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# ON SOME THEORETIC-FUNCTIONAL RESULTS CONCERNING THE THEORY OF EXTREMALITY AND THEIR APPLICATION

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**Abstract**. In this paper we consider some questions of measure theory and the theory of oscillatory integrals which have applications in the theory of extremality of manifolds. We obtain new results concerning these questions.

## 1. Introduction

Measure Theory has many applications in various questions of the Mathematical and Functional Analysis. This theory stands on the base of many fundamental results in almost all branches of mathematics. Some important properties of real numbers concerning their approximation by rational numbers are satisfied for almost all real numbers in the Lebesgue meaning only.

Let's consider at first the basic notions of the theory of extremal manifolds. Let we are given some set of real functions i. e. $f_1(x), ..., f_n(x)$ defined on all real axes or on its subset. In the theory of Diophantine Approximations one considers the question on simultaneous approximation of the values of these functions by rational fractions with fixed denominators. It is best known (see [4], [8]) that for every real number $\alpha$ it is possible to pick up a non-reducible rational number $a/q$ such that

$$\left|\alpha - \frac{a}{q}\right| \leq \frac{1}{q^2}. \tag{1.1}$$

Denoting by $\|\alpha\|$ a distance from $\alpha$ to the nearest integral number, we can write (1.1) as follows

$$\|q\alpha\| \leq q^{-1}.$$

For various real numbers there are own values for the denominator $q$. Let now we are given several real numbers $\alpha_1 ..., \alpha_n$. If we take approximations of a view (1.1), with one and the same denominator for the fractions, we cannot await so exact approximations.

In the work (see [7]) A. Khintschine showed that the system of inequalities

$$\max\left(\|tq\|, \left\|t^2 q\right\|, ..., \|t^n q\|\right) < \delta q^{-1/n}$$

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has infinite set of solutions in positive integral numbers $q > 0$ for almost all real $t$ in the Lebesgue sense. It means that for almost all real $t$ there are infinitely many natural numbers $q_1, q_2, ..., q_m, ...$ such that for every natural $m$ the following inequalities are satisfied:

$$\left| t^k - \frac{a_k}{q_m} \right| \leq \delta q_m^{-1-1/n},\ k = 1, ..., n.$$

To define the basic notions of the theory, we consider the system of inequalities

$$\max\left(\|\alpha_1 q\|, \|\alpha_2 q\|, ..., \|\alpha_n q\|\right) < q^{-u}, u > 0, \tag{1.2}$$

Let $u(\alpha_1, ..., \alpha_n)$ be defined as a *sup* of such $u > 0$ for which (1.2) is satisfied for infinite set of natural numbers $q$. It is not difficult to show that $u(\alpha_1, ..., \alpha_n) \geq 1/n$ (see [8]). From this definition it follows that the inequality (1.2) is satisfied for infinitely many natural numbers $q$ when $u < 1/n$. When $u(\alpha_1, ..., \alpha_n) = 1/n$ for almost all points of the variety $(\alpha_1, ..., \alpha_n) \in R^n$ of less dimension, then we call it as an *extremal manifold.*

## 2. Auxiliary results

The following lemma is known as Borel-Cantelli's lemma and plays an important role in the questions concerning extremality of manifolds ([8]).

**Lemma 2.1.** *Let $A_q (q = 1, 2, ...)$ be a sequence of measurable sets in $\mathbb{R}^n$, and*

$$\sum_{q=1}^{\infty} mesA_q < \infty.$$

*Then the measure of the set $E$ of points $x \in \mathbb{R}^n$ which fall into infinite number of sets $A_q$ equals to zero.*

*Proof.* For every $x \in E \subset \mathbb{R}^n$ and natural $n$ there is a natural number $m > n$ for which $x \in A_m$. Then, for any $x \in E$ and natural number $n \in \mathbb{N}$

$$x \in \bigcup_{k=n}^{\infty} A_k.$$

So,

$$E \subset \bigcup_{k=n}^{\infty} A_k.$$

Since the series of measures is convergent, then for arbitrary $\varepsilon > 0$ there exist a number $n$ such that

$$mes \bigcup_{k=n}^{\infty} A_k \leq \sum_{k=n}^{\infty} mesA_k < \varepsilon.$$

So, $mesE = 0$. Lemma 2.1 is proven. □

Below we will use the symbol $<<$ introduced by Vinogradov I. M. We write $A << B$ if one can find a constant $c$ such that $A \leq cB$.

The following lemma belongs to E.I. Kavaleuskaya (see [3], [6], [8]).

