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CUBIC BEZIER-LIKE TRANSITION CURVES WITH NEW BASIS FUNCTIONS

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Abstract. A cubic trigonometric Bézier-like curve similar to the cubic Bézier curve, with a shape parameter, is presented in this work. With the suitable shape parameter, the cubic trigonometric Bézier-like curves can be made close to the cubic Bézier curves. Then we investigate the condition of smooth connection between non-adjacent cubic trigonometric Bézier-like curves.

1. Introduction

Curve and surface design is very important because it finds many applications in different areas. Since Bezier and B-spline curves are insufficient to identify some curves, it is necessary to find new curve families. In this sense new curve families have been defined by the use of trigonometric functions [2],[7],[6] and [5].

The first objective of this study is to examine the basic properties of a new Bezier-like curve. For $\lambda = 1$ these new basis functions are the same as basis functions defined in [6]. Therefore our new basis functions can be considered a generalization of the basis functions defined by Sharma in [6].

On the other hand, since the transition curve has been used in road, railway and satellite orbit widely, it is an important research area. There is a wide range of literature available on this field, see [3] and [4] and references therein. For transition curves, it is important to find continuity conditions for such curves.For G^0 continuity, the endpoint points of two curves at the joint must have the same value. For G^1 continuity, the tangents of the two curves at the joint must have the same direction. For G^2, the two curves must have the same curvature at the linked point. The second purpose of this work is to investigate appropriate G^0, G^1 and G^2 continuity conditions for curves defined by these new basis functions. And finally, by using these basis functions we have shown the conditions which are the transition curve connecting two non-adjacent cubic curves.

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2. Cubic trigonometric Bézier-like basis function

Definition 2.1. For a arbitrarily selected real value of λ, where $\lambda \in [0,1]$, the following four functions of t ($t \in [0,1]$) are defined as cubic trigonometric Bézier-like basis functions with a shape parameters λ :

$$\begin{aligned} b_0(t) &= \frac{1}{4}(1-\lambda \sin \frac{\pi}{2}t)(1-\sin \frac{\pi}{2}t)^2 \\ b_1(t) &= \frac{1}{2}\left\{1-\frac{1}{2}(1-\lambda \cos \frac{\pi}{2}t)(1-\cos \frac{\pi}{2}t)^2\right\} \\ b_2(t) &= \frac{1}{2}\left\{1-\frac{1}{2}(1-\lambda \sin \frac{\pi}{2}t)(1-\sin \frac{\pi}{2}t)^2\right\} \\ b_3(t) &= \frac{1}{4}(1-\lambda \cos \frac{\pi}{2}t)(1-\cos \frac{\pi}{2}t)^2 \end{aligned} \tag{2.1}$$

Remark 2.1. It is easy to see that we obtain basis functions defined in [6] for $\lambda = 1$ in Definition 2.1.

Theorem 2.1. *The basis functions (2.1) have the following properties:*

(a) Nonnegativity : $b_i(t) \geq 0$ for $i = 0,1,2,3$.
(b) Partition of unity: $\sum_{i=0}^{3} b_i(t) = 1$
(c) Monotonicity: For a given parameter t, as the shape parameter λ increases, $b_0(t)$ and $b_3(t)$ decreases and as the shape parameter λ decreases, $b_1(t)$ and $b_2(t)$ increases.
(d) Symmetry: $b_i(t;\lambda) = b_{3-i}(1-t;\lambda)$, for $i = 0,1,2,3$.

Proof. (a) For $t \in [0,1]$ and $\lambda \in [0,1]$, then $1-\sin\frac{\pi}{2}t \geq 0, 1-\cos\frac{\pi}{2}t \geq 0, 0 \leq (1-\lambda\sin\frac{\pi}{2}t) \leq 1, 0 \leq (1-\lambda\cos\frac{\pi}{2}t) \leq 1, 0 \leq \frac{1}{4}(1-\lambda\sin\frac{\pi}{2}t) \leq 1$ and $0 \leq \frac{1}{4}(1-\lambda\cos\frac{\pi}{2}t) \leq 1$. It is obvious that $b_i(t) \geq 0$ for $i = 0,1,2,3$.

