_1(t)\,,$$

$$\dot{D}_1(t) = G_1(t)\sum_{i=1}^{l_1}\left[\chi(\bar{t}_{2i}) - \chi(\bar{t}_{2i-1})\right]\bar{D}_i(t), \tag{4.13}$$

The functions $G_1(t),\ K_1(t),\ D_1(t)$, which are the solution to Cauchy problems (4.13), (4.6), satisfy condition (4.7), i.e. they shift conditions (4.2) from the point $\tilde{t}_1 = t_0$ to the point $\tilde{t}_2$. But numerical solution to Cauchy problems (4.13), (4.6), as is known, confronts with instability due to the presence of fast increasing components. This is because the matrix $A_1(t)$ often has both positive and negative eigenvalues. That is why we try to find shifting functions that satisfy condition (4.12).

Multiply both parts of (4.7) by an arbitrary matrix function $Q(t)$ such that

$$Q(t_0) = I_{n\times n}, \quad rang\ Q(t) = n, \quad t \in [\tilde{t}_1, \tilde{t}_2),$$

and introduce the notations

$$g(t) = Q(t)G_1(t), \quad q(t) = Q(t)D_1(t), \quad r(t) = Q(t)K_1(t). \tag{4.14}$$

From (4.7), it follows that:

$$g(t)\psi(t) = r(t) + q(t)\lambda. \tag{4.15}$$

Differentiating (4.14) and taking (4.13) into account, we obtain:

$$\dot{g}(t) = \dot{Q}(t)G_1(t) + Q(t)\dot{G}_1(t) = \dot{Q}(t)Q^{-1}(t)g(t) - g(t)A_1(t), \tag{4.16}$$

$$\dot{q}(t) = \dot{Q}(t)D_1(t) + Q(t)\dot{D}_1(t) = \dot{Q}(t)Q^{-1}(t)q(t)+$$

$$+g(t)\sum_{i=1}^{l_1}\left[\chi(\bar{t}_{2i}) - \chi(\bar{t}_{2i-1})\right]\bar{D}_i(t), \tag{4.17}$$

$$\dot{r}(t) = \dot{Q}(t)K_1(t) + Q(t)\dot{K}_1(t) = \dot{Q}(t)Q^{-1}(t)r(t) + g(t)C_1(t). \tag{4.18}$$

Transposing relations (4.16)-(4.18), we obtain

$$\dot{g}^*(t) = g^*(t)(\dot{Q}(t)Q^{-1}(t))^* - A_1^*(t)g^*(t), \tag{4.19}$$

$$\dot{q}^*(t) = q^*(t)(\dot{Q}(t)Q^{-1}(t))^* + \sum_{i=1}^{l_1}\left[\chi(\bar{t}_{2i}) - \chi(\bar{t}_{2i-1})\right]\bar{D}_i^*(t)\,g^*(t), \tag{4.20}$$

$$\dot{r}^*(t) = r^*(t)(\dot{Q}(t)Q^{-1}(t))^* + C_1^*(t)g^*(t). \tag{4.21}$$

Choose the matrix functions $Q(t)$ such that the following relation holds

$$g(t)g^*(t) + q(t)q^*(t) + r(t)r^*(t) = const.$$

Differentiating it, we obtain

$$\dot{g}(t)g^*(t) + g(t)\dot{g}^*(t) + \dot{q}(t)q^*(t) + q(t)\dot{q}^*(t) + \dot{r}(t)r^*(t) + r(t)\dot{r}^*(t) = 0. \tag{4.22}$$

Substituting (4.16)-(4.19) into (4.22), after grouping, we obtain:

$$\big[Q(t)Q^{-1}(t)\left(g(t)g^*(t) + q(t)q^*(t) + r(t)r^*(t)\right)+$$

$$+\left(-g(t)A_1(t)g^*(t) + g(t)\sum_{i=1}^{l_1}\left[\chi(\bar{t}_{2i}) - \chi(\bar{t}_{2i-1})\right]\bar{D}_i(t)q^*(t) + g(t)C_1(t)r^*(t)\right)\Bigg]+$$

$$+\big[Q(t)Q^{-1}(t)\left(g(t)g^*(t) + q(t)q^*(t) + r(t)r^*(t)\right)+$$

$$+\left(-g(t)A_1(t)g^*(t) + g(t)\sum_{i=1}^{l_1}\left[\chi(\bar{t}_{2i}) - \chi(\bar{t}_{2i-1})\right]\bar{D}_i(t)q^*(t) + g(t)C_1(t)r^*(t)\right)\Bigg]^* = 0.$$

Assume that the expression in both square brackets equals 0:

$$\big[Q(t)Q^{-1}(t)\left(g(t)g^*(t) + q(t)q^*(t) + r(t)r^*(t)\right)+$$

$$+\left(-g(t)A_1(t)g^*(t)+g(t)\sum_{i=1}^{l_1}\left[\chi(\bar t_{2i})-\chi(\bar t_{2i-1})\right]\bar D_i(t)q^*(t)+g(t)C_1(t)r^*(t)\right)\Bigg]=0.$$

