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# ON THE BIVARIATE GENERALIZATION OF BERNSTEIN-FOMIN OPERATORS

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**Abstract**. This paper deals with an extension of the bivariate generalized Fomin operators based on Stancu's method on triangular and rectangular domain. For these operators we obtain Korovkin type convergence theorem on a rectangular domain.

## 1. Introduction

There are many investigations devoted to the generalization of the classical Bernstein polynomial operators. One of these generalization is given by M.A. Fomin that possesses interesting properties [5, 6]. For $f \in C[0,1]$, Fomin [5] introduced a generalization of the Bernstein operators, for given $n, m \in \mathbb{N}$, as follows:

$$F_{n,m}(f;x) = \Big(1+(m-1)x\Big)\sum_{k=0}^{[\frac{n}{m}]} f\Big(\frac{k}{n-(m-1)k}\Big)P^k_{n,m}(x), \tag{1.1}$$

where $P^k_{n,m}(x) = C^k_{n-(m-1)k}\ x^k \times (1-x)^{n-mk}$ and $[x]$, as usual, denotes the greatest integer less than $x$.

D.D. Stancu in [9] gave a method for obtaining polynomials of Bersntein type of two variables. The approximation by Bernstein-type polynomial operators on rectangular and triangular domain were largely studied due to their applications in many branch of mathematics see [1, 3, 4, 7]. In the spirit of Fomin polynomial operators we shall derive Fomin-type operators on square $S := [0,1]\times[0,1]$ and on triangular $\Delta := \Big\{x+y \le 1;\ x \ge 0, y \ge 0\Big\}$. Then their Korovkin-type approximation properties are investigated. More precisely, we will show Fomin type operator on rectangular domain defined by

$$\begin{aligned} &F_{n,m\ n',m'}\Big(f;x,y\Big) \\ &\quad = \Big(1+\Big(m-1\Big)x\Big)\Big(1+\Big(m'-1\Big)y\Big) \\ &\quad \times \sum_{i=0}^{[\frac{n}{m}]}\sum_{j=0}^{[\frac{n'}{m'}]} P^i_{n,m}(x)P^j_{n',m'}(y) f\Big(\frac{i}{n-(m-1)i}, \frac{j}{n'-(m'-1)j}\Big); \end{aligned} \tag{1.2}$$

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and on the triangular domain defined by

$$F_{n,m}\Big(f;x,y\Big) =$$

$$= \Big(1+\Big(m-1\Big)x\Big)\sum_{i=0}^{[\frac{n}{m}]}\sum_{j=0}^{[\frac{n}{m}]-i}\Big(1+\Big(m-1\Big)\frac{n-(m-1)i}{n-im}y\Big)$$

$$\times P_{n,m}^{i}(x)P_{n-im,m}^{j}(\frac{n-(m-1)i}{n-im}y)$$

$$\times f\Big(\frac{i}{n-(m-1)i},\frac{(n-im)j}{(n-(m-1)i)(n-im-(m-1)j)}\Big); \quad (1.3)$$

then for investigating convergence problem of generalized Fomin operators we use the Korovkin theorem for two variable functions. We refer the readers to [2, 8], where the Krovkin theorem and its generalizations are given. We recall the following theorem on convergence of the bivariate linear positive operators for functions of two variables proved in [10].

Let $C\big([0,1]\times[0,1]\big)$ denote the set of continuous functions on the $[0,1]\times[0,1]$. $C\big([0,1]\times[0,1]\big)$ is a linear normed space with the norm

$$\|f\|_{C\big([0,1]\times[0,1]\big)} = \sup_{(x,y)\in[0,1]\times[0,1]} |f(x,y)|.$$

**Theorem A.** *If $\{L_n\}$ is a sequence of linear positive operators satisfying the conditions*

$$\lim_{n\to\infty}\|L_n(1;x_1,x_2)-1\|_{C[0,1]\times C[0,1]}=0; \quad (1.4)$$

$$\lim_{n\to\infty}\|L_n(t_j;x_1,x_2)-x_j\|_{C[0,1]\times C[0,1]}=0, \qquad j=1,2; \quad (1.5)$$

$$\lim_{n\to\infty}\|L_n(t_1^2+t_2^2;x_1,x_2)-x_1^2+x_2^2\|_{C[0,1]\times C[0,1]}=0. \quad (1.6)$$

*Then for any function $f\in C[0,1]\times C[0,1]$ which is bounded in $\mathbb{R}\times\mathbb{R}$, we have*

$$\lim_{n\to\infty}\|L_n(f;x_1,x_2)-f(x_1,x_2)\|_{C[0,1]\times C[0,1]}=0.$$

## 2. Obtaining an extension of the bivariate generalized Fomin operators

For deriving bivariate generalized Fomin operators, we employ the technique used by D.D. Stancu [9] for Bernstein operators. Let $\psi_1=\psi_1(x)$ and $\psi_2=\psi_2(x)$ be two polynomials such that on the interval $[0,1]$ have property $0\le\psi_1\le\psi_2$. Let $f(x,y)$ is a function of two variables with domain $D$, which $D$ restricted by lines $s=0, s=1$ and two curve $y=\psi_1, y=\psi_2$.