**Lemma 2.2.** *Let $m \leq N, q$ be natural numbers, $f_j(\bar{x}), j = 1, ..., N$ be a real measurable functions defined in the cube $\Omega = [0,1]^m$. Denote by $\mu(q)$ the measure of a set of that $\bar{x} \in \Omega = [0,1]^m$ for which*

$$\|f_j(\bar{x})\| < q^{-r_j} \, (1 \leq j \leq N).$$

*Then,*

$$\mu(q) << q^{-r} \sum_{|c_1|<q^{r_1}} \cdots \sum_{|c_N|<q^{r_N}} \left| \int_\Omega e^{2\pi i (c_1 f_1(\bar{x}) + \cdots + c_N f_N(\bar{x}))} d\bar{x} \right|;$$

*here $r = r_1 + \cdots + r_N$, and the constant in the symbol $<<$ depends on $N$ only.*

*Proof.* Denote by $E$ the set of the points $\bar{x} \in \Omega = [0,1]^m$ for which the conditions

$$\|f_j(\bar{x})\| < q^{-r_j} \, (1 \leq j \leq N)$$

are satisfied. Note that for every integral $c > 0$ and real $\alpha$ such that $\|\alpha\| \leq \delta$ we have also $\|c\alpha\| \leq c\delta$. Really, the condition $\|\alpha\| \leq \delta$ means that for some integral $a$ $\alpha = a + \varepsilon, 0 \leq |\varepsilon| \leq \delta$, then

$$c\alpha = ca + c\varepsilon.$$

Then for integral $c_j$, $|c_j| < (4\pi N)^{-1} q^{r_j}$ we have at $\bar{x} \in E$

$$|c_1 f_1(\bar{x}) + \cdots + c_N f_N(\bar{x})| < |c_1 f_1(\bar{x})| + \cdots + |c_N f_N(\bar{x})| \leq$$

$$\leq \|c_1 f_1(\bar{x})\| + \cdots + \|c_N f_N(\bar{x})\| < \sum_{j=1}^{N} (4\pi N)^{-1} \leq \frac{1}{4\pi}.$$

Therefore,

$$\left| e^{2\pi i (c_1 f_1(\bar{x}) + \cdots + c_N f_N(\bar{x}))} - 1 \right| < 2\pi \left( |c_1 f_1(\bar{x})| + \cdots + |c_N f_N(\bar{x})| \right) \leq \frac{1}{2}.$$

Consequently,

$$\left| \sum_{|c_1| \leq (4\pi N)^{-1} q^{r_1}} \cdots \sum_{|c_N| \leq (4\pi N)^{-1} q^{r_N}} e^{2\pi i (c_1 f_1(\bar{x}) + \cdots + c_N f_N(\bar{x}))} \right| =$$

$$= \left| \, (2\pi N)^{-N} q^r + \sum_{|c_1| \leq (4\pi N)^{-1} q^{r_1}} \cdots \sum_{|c_N| \leq (4\pi N)^{-1} q^{r_N}} 1 \times \right.$$

$$\times \left. \left( e^{2\pi i (c_1 f_1(\bar{x}) + \cdots + c_N f_N(\bar{x}))} - 1 \right) \right| \geq (2\pi N)^{-N} q^r -$$

$$- \left| \sum_{|c_1| \leq (4\pi N)^{-1} q^{r_1}} \cdots \sum_{|c_N| \leq (4\pi N)^{-1} q^{r_N}} \left( e^{2\pi i (c_1 f_1(\bar{x}) + \cdots + c_N f_N(\bar{x}))} - 1 \right) \right| \geq$$

$$\geq (2\pi N)^{-N} q^r - \sum_{|c_1| \leq (4\pi N)^{-1} q^{r_1}} \cdots \sum_{|c_N| \leq (4\pi N)^{-1} q^{r_N}} 1 \times$$

$$\times \left| e^{2\pi i (c_1 f_1(\bar{x}) + \cdots + c_N f_N(\bar{x}))} - 1 \right| > \frac{1}{2} (2\pi N)^{-N} q^r.$$

We can write this inequality as follows

$$\frac{1}{2} (2\pi N)^{-N} q^r \leq$$

$$\le \left| \sum_{|c_1|\le(4\pi N)^{-1}q^{r_1}} \cdots \sum_{|c_N|\le(4\pi N)^{-1}q^{r_N}} e^{2\pi i(c_1 f_1(\bar{x})+\cdots+c_N f_N(\bar{x}))} \right|.$$