From this, it follows that

$$Q(t)Q^{-1}(t)=Q^0(t), \tag{4.23}$$

where

$$Q^0(t)=\left[g(t)A_1(t)g^*(t)-g(t)\sum_{i=1}^{l_1}\left[\chi(\bar t_{2i})-\chi(\bar t_{2i-1})\right]\bar D_i(t)q^*(t)-g(t)C_1(t)r^*(t)\right]\times$$
$$\times\left[g(t)g^*(t)+q(t)q^*(t)+r(t)r^*(t)\right]^{-1}.$$

Substituting (4.23) into (4.16)-(4.18) and renaming the functions $g(t)$ in $G_1(t)$, $q(t)$ and $D_1(t)$, $r(t)$ in $K_1(t)$, we obtain the statement of the theorem. □

It is not difficult to carry on similar considerations and to obtain formulas for shifting condition (4.3) successively to the left.

Thus to implement iterative procedure (3.12), it is necessary to go through the following steps on every iteration for given $u(t)=u^k(t), t\in[t_0,T], k=0,1,...$:

1) to solve the problem (2.1), (2.2) by using the numerical scheme proposed in [9], and to determine the phase trajectory $x(t),\ \ t\in[t_0,T]$;

2) to solve problem (4.1)–(4.5) with the use of shift procedure (4.7) applied to the boundary conditions, and to determine the adjoint vector-function $\psi(t), t\in[t_0,T]$ and the vector of dual variables λ;

3) to substitute the obtained values of $x(t)$, $\psi(t)$, $t\in[t_0,T]$ into formula (3.11) and to determine the value of the gradient of the functional.

Instead of gradient projection method (3.12), other efficient first order numerical optimization methods can be used (see [29]).

5. Numerical experiments

Problem. Consider the following optimal control problem for $t\in[0;1]$, $n=2$, $r=1$, $U\equiv E^1$:

$$\begin{cases}\dot x_1(t)=4tx_1(t)-x_2(t)+tu-5t^2+5t+3\,,\\ \dot x_2(t)=3x_1(t)+2tx_2(t)-2t^3-6t+3.\end{cases} \tag{5.1}$$

$$\int_0^{0.25}\begin{pmatrix}\tau & -2\\ 0 & 3\end{pmatrix}x(\tau)d\tau+\begin{pmatrix}5 & 1\\ 2 & 3\end{pmatrix}x(0.5)+$$
$$+\int_{0.8}^{1}\begin{pmatrix}\tau-1 & 2\\ 1 & 0\end{pmatrix}x(\tau)d\tau=\begin{pmatrix}-0.3025\\ 4.6756\end{pmatrix}. \tag{5.2}$$

$$\bar D_1(t)=\begin{pmatrix}t & -2\\ 0 & 3\end{pmatrix},\quad \bar D_2(t)=\begin{pmatrix}t-1 & 2\\ 1 & 0\end{pmatrix},\quad \tilde D_1=\begin{pmatrix}5 & 1\\ 2 & 3\end{pmatrix}.$$

$$J(u)=\int_0^1 [x_1(t)-u(t)+2]^2\,dt+x_1^2(0,5)+$$

$$+[x_2(0,5)-1.25]^2+[x_1(1)-1]^2+[x_2(1)-2]^2\,. \qquad (5.3)$$

The exact solution to the problem are the following functions: $u^*(t)=2t+1,\quad x_1^*(t)=2t-1,\quad x_2^*(t)=t^2+1$; the minimal value of the functional is $J(u^*)=0$.

According to formulas (3.4)-(3.8), the adjoint problem is as follows:

$$\dot\psi_1(t)=-4t\psi_1(t)-3\psi_2(t)+(\chi(0.25)-\chi(0))\,(t\lambda_1)+$$

$$+(\chi(1)-\chi(0.8))((t-1)\lambda_1+\lambda_2)+2(x_1(t)-u(t)+2),$$

$$\dot\psi_2(t)=\psi_1(t)-2t\psi_2(t)+(\chi(0.25)-\chi(0))\,(-2\lambda_1+3\lambda_2)+$$

$$+(\chi(1)-\chi(0.8))(2\lambda_1),$$

$$\psi_1(0)=0\ ,\ \psi_2(0)=0\ ,$$
$$\psi_1(1)=-2\,[x_1(1)-1],\ \psi_2(1)=-2\,[x_2(1)-2]\,,$$

$$\psi_1^+(0.5)-\psi_1^-(0.5)=2x_1(0.5)+5\lambda_1+2\lambda_2\ ,$$

$$\psi_2^+(0.5)-\psi_2^-(0.5)=2\,[x_2(0.5)-1.25]+\lambda_1+3\lambda_2\ .$$

The gradient of the functional is determined as follows:

$$\nabla J(u)=-\,[\,x_1(t)-u(t)+2\,]-t\psi_1(t).$$

Numerical experiments were carried out for different initial controls $u^0(t)$ and for different numbers N of partition of time interval. Fourth order Runge-Kutta method and conjugate gradient method were used. In figure 1, we give the results of solution to system (5.1), (5.2) and to the corresponding adjoint system. We also show the values of the components of the normalized gradients ($\nabla^{norm.}_{analyt.}J$) calculated by the proposed formulas (3.11) and the values of the components of the normalized gradients ($\nabla^{norm.}_{approx.}J$) obtained by finite difference approximation:

$$\partial J(u)/\partial u_j\approx(J(u+\delta e_j)-J(u))/\delta, \qquad (5.4)$$

where u_j is the value of the control $u=(u_1,u_2,...,u_N)$ at thej^{th} discretization point; e_j is the N- dimensional unit vector consisting of zeros except for the j^{th} component. The value of δ equals 0.01 and 0.001.