By considering the following change of variable

$$y=\big(\psi_2-\psi_1\big)t+\psi_1; \qquad t\in[0,1]$$

we have

$$f(x,y)=f\Big(x,\big(\psi_2(x)-\psi_1(x)\big)t+\psi_1(x)\Big)=\psi(x,t).$$

We consider the function $g(x)$ as follows:

$$g(x) := f\Big(x \ ; \ \big(\psi_2(x) - \psi_1(x)\big)t + \psi_1(x)\Big).$$

Then for any fixed $t \in [0,1]$, thanks to (1.1), we have

$$\begin{aligned}
g(x) &\cong \Big(1+(m-1)x\Big) \sum_{i=0}^{[\frac{n}{m}]} P^i_{n,m}(x) \quad g\Big(\frac{i}{n-(m-1)i}\Big) \\
&= \Big(1+(m-1)x\Big) \sum_{i=0}^{[\frac{n}{m}]} P^i_{n,m}(x) \times \\
&\times f\Big(\frac{i}{n-(m-1)i} \ , \ \Big(\psi_2(\frac{i}{n-(m-1)i}) - \psi_1(\frac{i}{n-(m-1)i})\Big)t + \psi_1(\frac{i}{n-(m-1)i})\Big) \\
&= \Big(1+(m-1)x\Big) \sum_{i=0}^{[\frac{n}{m}]} P^i_{n,m}(x)\psi\left(\frac{i}{n-(m-1)i}, t\right) \qquad (2.1)
\end{aligned}$$

On the other hands, for any $i \in \big\{0,1,2,...,[\frac{n}{m}]\big\}$, one can write Fomin expansion for one variable function $\psi\left(\frac{i}{n-(m-1)i}, t\right)$. Then we obtain

$$\begin{aligned}
&\psi\left(\frac{i}{n-(m-1)i} \ , \ t\right) \\
&\cong \Big(1+(m_i-1)t\Big) \sum_{j=0}^{[\frac{n_i}{m_i}]} P^j_{n_i,m_i}(t) \ \psi\left(\frac{i}{n-(m-1)i}, \frac{j}{n_i-(m_i-1)j}\right). \qquad (2.2)
\end{aligned}$$

By (2.1) and (2.2), we have

$$\begin{aligned}
&\forall x, t \in [0,1); \\
&f\Big(x \ , \ (\psi_2(x) - \psi_1(x))t + \psi_1(x)\Big) \\
&\cong \Big(1+(m-1)x\Big) \sum_{i=0}^{[\frac{n}{m}]} P^i_{n,m}(x)\Big(1+(m_i-1)t\Big) \qquad (2.3) \\
&\times \sum_{j=0}^{[\frac{n_i}{m_i}]} P^j_{n_i,m_i}(t)\psi\left(\frac{i}{n-(m-1)i}, \frac{j}{n_i-(m_i-1)j}\right).
\end{aligned}$$

Now by changing of variable we can write Fomin type expansion for bivariate function $f(x,y)$ as follow:

$$f\Big(x\ ,\ y\Big)$$
$$\cong \Big(1+(m-1)x\Big)\sum_{i=0}^{[\frac{n}{m}]} P^i_{n,m}(x)\Big(1+(m_i-1)\frac{y-\psi_1(\frac{i}{n-(m-1)i})}{\psi_2(\frac{i}{n-(m-1)i})-\psi_1(\frac{i}{n-(m-1)i})}\Big)$$
$$\times\sum_{j=0}^{[\frac{n_i}{m_i}]} P^j_{n_i,m_i}\left(\frac{y-\psi_1(\frac{i}{n-(m-1)i})}{\psi_2(\frac{i}{n-(m-1)i})-\psi_1(\frac{i}{n-(m-1)i})}\right)\psi\left(\frac{i}{n-(m-1)i},\frac{j}{n_i-(m_i-1)j}\right).$$

Now by particularizing $\psi_1$, $\psi_2$ and $m_i$, $n_i$ for any $i\in\{1,2,3,...,[\frac{n}{m}]\}$, we can find our desired polynomials on triangles and rectangular domain. And the three following cases are remarkable instances of this:

(1) If we set $\psi_1=0$, $\psi_2=1$ and $m_i=m'$, $n_i=n'$. In this case the domain $D$ becomes the square $[0,1]\times[0,1]$ and we get the (1.2);
(2) If we set $\psi_1=0$, $\psi_2=1-x$ and $m_i=m$, $n_i=n-im$. In this case the domain $D$ becomes the triangles $\triangle$ and we get the (1.3);
(3) If we set $\psi_1=0$, $\psi_2=x$ and $m_i=m$, $n_i=i$ we get the formula

$$F_{n,m}(f;x,y)=\Big(1+(m-1)x\Big)\sum_{i=0}^{[\frac{n}{m}]} P^i_{n,m}(x)\Big(1+(m-1)\frac{y(n-(m-1)i)}{i}\Big)$$
$$\times\sum_{j=0}^{[\frac{i}{m}]} P^j_{i,m}\left(\frac{y(n-(m-1)i)}{i}\right) f\left(\frac{i}{n-(m-1)i},\frac{ji}{(n-(m-1)i)(i-(m-1)j)}\right). \tag{2.4}$$