(b) $\sum_{i=0}^{3} b_i(t) = \frac{1}{4}(1-\lambda\sin\frac{\pi}{2}t)(1-\sin\frac{\pi}{2}t)^2 + \frac{1}{2}\left\{1-\frac{1}{2}(1-\lambda\cos\frac{\pi}{2}t)(1-\cos\frac{\pi}{2}t)^2\right\} + \frac{1}{2}\left\{1-\frac{1}{2}(1-\lambda\sin\frac{\pi}{2}t)(1-\sin\frac{\pi}{2}t)^2\right\} + \frac{1}{4}(1-\lambda\cos\frac{\pi}{2}t)(1-\cos\frac{\pi}{2}t)^2 = 1$.

The remaining cases follow obviously.

In Fig. 1 cubic trigonometric Bézier-like basis function are plotted by taking t on x axis and $b_i(t)$ on y axis, for $\lambda = 0$ (dash-dotted lines) , $\lambda = 0.5$ (solid lines) and $\lambda = 1$ (dashed lines) where $b_0(t), b_1(t), b_2(t)$ and $b_3(t)$ are denoted by blue lines, red lines, green lines and black lines respectively.

□

3. Cubic trigonometric Bézier-like curve

Definition 3.1. Given points $P_i (i = 0,1,2,3)$ in $\mathbb{R}^2$ or $\mathbb{R}^3$, then

$$r(t) = \sum_{i=0}^{3} P_i b_i(t), \quad t \in [0,1], \lambda \in [0,1] \tag{3.1}$$

is called a cubic trigonometric Bézier-like curve with a shape parameter λ.

Theorem 3.1. *The cubic trigonometric Bézier-like curves (3.1) have the following properties:*

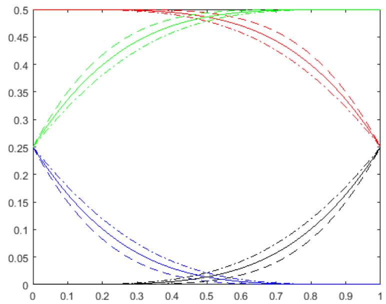

FIGURE 1. Cubic trigonometric Bézier-like basis function with different values of shape parameter

(a) Terminal properties :

$$\begin{aligned}
r(0) &= \frac{1}{4}P_0 + \frac{1}{2}P_1 + \frac{1}{4}P_2 \qquad (3.2)\\
r(1) &= \frac{1}{4}P_1 + \frac{1}{2}P_2 + \frac{1}{4}P_3\\
r'(0) &= (\frac{\lambda\pi}{8} + \frac{\pi}{4})(P_2 - P_0) + \frac{1}{2}(P_2 + P_1)\\
r'(1) &= (\frac{\lambda\pi}{8} + \frac{\pi}{4})(P_3 - P_1) + \frac{1}{2}(P_2 + P_1)\\
r''(0) &= (\frac{\lambda\pi^2}{4} + \frac{\pi^2}{8})(P_0 - P_2) + \frac{1}{2}(P_2 + P_1)\\
r''(1) &= (\frac{\lambda\pi^2}{4} + \frac{\pi^2}{8})(P_3 - P_1) + \frac{1}{2}(P_2 + P_1)
\end{aligned}$$

(b) Symmetry : P_0, P_1, P_2, P_3 *and* P_3, P_2, P_1, P_0 *define the same cubic trigonometric Bézier-like curve in different parametrizations, i.e ,*

$$r(t;\lambda;P_0,P_1,P_2,P_3) = r(1-t;\lambda;P_3,P_2,P_1,P_0), \quad t \in [0,1], \lambda \in [0,1] \qquad (3.3)$$

(c) Geometric invariance : The shape of the cubic trigonometric Bézier-like curve is independent of the choice of coordinates, i.e.,(3.1) satisfies the following two equations:

$$\begin{aligned}
r(t;\lambda;P_0+q,P_1+q,P_2+q,P_3+q) &= r(t;\lambda;P_0,P_1,P_2,P_3)+q, \qquad (3.4)\\
r(t;\lambda;P_0*T,P_1*T,P_2*T,P_3*T) &= r(t;\lambda;P_0,P_1,P_2,P_3)*T,
\end{aligned}$$

where q is arbitrary vector in $\mathbb{R}^2$ or $\mathbb{R}^3$, and T is an arbitrary $d \times d$ matrix, $d = 2$ or 3.