The initial value of the functional is $J(u^0)=56.28717,\ \lambda_1=0.2387,\quad \lambda_2=0.1781$. The values of the functional obtained in the course of the iterations are as follows:

$$J(u^1)=1.93187,\quad J(u^2)=0.10445,\quad J(u^3)=0.00868,$$
$$J(u^4)=0.00023,\quad J(u^5)=0.00004.$$

Figure 1. **Initial values of the controls, of the phase variables, and of the normalized gradients calculated using both the proposed formulas and (5.4)**

t	$u^{(0)}(t)$	$x_1^{(0)}(t)$	$x_2^{(0)}(t)$	$\psi_1^{(0)}(t)$	$\psi_2^{(0)}(t)$	$\nabla^{norm.}_{analyt.}J$	$\nabla^{norm.}_{approx.}J$	
							$\delta = 10^{-2}$	$\delta = 10^{-3}$
0	1.000	1.5886	1.2034	-9.2836	-6.3653	-0.0161	-0.0156	-0.0161
20	2.000	1.5641	2.4702	-0.5626	3.2122	-0.0106	-0.0102	-0.0107
40	3.000	1.2382	3.5937	2.9294	14.8639	-0.0053	-0.0049	-0.0052
60	4.000	0.6657	4.4538	-4.5081	14.2492	0.0147	0.0147	0.0145
80	5.000	-0.0781	4.9528	-10.9008	11.4433	0.0411	0.0413	0.0410
100	6.000	-0.8973	5.0218	-15.1806	7.0519	0.0668	0.0688	0.0667
120	7.000	-1.6767	4.6295	-21.2792	9.1844	0.0996	0.0990	0.0994
140	8.000	-2.2835	3.7919	-22.7128	2.1275	0.1169	0.1164	0.1168
160	9.000	-2.5710	2.5845	-14.7216	-0.0383	0.1039	0.1034	0.1038
180	10.000	-2.3853	1.1544	-6.8798	-0.6530	0.0827	0.0823	0.0826
200	11.000	-1.5759	-0.2660	0.0000	-0.0000	0.0800	0.0795	0.0800

On the sixth iteration of conjugate gradient method, we obtain the results given in figure 2 with the minimal value of the functional $J(u^6)$ equal to 10^{-6}.

Figure 2. **The exact solution to the problem and the solution obtained after the sixth iteration**

t	Solution obtained					Exact solution		
	$u^{(6)}(t)$	$x_1^{(6)}(t)$	$x_2^{(6)}(t)$	$\psi_1^{(6)}(t)$	$\psi_2^{(6)}(t)$	$u^*(t)$	$x_1^*(t)$	$x_2^*(t)$
0	0.9999	-1.0000	1.0001	0.0059	-0.0040	1.0000	-1.0000	1.0000
20	1.2000	-0.8001	1.0101	0.0095	0.0027	1.2000	-0.8000	1.0100
40	1.4001	-0.6001	1.0401	0.0089	0.0098	1.4000	-0.6000	1.0400
60	1.6001	-0.4001	1.0902	0.0038	0.0114	1.6000	-0.4000	1.0900
80	1.7999	-0.2001	1.1602	-0.0012	0.0114	1.8000	-0.2000	1.1600
100	1.9999	-0.0001	1.2500	-0.0049	0.0099	2.0000	0.0000	1.2500
120	2.1999	0.1999	1.3599	-0.0055	0.0044	2.2000	0.2000	1.3600
140	2.3998	0.4000	1.4899	-0.0053	0.0025	2.4000	0.4000	1.4900
160	2.6001	0.6001	1.6399	-0.0026	0.0014	2.6000	0.6000	1.6400
180	2.8001	0.8001	1.8101	-0.0008	0.0006	2.8000	0.8000	1.8100
200	3.0001	1.0001	2.0001	0.0000	-0.0000	3.0000	1.0000	2.0000

6. Conclusion

In the work, we propose the technique for numerical solution to optimal control problems for ordinary differential equations systems involving non-separated multipoint and integral conditions. Note that a mere numerical solution to the differential systems presents certain difficulties. The adjoint problem also has a specific character which lies both in the equation itself and in the presence of an unknown vector of Lagrange coefficients in the conditions.

The formulas proposed in the work, as well as the computational schemes make it possible to take into account all the specific characters which occur when calculating the gradient of the functional. Overall, the proposed approach allows us to use a rich arsenal of first order optimization methods and the corresponding standard software to solve the considered optimal control problems.

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Received: December 19, 2017; Revised: April 22, 2018; Accepted: May 31, 2018