## 3. Convergence of $F_{n,m,n',m'}(f;x,y)$ on Rectangular Domain

In this section we give some properties of polynomial operators $F_{n,m,n',m'}(f;x,y)$. Let us consider

$$E_{n,m,n',m'}(x,y)=\frac{F_{n,m,n',m'}(1;x,y)}{\Big(1+(m-1)x\Big)\Big(1+(m'-1)y\Big)}. \tag{3.1}$$

It is clear that $E_{n,m,n',m'}(x,y)=A_{n,m}(x)A_{n',m'}(y)$, where $A_{n,m}(x)=\frac{F_{n,m}(1;x)}{(1+(m-1)x)}$.

**Lemma 3.1.** *For $n\geq m$ and $n'\geq m'$ and for any $(x,y)\in[0,1]\times[0,1]$, the following statements are valid.*

◇ $$E_{n,m,n',m'}(x,y)=(1-x)\ E_{n-1,m,n',m'}(x,y)+x\ E_{n-m,m,n',m'}(x,y) \tag{3.2}$$

◇ $$E_{n,m,n',m'}(x,y)=(1-y)\ E_{n,m,n'-1,m'}(x,y)+y\ E_{n,m,n'-m',m'}(x,y) \tag{3.3}$$

$$\begin{aligned}\diamond \quad & E_{n,m,n',m'}(x,y) \qquad (3.4)\\ &= (1-y)(1-x)\ E_{n-1,m,n'-1,m'}(x,y) + (1-x)y\ E_{n-1,m,n'-m',m'}(x,y)\\ &+ (1-y)x\ E_{n-m,m,n'-1,m'}(x,y) + xy\ E_{n-m,m,n'-m',m'}(x,y)\end{aligned}$$

*Proof.* By [[6], Proposition 1], The proof of this lemma is trivial. □

**Proposition 3.1.** *Let*

$$\begin{aligned}&\kappa_{n,m,n',m'}(x,y)\\ &= E_{n,m,n',m'}(x,y) + y\sum_{k'=1}^{m'-1} E_{n,m,n'-k',m'}(x,y) + x\sum_{k=1}^{m-1} E_{n-k,m,n',m'}(x,y)\\ &+ xy\sum_{k=1}^{m-1}\sum_{k'=1}^{m'-1} E_{n-k,m,n'-k',m'}(x,y). \qquad (3.5)\end{aligned}$$

*Then* $\kappa_{n,m,n',m'}(x,y) \equiv 1$ *for any* $(x,y) \in [0,1]\times[0,1]$.

*Proof.* We use the mathematical induction. First we show that $\kappa_{2,2,2,2}(x,y) \equiv 1$. By (1.1), (3.1) and (3.5) we have

$$\begin{aligned}\kappa_{2,2,2,2}(x,y) &= E_{2,2,2,2}(x,y) + y\sum_{k'=1}^{m'-1} E_{2,2,2-k',2}(x,y) + x\sum_{k=1}^{m-1} E_{2-k,2,2,2}(x,y)\\ &+ xy\sum_{k=1}^{m-1}\sum_{k'=1}^{m'-1} E_{2-k,2,2-k',2}(x,y) = \Big((1-x)^2+x\Big)\Big((1-y)^2+y\Big)\\ &+ y\Big((1-x)^2+x\Big)(1-y) + x(1-x).\Big((1-y)^2+y\Big)\\ &+ xy(1-x)(1-y) \equiv 1\end{aligned}$$

Now assume that for any $m, m' > 2$, we have $\kappa_{m,m,m',m'}(x,y) \equiv 1$. By (1.1) and (3.1) for every $(x,y) \in [0,1]\times[0,1]$ we have

$$\begin{aligned}E_{m,m,m',m'}(x,y) &= \Big((1-x)^m + x\Big)\Big((1-y)^{m'} + y\Big)\\ &= (1-x)^m(1-y)^{m'} + y(1-x)^m + x(1-y)^{m'} + xy; \qquad (3.6)\end{aligned}$$

$$E_{m-k,m,m'-k',m'}(x,y) = \big(1-x\big)^{m-k}\big(1-y\big)^{m'-k'}; \qquad (3.7)$$

$$E_{m,m,m'-k',m'}(x,y) = \Big((1-x)^m + x\Big)\Big((1-y)^{m'-k'}\Big); \qquad (3.8)$$

and

$$E_{m-k,m,m',m'}(x,y) = \big(1-x\big)^{m-k}\Big(\big(1-y\big)^{m'} + y\Big). \qquad (3.9)$$