(d) Convex hull property: The entire cubic trigonometric Bézier-like curve segment lies inside its control polygon spanned by P_0, P_1, P_2, P_3.

4. Shape Control of the Cubic trigonometric Bézier-like curve

For $t \in [0,1]$, we rewrite (3.1) as follows:

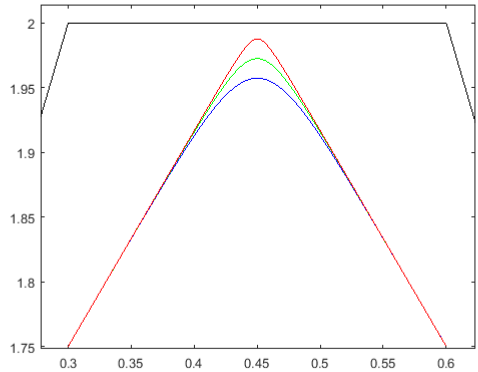

FIGURE 2. Effect of shape parameter on the cubic trigonometric Bézier-like curves

$$r(t) = \sum_{i=0}^{3} P_i c_i(t) + \frac{1}{4}(\sin\frac{\pi}{2}t)(\sin\frac{\pi}{2}t - 2)(1 - \lambda\sin\frac{\pi}{2}t)(P_0 - P_2) \qquad (4.1)$$
$$+ \frac{1}{4}(\cos\frac{\pi}{2}t)(\cos\frac{\pi}{2}t - 2)(1 - \lambda\cos\frac{\pi}{2}t)(P_3 - P_1)$$

where $c_0(t) = \frac{1}{4}(1 - \lambda\sin\frac{\pi}{2}t), c_1(t) = \frac{1}{2}(1 - \frac{1}{2}(1 - \lambda\cos\frac{\pi}{2}t)), c_2(t) = \frac{1}{2}(1 - \frac{1}{2}(1 - \lambda\sin\frac{\pi}{2}t)), c_3(t) = \frac{1}{4}(1 - \lambda\cos\frac{\pi}{2}t)$.

Obviously, shape parameter λ affect curves only on the control points $(P_0 - P_2)$ and $(P_3 - P_1)$ respectively. As λ increases, the curve moves in the direction of the control points $(P_0 - P_2)$ and $(P_3 - P_1)$ and as λ decreases the curve moves in the opposite direction to the control points $(P_0 - P_2)$ and $(P_3 - P_1)$.In Fig. 2 the effect of shape parameter λ on the cubic trigonometric Bézier-like curve for $\lambda = 1$ (blue line), $\lambda = 0.5$ (green line), $\lambda = 0$ (red line) is illustrated.

5. Approximability

Here we show the relations of the cubic trigonometric Bézier-like curves and cubic Bézier curves corresponding to their control polygon.

Suppose P_0, P_1, P_2 and P_3 are not collinear ; the relationship between cubic trigonometric Bézier-like curve $r(t)$, and the cubic Bézier curve $B(t) = \sum_{i=0}^{3} P_i \left(\begin{array}{c} 3 \\ i \end{array}\right)(1-t)^{3-i}t^i$ with the same control points $P_i (i = 0, 1, 2, 3)$ are as follows:

$r(0) = \frac{1}{4}\left[P_0 + 2P_1 + P_2\right], r(1) = \frac{1}{4}\left[P_1 + 2P_2 + P_3\right], P_0 = B(0), P_3 = B(1)$.

Where $P^* = \frac{1}{2}(P_1 + P_2)$ and $0 < (\frac{1}{2} - \frac{\lambda\sqrt{2}}{4}) < 1$ for $\lambda \in [0, 1]$,

$B(\frac{1}{2}) - P^* = \frac{1}{8}(P_0 - P_1 - P_2 + P_3)$ and we have

$$\begin{aligned} r(\frac{1}{2}) - P^* &= \frac{1}{16}(2-\lambda\sqrt{2})(3-2\sqrt{2})(P_0 - P_1 - P_2 + P_3) \qquad (5.1)\\ &= \frac{1}{4}(\frac{1}{2} - \frac{\lambda\sqrt{2}}{4})(3-2\sqrt{2})(P_0 - P_1 - P_2 + P_3)\\ &\leq (\frac{1}{2} - \frac{\lambda\sqrt{2}}{4})\frac{1}{8}(P_0 - P_1 - P_2 + P_3)\\ &\leq (\frac{1}{2} - \frac{\lambda\sqrt{2}}{4})(B(\frac{1}{2}) - P^*). \end{aligned}$$

From here we conclude that cubic trigonometric Bézier-like curves are closer to the control polygon than the cubic Bézier curves.