Quadrating this inequality and integrating over all points $\bar{x}\in E$ we get

$$\frac{\mu(q)}{4}(2\pi N)^{-2N}q^{2r} \le$$

$$\le \int_E d\bar{x} \left| \sum_{|c_1|\le(4\pi N)^{-1}q^{r_1}} \cdots \sum_{|c_N|\le(4\pi N)^{-1}q^{r_N}} e^{2\pi i(c_1 f_1(\bar{x})+\cdots+c_N f_N(\bar{x}))} \right|^2. \tag{2.1}$$

Estimate the right hand side of this inequality. We have

$$\int_E d\bar{x} \left| \sum_{|c_1|\le(4\pi N)^{-1}q^{r_1}} \cdots \sum_{|c_N|\le(4\pi N)^{-1}q^{r_N}} e^{2\pi i(c_1 f_1(\bar{x})+\cdots+c_N f_N(\bar{x}))} \right|^2 =$$

$$= \int_E d\bar{x} \sum_{|c_1|\le(4\pi N)^{-1}q^{r_1}} \cdots \sum_{|c_N|\le(4\pi N)^{-1}q^{r_N}} \sum_{|c_1'|\le(4\pi N)^{-1}q^{r_1}} \cdots \sum_{|c_N'|\le(4\pi N)^{-1}q^{r_N}} 1\times$$

$$\times e^{2\pi i((c_1-c_1')f_1(\bar{x})+\cdots+(c_N-c_N')f_N(\bar{x}))} \le$$

$$\le \sum_{|c_1|\le(4\pi N)^{-1}q^{r_1}} \cdots \sum_{|c_N|\le(4\pi N)^{-1}q^{r_N}} \sum_{|c_1'|\le(4\pi N)^{-1}q^{r_1}} \cdots \sum_{|c_N'|\le(4\pi N)^{-1}q^{r_N}} 1\times$$

$$\times \left| \int_E e^{2\pi i((c_1-c_1')f_1(\bar{x})+\cdots+(c_N-c_N')f_N(\bar{x}))} d\bar{x} \right| \le$$

$$\le (2\pi N)^{-N} q^r \sum_{|c_1|\le(2\pi N)^{-1}q^{r_1}} \cdots \sum_{|c_N|\le(4\pi N)^{-1}q^{r_N}} 1\times$$

$$\times \left| \int_E e^{2\pi i(c_1 f_1(\bar{x})+\cdots+c_N f_N(\bar{x}))} d\bar{x} \right|.$$

Returning to the inequality (2.1), we get

$$\frac{\mu(q)}{4}(2\pi N)^{-2N}q^{2r} \le (2\pi N)^{-N} q^r \times$$

$$\times \sum_{|c_1|\le(2\pi N)^{-1}q^{r_1}} \cdots \sum_{|c_N|\le(2\pi N)^{-1}q^{r_N}} \left| \int_E e^{2\pi i(c_1 f_1(\bar{x})+\cdots+c_N f_N(\bar{x}))} d\bar{x} \right|,$$

or

$$\mu(q) \le 4(2\pi N)^N q^{-r} \times$$

$$\times \sum_{|c_1|\le(2\pi N)^{-1}q^{r_1}} \cdots \sum_{|c_N|\le(2\pi N)^{-1}q^{r_N}} \left| \int_E e^{2\pi i(c_1 f_1(\bar{x})+\cdots+c_N f_N(\bar{x}))} d\bar{x} \right| <<$$

$$<< q^{-r} \sum_{|c_1|<q^{r_1}} \cdots \sum_{|c_N|<q^{r_N}} \left| \int_\Omega e^{2\pi i(c_1 f_1(\bar{x})+\cdots+c_N f_N(\bar{x}))} d\bar{x} \right|.$$

Here the last chain of these inequalities was got by extending the region of summation. Lemma 2.2 is proven. □

## 3. Main results

Consider now two applications of the obtained results. Let we are given some continuously differentiable $m$-dimensional manifold $\Gamma = (f_1(\bar{x}), ..., f_N(\bar{x}))$, $\bar{x} \in \Omega = [0,1]^m, m < N$. Taking natural number such that $mh > N$ consider the map

$$\varphi_j : \Omega^h \to R^N$$

defined by the equalities

$$\varphi_j(\bar{x}) = \varphi_j(\bar{x}_1, ..., \bar{x}_h) = f_j(\bar{x}_1) + \cdots + f_j(\bar{x}_h);\ \bar{x}_s = (x_{s1}, ..., x_{sm}).$$