Then by (3.5) we have

$$\begin{aligned}
&\kappa_{m+1,m+1,m'+1,m'+1}(x,y)\\
&= E_{m+1,m+1,m'+1,m'+1}(x,y) + y\sum_{k'=1}^{m'-1} E_{m+1,m+1,m'+1-k',m'+1}(x,y)\\
&+ x\sum_{k=1}^{m-1} E_{m+1-k,m+1,m'+1,m'+1}(x,y) + xy\sum_{k=1}^{m-1}\sum_{k'=1}^{m'-1} E_{m+1-k,m+1,m'+1-k',m'+1}(x,y)
\end{aligned}$$

Now by (3.4) and index shifting we obtain

$$\begin{aligned}
&= (1-y)(1-x)\ E_{m,m+1,m',m'+1}(x,y) + (1-x)y\ E_{m,m+1,0,m'+1}(x,y)\\
&+ (1-y)x\ E_{0,m+1,m',m'+1}(x,y) + xy\ E_{0,m+1,0,m'+1}(x,y)\\
&+ y(1-x)E_{m,m+1,m',m'+1}(x,y) + y(1-x)\sum_{k'=1}^{m'-1} E_{m,m+1,m'-k',m'+1}(x,y)\\
&+ yx\ E_{0,m+1,m',m'+1}(x,y) + yx\ \sum_{k'=1}^{m'-1} E_{0,m+1,m'-k',m'+1}(x,y)\\
&+ x(1-y)E_{m,m+1,m',m'+1}(x,y) + x(1-y)\sum_{k=1}^{m-1} E_{m-k,m+1,m',m'+1}(x,y)\\
&+ xy\ E_{m,m+1,0,m'+1}(x,y) + xy\sum_{k=1}^{m-1} E_{m-k,m+1,0,m'+1}(x,y)\\
&+ xyE_{m,m+1,m',m'+1}(x,y) + xy\sum_{k=1}^{m-1} E_{m-k,m+1,m',m'+1}\\
&+ xy\sum_{k'=1}^{m'-1} E_{m,m+1,m'-k',m'+1} + xy\sum_{k=1}^{m-1}\sum_{k'=1}^{m'-1} E_{m-k,m+1,m'-k',m'+1}(x,y),
\end{aligned}$$

using (3.1), after simple calculations, we have

$$\begin{aligned}
&= \big(1-x\big)^{m}\big(1-y\big)^{m'} + y\ (1-x)^{m} + x\ (1-y)^{m'} + xy\\
&+ y\sum_{k'=1}^{m'-1}\Big((1-x)^{m} + x\Big)(1-y)^{m'-k'}\\
&+ x\sum_{k=1}^{m-1}(1-x)^{m-k}\Big((1-y)^{m'} + y\Big)\\
&+ xy\sum_{k'=1}^{m'-1}(1-x)^{m}(1-y)^{m'-k'} + xy\sum_{k=1}^{m-1}(1-x)^{m-k}(1-y)^{m'}
\end{aligned}$$

$$+ xy \sum_{k=1}^{m-1} \sum_{k'=1}^{m'-1} (1-x)^{m-k}(1-y)^{m'-k'} - xy \sum_{k'=1}^{m'-1} (1-x)^{m}(1-y)^{m'-k'}$$
$$- xy \sum_{k=1}^{m-1} (1-x)^{m-k}(1-y)^{m'} = \kappa_{m,m,m',m'}(x,y) \equiv 1,$$

and for $n = m \geq 2$ and $n' = m' \geq 2$, the proof is completed. Now in general assume for any $\forall (x,y) \in [0,1] \times [0,1]$, $\kappa_{n.m,n',m'}(x,y) \equiv 1$ for some $n \geq m$ and $n' \geq m'$. Point of view (3.2), (3.5) and index shifting we have

$$\begin{aligned}
&\kappa_{n+1,m,n',m'}(x,y) \\
&= E_{n+1,m,n',m'}(x,y) + y \sum_{k'=1}^{m'-1} E_{n+1,m,n'-k',m'}(x,y) \\
&+ x \sum_{k=1}^{m-1} E_{n+1-k,m,n',m'}(x,y) + xy \sum_{k=1}^{m-1} \sum_{k'=1}^{m'-1} E_{n+1-k,m,n'-k',m'}(x,y) \\
&= (1-x)\ E_{n,m,n',m'}(x,y) + x\ E_{n+1-m,m,n',m'}(x,y) \\
&+ y(1-x) \sum_{k'=1}^{m'} E_{n,m,n'-k',m'}(x,y) - y(1-x) E_{n,m,n'-m',m'}(x,y) \\
&+ yx \sum_{k'=1}^{m'} E_{n+1-m,m,n'-k',m'}(x,y) - yx E_{n+1-m,m,n'-m',m'}(x,y) \\
&+ x \sum_{k=1}^{m} E_{n+1-k,m,n',m'}(x,y) - x E_{n+1-m,m,n',m'}(x,y) \\
&+ xy \sum_{k=1}^{m} \sum_{k'=1}^{m'} E_{n+1-k,m,n'-k',m'}(x,y) + xy E_{n+1-m,m,n'-m',m'}(x,y) \\
&- xy \sum_{k=1}^{m} E_{n+1-k,m,n'-m',m'}(x,y) - xy \sum_{k'=1}^{m'} E_{n+1-m,m,n'-k',m'}(x,y).
\end{aligned}$$