6. Transition Curves Between Non-adjacent Cubic Trigonometric Bézier-like curves

Let

$$r_1(t) = P_0 b_0(t) + P_1 b_1(t) + P_2 b_2(t) + P_3 b_3(t) \qquad (6.1)$$

$$r_2(t) = Q_0 b_0(t) + Q_1 b_1(t) + Q_2 b_2(t) + Q_3 b_3(t) \qquad (6.2)$$

be two nonadjacent cubic trigonometric Bézier-like curves. We can smoothly connect non-adjacent cubic trigonometric Bézier-like curves a transition curves defined by

$$r_3(t) = R_0 b_0(t) + R_1 b_1(t) + R_2 b_2(t) + R_3 b_3(t) \qquad (6.3)$$

The method of transition curve construction between nonadjacent cubic trigonometric Bézier-like curves as shown in Fig. 3.

For the G^0 smooth connect conditions

$$\begin{aligned} r_1(1) &= r_3(0) \qquad (6.4)\\ r_3(1) &= r_2(0) \end{aligned}$$

and for the G^1 smooth connect conditions we need the show

$$\begin{aligned} r_1'(1) &= r_3'(0) \qquad (6.5)\\ r_3'(1) &= r_2'(0) \end{aligned}$$

Theorem 6.1. *Let two nonadjacent cubic trigonometric Bézier-like curves $r_1(t)$ and $r_2(t)$ defined by (6.1) and (6.2) respectively. If we choose $P_1 = -P_2, Q_1 = -Q_2$, $R_1 = -R_2$ and $m = n$,$r_3(t)$ is a cubic trigonometric Bézier-like transition curve between $r_1(t)$ and $r_2(t)$ curves.*

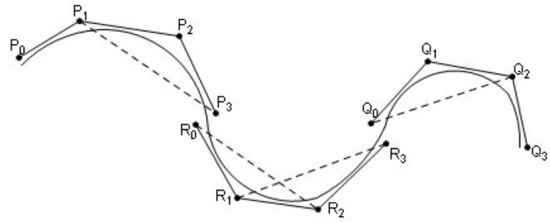

FIGURE 3. [1] Transition curve with non-adjacent cubic trigonometric Bézier-like curves

Proof. The endpoint points properties of $r_1(1)$ and $r_3(0)$ should satisfy G^0, G^1 smooth connect conditions. And we can get the two equations from (6.4) and (6.5) as follows :

$$P_1 + 2P_2 + P_3 = R_0 + 2R_1 + R_2 \tag{6.6}$$

$$R_2 - R_0 = m(P_3 - P_1) \tag{6.7}$$

The endpoint points properties of $r_3(1)$ and $r_2(0)$ should satisfy G^0, G^1 smooth connect conditions. And we can get the two equations from (6.4) and (6.5) as follows :

$$R_1 + 2R_2 + R_3 = Q_0 + 2Q_1 + Q_2 \tag{6.8}$$

$$R_3 - R_1 = m(Q_2 - Q_0). \tag{6.9}$$

And for the G^2 smooth connect conditions

$$\kappa_1(1) = \kappa_3(0) = 0 \tag{6.10}$$

$$\kappa_3(1) = \kappa_2(0) = 0$$

where $\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$ and $\kappa_1(t)$ is curvature of $r_1(t), \kappa_2(t)$ is curvature of $r_2(t)$ and $\kappa_3(t)$ is curvature of $r_3(t)$. So we prove G^2 smooth connect conditions. Combined equations (6.6)-(6.9),we can get an equation system which contains 4 equations and 4 unknowns. If we solve these equations then we have that $R_0, R_1 = -R_2, R_3, m = n$. □

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