Let the Jacoby matrix of the map $(\bar{x}_1, ..., \bar{x}) \mapsto (\varphi_1(\bar{x}), ..., \varphi_h(\bar{x}))$ i. e. the matrix composed of the gradients of the functions $\varphi_1(\bar{x}), ..., \varphi_h(\bar{x})$ be the matrix of maximal rank. It is easy to see that the Jacoby matrix has a view

$$\begin{pmatrix} \frac{\partial \varphi_1}{\partial x_{11}} & \cdots & \frac{\partial \varphi_1}{\partial x_{hm}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \varphi_N}{\partial x_{11}} & \cdots & \frac{\partial \varphi_N}{\partial x_{hm}} \end{pmatrix}.$$

**Theorem 3.1.** *If the Jacoby matrix of the map $(\bar{x}_1, ..., \bar{x}_h) \mapsto (\varphi_1(\bar{x}), ..., \varphi_h(\bar{x}))$ has a maximal rank everywhere in $\Omega^h$ for some natural $h$ then the differentiable manifold $\Gamma$ is extremal.*

*Proof.* As it is clear from the proof of the lemma 2.1 the set $A$ of points $x \in \mathbb{R}^n$ which fall into infinite number of sets $A_q$ is possible represent as $A = \bigcap_{k=1}^{\infty} \bigcup_{q=k}^{\infty} A_q$. Since the sets $A_q$ are measurable, then the set $A$ is measurable also, independent of the convergence of the series of this lemma. If now the set $A$ is not of zero measure then $mesA > 0$ . In this case the set

$$A^{2h} = A \times \cdots \times A$$

has positive measure in $\Omega^{2h}$ . So, if we have proven that $mesA^{2h} = 0$ then our assumption $mesA > 0$ is false. We will use this reasoning to prove the theorem 1. Define now the the set $A_q$ by the condition $\|qf_j(\bar{x})\| < (2h)^{-1} q^{-r_j}$ . Then the set of points in $\Omega^{2h}$ for which $\|q(\varphi_j(\bar{x}) - \varphi_j(\bar{y}))\| < (q)^{-r_j}$ will contain the set $(A_q)^{2h}$.

In the lemma 2.2 we take $r_j = 1/N + \delta$, $\delta > 0$, $1 \le j \le N$ and $q(\varphi_j(\bar{x}) - \varphi_j(\bar{y}))$ instead of $f_j(\bar{x})$, where $q > 0$ is a fixed sufficient large number. Then by the lemma 2.1 we have for the measure $\mu_q$ of the set of such $(\bar{x}, \bar{y}) \in \Omega^{2h}$ for which $\|q(\varphi_j(\bar{x}) - \varphi_j(\bar{y}))\| < q^{-r_j}$:

$$(\mu(q))^{2h} \le \mu_q << q^{-r-N\delta} \times$$
$$\times \sum_{|c_1| < q^{r_1+\delta}} \cdots \sum_{|c_N| < q^{r_N+\delta}} \left| \left( \int_{\Omega} e^{2\pi i (qc_1 f_1(\bar{x}) + \cdots + qc_N f_N(\bar{x}))} d\bar{x} \right)^h \right|^2 =$$
$$= q^{-r-N\delta} \sum_{|c_1| < q^{r_1+\delta}} \cdots \sum_{|c_N| < q^{r_N+\delta}} \left| \int_{\Omega} \cdots \int_{\Omega} e^{2\pi i (qc_1 \varphi_1(\bar{x}) + \cdots + qc_N \varphi_N(\bar{x}))} d\bar{x}_1 \cdots d\bar{x}_h \right|^2. \tag{3.1}$$

We have:

$$\left( \int_{\Omega} e^{2\pi i (qc_1 f_1(\bar{x}) + \cdots + qc_N f_N(\bar{x}))} d\bar{x} \right)^h =$$

$$= \int_\Omega \cdots \int_\Omega e^{2\pi i(qc_1(f_1(\bar{x}_1)+\cdots+f_1(\bar{x}_h))+\cdots+qc_N(f_N(\bar{x}_1)+\cdots+f_N(\bar{x}_h))}d\bar{x}_1\cdots d\bar{x}_h =$$

$$= \int_\Omega \cdots \int_\Omega e^{2\pi i(qc_1u_1+\cdots+qc_Nu_N)}d\bar{x}_1\cdots d\bar{x}_h;$$

here we have introduced the notations $u_j = \varphi_j(\bar{x}) = f_j(\bar{x}_1) + \cdots + f_j(\bar{x}_h)$; $\bar{x}_s = (x_{s1}, ..., x_{sm})$. Applying the consequence of the lemma 2.1 from the works [2, 5], we can represent the last integral as below