Then

$$\begin{aligned}
\kappa_{n+1,m,n',m'}(x,y) &= E_{n,m,n',m'}(x,y) + y \sum_{k'=1}^{m'-1} E_{n,m,n'-k',m'}(x,y) \\
&+ x \sum_{k=1}^{m-1} E_{n-k,m,n',m'}(x,y) + xy \sum_{k=1}^{m-1} \sum_{k'=1}^{m'-1} E_{n-k,m,n'-k',m'}(x,y) \\
&= \kappa_{n,m,n',m'}(x,y) \equiv 1.
\end{aligned}$$

Also by using (3.3) and (3.5) we obtain

$$\kappa_{n,m,n'+1,m'}(x,y) = \kappa_{n,m,n',m'}(x,y) \equiv 1.$$

Then the proof is completed. □

**Proposition 3.2.** *Let*

$$\Delta_{n,m,n',m'}(x,y) = E_{n,m,n',m'}(x,y) - E_{n,m,n'-1,m'}(x,y) - E_{n-1,m,n',m'}(x,y) + E_{n-1,m,n'-1,m'}(x,y). \tag{3.10}$$

*Then for all $(x,y) \in [0,1]\times[0,1]$ and $n > m > 1$ and $n' > m' > 1$, the following inequality are valid.*

$$\left|\Delta_{n,m,n',m'}(x,y)\right| \leq \{x(m-1)\}^{\left[\frac{n-m}{m-1}\right]} \{y(m'-1)\}^{\left[\frac{n'-m'}{m'-1}\right]}. \tag{3.11}$$

*Proof.* From (3.2),(3.3),(3.4) and (3.10), we have

$$\Delta_{n,m,n',m'}(x,y) = xy\,\Big[E_{n-1,m,n'-1,m'}(x,y) - E_{n-1,m,n'-m',m'}(x,y) - E_{n-m,m,n'-1,m'}(x,y) + E_{n-m,m,n'-m',m'}(x,y)\Big]$$

and consequently we have

$$\Delta_{n,m,n',m'}(x,y) = xy\sum_{k=1}^{m-1}\sum_{k'=1}^{m'-1}\Big[E_{n-k,m,n'-k',m'} - E_{n-k,m,n'-k'-1,m'} - E_{n-k-1,m,n'-k',m'} + E_{n-k-1,m,n'-k'-1,m'}\Big]. \tag{3.12}$$

on the other hand, from (3.10) we obtain

$$\Delta_{n-k,m,n'-k',m'}(x,y) = E_{n-k,m,n'-k',m'}(x,y) - E_{n-k,m,n'-k'-1,m'}(x,y) - E_{n-k-1,m,n'-k',m'}(x,y) + E_{n-k-1,m,n'-k'-1,m'}(x,y).$$

where $k < m-n$ and $k' < m'-n'$. Then we have

$$\Delta_{n,m,n',m'}(x,y) = xy\sum_{k=1}^{m-1}\sum_{k'=1}^{m'-1}\Delta_{n-k,m,n'-k',m'}(x,y). \tag{3.13}$$

Taking in (3.13) $n-1, n-2, ..., n-m+1 \quad n'-1, n'-2, ..., n'-m'+1$ instead of $n,\ n'$, we successively obtain that

$$\Delta_{n-k,m,n'-k',m'}(x,y) = xy\sum_{l=1}^{m-1}\sum_{l'=1}^{m'-1}\Delta_{n-k-l,m,n'-k'-l',m'}(x,y) \tag{3.14}$$

where $k = 1,2,...,m-1 \quad k' = 1,2,...,m'-1$. Applying (3.14) in (3.13) $\left[\frac{n-m}{m-1}\right]$-times and taking into account that $|\Delta_{n,m,n',m'}(x,y)| \leq 1$ (by (3.5) and $E_{n,m,n',m'}(x,y) \geq 0$) we have

$$\left|\Delta_{n,m,n',m'}(x,y)\right| \leq \{x(m-1)\}^{\left[\frac{n-m}{m-1}\right]} \{y(m'-1)\}^{\left[\frac{n'-m'}{m'-1}\right]}.$$

□

**Corollary 3.1.** *Let the conditions of Proposition 3.2 be valid. We have*

$$E_{n-1,m,n'-1,m'}(x,y) = E_{n,m,n'-1,m'}(x,y) + E_{n-1,m,n',m'}(x,y) - E_{n,m,n',m'}(x,y) + \gamma'_{n,n'}(x,y) \tag{3.15}$$

$$E_{n,m,n',m'}(x,y) = \frac{1}{[1+(m-1)x][1+(m'-1)y]} + \gamma_{n,n'}(x,y). \tag{3.16}$$

*where* $\lim_{n\to\infty}\lim_{n'\to\infty}\gamma_{n,n'}(x,y) = 0$ *uniformly on the closed square* $[0,\alpha]\times[0,\beta]$ *such that* $0<\alpha<\frac{1}{m-1}$ $\quad 0<\beta<\frac{1}{m'-1}$.

*Proof.* Equation (3.15) follows from (3.11) directly. By relation (14) in [5], we have

$$E_{n,m,n'-1,m'}(x,y) = E_{n,m,n',m'}(x,y) + A_{n,m}(x)\beta_{n'}(y) \tag{3.17}$$

$$E_{n-1,m,n',m'}(x,y) = E_{n,m,n',m'}(x,y) + A_{n',m'}(y)\alpha_n(x), \tag{3.18}$$

where $\beta_{n'}(y)$ and $\alpha_n(x)$ tend to zero uniformly as $n,n'\to\infty$ on $[0,\alpha]$ and $[0,\beta]$, respectively. Then the equality (3.15), is as following form

$$\begin{aligned} &E_{n-1,m,n'-1,m'}(x,y) = \\ &E_{n,m,n',m'}(x,y) + A_{n,m}(x)\beta_{n'}(y) + A_{n',m'}(y)\alpha_n(x) + \alpha_n(x)\beta_{n'}(y) \end{aligned} \tag{3.19}$$

Because of (3.17),(3.18) and (3.19) the equality (3.5) may be written (for large $n,n'$) in the form

$$\begin{aligned} &E_{n,m,n',m'}(x,y) + y(m'-1)\left\{E_{n,m,n',m'}(x,y) + A_{n,m}(x)\delta_{n'}(y)\right\} \\ &+ x(m-1)\left\{E_{n,m,n',m'}(x,y) + A_{n',m'}(y)\eta_n(x)\right\} \\ &+ xy(m-1)(m'-1)\left\{E_{n,m,n',m'}(x,y) + A_{n,m}(x)\delta_{n'}(y) + A_{n',m'}(y)\eta_n(x)\right. \\ &\left.+\eta_n(x)\delta_{n'}(y)\right\} \equiv 1, \end{aligned}$$

where thereupon $\delta_{n'}(y)$ and $\eta_n(x)$ tend to zero uniformly as $n,n'\to\infty$. So

$$\begin{aligned} &E_{n,m,n',m'}(x,y) + y(m'-1)E_{n,m,n',m'}(x,y) + x(m-1)E_{n,m,n',m'}(x,y) \\ &+ xy(m-1)(m'-1)E_{n,m,n',m'}(x,y) + (y(m'-1))A_{n',m'}(y)\eta_n(x) \\ &+ (x(m-1))A_{n,m}(x) \\ &\times \delta_{n'}(y) + (xy(m-1)(m'-1))\Big(A_{n,m}(x)\delta_{n'}(y) + A_{n',m'}(y)\eta_n(x) + \eta_n(x)\delta_{n'}(y)\Big) \equiv 1. \end{aligned}$$

Then, we get

$$\begin{aligned} E_{n,m,n',m'}(x,y) &= \frac{1}{[1+(m-1)x][1+(m'-1)y]} - \frac{((m'-1)y)A_{n',m'}(y)\eta_n(x)}{[1+(m'-1)y]} \\ &- \frac{((x(m-1)))A_{n,m}(x)\delta_{n'}(y)}{[1+(m-1)x]} - \frac{(xy(m-1)(m'-1))\eta_n(x)\delta_{n'}(y)}{[1+(m-1)x][1+(m'-1)y]} \end{aligned}$$

Now by considering,

$$\begin{aligned} &\gamma_{n,n'}(x,y) := \\ &\qquad -\Big[\frac{((m'-1)y)A_{n',m'}(y)\eta_n(x)}{[1+(m'-1)y]} + \frac{((x(m-1)))A_{n,m}(x)\delta_{n'}(y)}{[1+(m-1)x]} \\ &\qquad\qquad + \frac{(xy(m-1)(m'-1))\eta_n(x)\delta_{n'}(y)}{[1+(m-1)x][1+(m'-1)y]}\Big], \end{aligned}$$

we get

$$E_{n,m,n',m'}(x,y) = \frac{1}{[1+(m-1)x][1+(m'-1)y]} + \gamma_{n,n'}(x,y).$$

□

## 4. Main Result

In this section we give following theorem about convergence of a two variable generalized Bernstein polynomials on square as (1.2).

**Theorem 4.1.** *Let* $0 < \alpha < \frac{1}{m-1}$, $0 < \beta < \frac{1}{m'-1}$ *and* $m, m' > 1$. *For every* $f \in C_{[0,1]\times[0,1]}$, *the sequence* $F_{n,m,\ n',m'}\ (f; x, y)$, *is defined in* (1.2), *converges to* $f$, *that is,*

$$\lim_{n\to\infty}\lim_{n'\to\infty} \left\|F_{n,m,\ n',m'}\ (f; x_1, x_2) - f\right\|_{C_{[0,\alpha]\times[0,\beta]}} = 0$$

*Proof.* Using Theorem A, it is sufficient to verify the following conditions

$$\lim_{n\to\infty}\lim_{n'\to\infty} \|F_{n,m,\ n',m'}\ (1; x_1, x_2) - 1\|_{C_{[0,\alpha]\times[0,\beta]}} = 0; \tag{4.1}$$

$$\lim_{n\to\infty}\lim_{n'\to\infty} \|F_{n,m,\ n',m'}(t_j; x_1, x_2) - x_j\|_{C_{[0,\alpha]\times[0,\beta]}} = 0, \qquad j = 1,2; \tag{4.2}$$

$$\lim_{n\to\infty}\lim_{n'\to\infty} \|F_{n,m,\ n',m'}(t_1^2 + t_2^2; x_1, x_2) - x_1^2 + x_2^2\|_{C_{[0,\alpha]\times[0,\beta]}} = 0. \tag{4.3}$$

By (3.1) and (3.16), we have

$$\begin{aligned}
&\|F_{n,m,\ n',m'}\ (1; x_1, x_2) - 1\|_{C_{[0,\alpha]\times[0,\beta]}} \\
&\qquad = \sup_{(x_1,x_2)\in[0,\alpha]\times[0,\beta]} \Big([1+(m-1)x_1][1+(m'-1)x_2]\Big)|\gamma_{n,n'}(x_1,x_2)| \\
&\qquad \le 4 \sup_{(x_1,x_2)\in[0,\alpha]\times[0,\beta]} |\gamma_{n,n'}(x_1,x_2)|, \qquad (4.4)
\end{aligned}$$

where $\gamma_{n,n'}(x_1, x_2)$ uniformly converges to zero on $[0,\alpha]\times[0,\beta]$. Hence the condition of (4.1) is fulfilled. Now It is easy to see that

$$\begin{aligned}
&F_{n,m,\ n',m'}\ (t_1; x_1, x_2) \\
&= \Big(1 - \Big(m-1\Big)x_1\Big)\Big(1 - \Big(m'-1\Big)x_2\Big) \times \sum_{i=0}^{[\frac{n}{m}]}\sum_{j=0}^{[\frac{n'}{m'}]} P_{n,m}^i(x_1) P_{n',m'}^j(x_2) \Big(\frac{i}{n-(m-1)i}\Big).
\end{aligned}$$

Now using the identity (14) in [6], we have

$$\begin{aligned}
&F_{n,m,\ n',m'}\ (t_1; x_1, x_2) \\
&= \Big[A_{n',m'}(x_2)\big((1+(m'-1)x_2\big)\Big]\Big[\big(1+(m-1)x_1\big)x_1 A_{n-m,m}(x_1)\Big] \\
&= \Big[A_{n',m'}(x_2)\big((1+(m'-1)x_2\big)\Big]\Big[\big(1+(m-1)x_1\big)x_1\Big(A_{n,m}(x_1) + \alpha_n(x_1)\Big)\Big] \\
&= \big(1+(m-1)x_1\big)\big((1+(m'-1)x_2\big)x_1 E_{n,m,n',m'}(x_1,x_2) \\
&\qquad\qquad + A_{n',m'}(x_2)\big((1+(m'-1)x_2\big)\big(1+(m-1)x_1\big)x_1\alpha_n(x_1)
\end{aligned}$$

and thanks (3.16) and the (3.18) for large n, we have

$$\begin{aligned}&F_{n,m,\ n',m'}\ (t_1;x_1,x_2)\\&=x_1+\big(1+(m-1)x_1\big)\big((1+(m'-1)x_2\big)x_1\gamma_{n,n'}(x_1,x_2)\\&\qquad+A_{n',m'}(x_2)\big((1+(m'-1)x_2\big)\big(1+(m-1)x_1\big)x_1\alpha_n(x_1)\end{aligned}$$

Therefore we have

$$\begin{aligned}&\|F_{n,m,\ n',m'}\ (t_1;x_1,x_2)-x_1\|_{C_{[0,\alpha]\times[0,\beta]}}\\&=\sup_{(x_1,x_2)\in[0,\alpha]\times[0,\beta]}\Big[\big(1+(m-1)x_1\big)\big((1+(m'-1)x_2\big)x_1\Big]|\gamma_{n,n'}(x_1,x_2)|\\&\qquad+\Big[A_{n',m'}(x_2)\big((1+(m'-1)x_2\big)\big(1+(m-1)x_1\big)x_1\Big]|\alpha_n(x_1)|\end{aligned}\tag{4.5}$$

then the condition (4.2) for $j=1$ holds. Also by same method the condition (4.2) for $j=2$ holds. To complete the proof, consider

$$\begin{aligned}&F_{n,m\ n',m'}\Big(t_1^2+t_2^2;x_1,x_2\Big)\\&=\Big(1+\Big(m-1\Big)x_1\Big)\Big(1+\Big(m'-1\Big)x_2\Big)\\&\times\sum_{i=0}^{[\frac{n}{m}]}\sum_{j=0}^{[\frac{n'}{m'}]}P^i_{n,m}(x_1)P^j_{n',m'}(x_2)\Big[(\frac{i}{n-(m-1)i})^2+(\frac{j}{n'-(m'-1)j})^2\Big],\\&=\Big(1+\Big(m-1\Big)x_1\Big)\sum_{i=0}^{[\frac{n}{m}]}P^i_{n,m}(x_1)\Big[(\frac{i}{n-(m-1)i})^2\Big]\\&\times\Big(1+\Big(m'-1\Big)x_2\Big)\sum_{j=0}^{[\frac{n'}{m'}]}P^j_{n',m'}(x_2)+\Big(1+\Big(m-1\Big)x_1\Big)\sum_{i=0}^{[\frac{n}{m}]}P^i_{n,m}(x_1)\\&\times\Big(1+\Big(m'-1\Big)x_2\Big)\sum_{j=0}^{[\frac{n'}{m'}]}P^j_{n',m'}(x_2)\Big[(\frac{j}{n'-(m'-1)j})^2\Big].\end{aligned}\tag{4.6}$$

By letting

$$p_{n,m}(x)=\sum_{k=1}^{[\frac{n}{m}]}\frac{1}{n-(m-1)k}\ C^{k-1}_{n-(m-1)k-1}\ x^{k-1}(1-x)^{n-mk}$$

$$q_{n,m}(x)=\sum_{k=1}^{[\frac{n}{m}]}\frac{k-1}{n-(m-1)k}\ C^{k-1}_{n-(m-1)k-1}\ x^{k-1}(1-x)^{n-mk}.$$

The following facts are valid (see [6]):

$$\Big(1+\Big(m-1\Big)x\Big)\sum_{i=0}^{[\frac{n}{m}]} P^i_{n,m}(x)\Big[(\frac{i}{n-(m-1)i})^2\Big]$$
$$\leq x\frac{m}{n}+2x\ \frac{m}{n}\Big(\beta_n(x)+\zeta_n(x)\Big)+2x^2\Big(\beta_n(x)+\eta_n(x)\Big)+x^2$$

where $\zeta_n(x),\ \beta_n(x),\ \eta_n(x)$ tends to zero uniformly on $[0,\alpha]$ as $n\to\infty$. We apply above inequality for both parts of (4.6), then

$$F_{n,m\ n',m'}\Big(t_1^2+t_2^2;x_1,x_2\Big)$$
$$\leq \Big[\Big(1+\Big(m'-1\Big)x_2\Big)\ A_{n',m'}(x_2)\Big]\Big(x_1\frac{m}{n}+2x_1\ \frac{m}{n}\Big(\beta_n(x_1)+\zeta_n(x_1)\Big)$$
$$+2x_1^2\Big(\beta_n(x_1)+\eta_n(x_1)\Big)+x_1^2\Big)+\Big[\Big(1+\Big(m-1\Big)x_1\Big)\ A_{n,m}(x_1)\Big]$$
$$\times\Big(x_2\frac{m'}{n'}+2x_2\ \frac{m'}{n'}\Big(\beta_{n'}(x_2)+\zeta_{n'}(x_2)\Big)+2x_2^2\Big(\beta_{n'}(x_2)+\eta_{n'}(x_2)\Big)+x_2^2\Big)$$
$$\leq \Big[\Big(1+(m'-1)x_2\Big)\ A_{n',m'}(x_2)\Big]\Big(\phi_{n,m}(x_1)+x_1^2\Big)$$
$$+\Big[(1+\Big(m-1)x_1\Big)\ A_{n,m}(x_1)\Big]\Big(\psi_{n',m'}(x_2)+x_2^2\Big)$$
$$\leq\Big(1-\Big((m'-1)x_2\Big)\ \rho_{n'}(x_2)\Big)\Big(\phi_{n,m}(x_1)+x_1^2\Big)$$
$$+\Big(1-\Big(\Big(m-1\Big)x_1\Big)\ \varrho_n(x_1)\Big)\Big(\psi_{n',m'}(x_2)+x_2^2\Big).$$

Then one can write

$$\Big\|F_{n,m,\ n',m'}(t_1^2+t_2^2;x_1,x_2)-\big(x_1^2+x_2^2\big)|\Big\|_{C_{[0,\alpha]\times[0,\beta]}}$$
$$\leq \sup_{(x_1,x_2)\in[0,\alpha]\times[0,\beta]}\Big|\phi_{n,m}(x_1)-\Big((m'-1)x_2\Big)\ \rho_{n'}(x_2)\ x_1^2+\Big((m'-1)x_2\Big)\ \rho_{n'}(x_2)$$
$$\times\ \phi_{n,m}(x_1)+\psi_{n',m'}(x_2)+\Big(\Big(m-1\Big)x_1\Big)\ \varrho_n(x_1)\ x_2^2+\Big(\Big(m-1\Big)x_1\Big)\ \varrho_n(x_1)\psi_{n',m'}(x_2)\Big|.$$

Then by definition of $\phi_{n,m}(x_1)$ and $\psi_{n',m'}(x_2)$, we have

$$\lim_{n\to\infty}\lim_{n'\to\infty}\Big\|F_{n,m,\ n',m'}(t_1^2+t_2^2;x_1,x_2)-x_1^2+x_2^2\Big\|_{C_{[0,\alpha]\times[0,\beta]}}=0.$$

□

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