$$\int_\Omega \cdots \int_\Omega e^{2\pi i(qc_1u_1+\cdots+qc_Nu_N)}d\bar{x}_1\cdots d\bar{x}_h =$$

$$\int_{m_1}^{M_1} \cdots \int_{m_N}^{M_N} \left(\int_\Pi \frac{ds}{\sqrt{G}}\right) e^{2\pi i(qc_1u_1+\cdots+qc_Nu_N)}du_1\cdots du_N, \quad (3.2)$$

where $\Pi$ denotes the surface defined by the system of equations

$$f_j(\bar{x}_1) + \cdots + f_j(\bar{x}_h) = u_j,\ j = 1, ..., N$$

and the numbers $m_j$, $M_j$ denote the minimal and maximal values of the function $\varphi_j(\bar{x})$; here $G$ denotes a Gram determinant of gradients of the functions standing on the left hand sides of the equations of this system. Since the Jacoby matrix of the functions standing on the left hand sides of the equations of this system is bounded from above and below, then in the suitable neighborhood $\Omega(x_0, \bar{u})$ of any solution $x_0 = (\bar{x}_1, ..., \bar{x}_h)$ of the system the set of solutions can be described as a surface, by the theorem on implicit functions, as follows:

$$x_k = \psi_k(\bar{\xi}),$$

with independent variables $\bar{\xi} = (\xi_1, ..., \xi_{hm-N}) \in \Omega(x_0, \bar{u})$. Let us denote

$$\Omega(\bar{u}) = \cup\Omega(x_0, \bar{u}),$$

where the union is taken over all points $x_0$ on the surface $\Pi$. From the compactness of the cube if follows that actually the union consists of finite number of neighborhoods. Not breaking a generality, we assume that some of minors of the Jacoby matrix is non-zero everywhere in these neighborhoods. Now we denote

$$g(u_1, ..., u_N) = \int_{\Omega(\bar{u})} \frac{1}{|J|^{-1}} d\xi_1 \cdots d\xi_{hm-N},$$

where $|J|$ means a minor of the Jacoby matrix consisted of, say, the last $N$ columns. Taking $m'_j = [m_j]$ and $M'_j = [M_j] + 1$ we can continue this function to the product $[m'_1, M'_1] \times \cdots \times [m'_N, M'_N]$ taking $g(u_1, ..., u_N) = 0$ when $\bar{u}$ doesn't belong to $\Omega(\bar{u})$. Therefore, we can rewrite integral on the right side of the equality (3.2) as an integral

$$\int_{m'_1}^{M'_1} \cdots \int_{m'_N}^{M'_N} g(u_1, ..., u_N)e^{2\pi i(qc_1u_1+\cdots+qc_Nu_N)}du_1\cdots du_N;$$

here $m'_1 \leq m_1, M_j \leq M'_j$, and the numbers $m'_1, M'_j$ are integral. Dissecting the cube $[m'_1, M'_1] \times \cdots \times [m'_N, M'_N]$ into the union of unite cubes we get:

$$\int_{m'_1}^{M'_1} \cdots \int_{m'_N}^{M'_N} g(u_1, ..., u_N)e^{2\pi i(qc_1u_1+\cdots+qc_Nu_N)}du_1\cdots du_N =$$

$$= \sum_{m_1'\le i_1<M_1'-1} \cdots \sum_{m_N'\le i_N<M_N'-1} 1\times$$

$$\times \int_{i_1}^{i_1+1} \cdots \int_{i_N}^{i_N+1} g(u_1,...,u_N) e^{2\pi i(qc_1u_1+\cdots+qc_Nu_N)} du_1\cdots du_N =$$

$$= \sum_{m_1'\le i_1<M_1'-1} \cdots \sum_{m_N'\le i_N<M_N'-1} \bar{c}_{i_1\ldots i_N}(qc_1,...,qc_N);$$

here

$$\bar{c}_{i_1\ldots i_N}(c_1,...,c_N) = \int_{i_1}^{i_1+1} \cdots \int_{i_N}^{i_N+1} g(u_1,...,u_N) e^{2\pi i(c_1u_1+\cdots+c_Nu_N)} du_1\cdots du_N$$

is a Fourier coefficient of the function $g(u_1,...,u_N)$ in the cube $[i_1,i_1+1]\times\cdots\times[i_N,i_N+1]$ and here the function under integral is equal to zero when

$$(u_1,...,u_N)\notin[m_1,M_1]\times\cdots\times[m_N,M_N].$$

Applying the Cauchy inequality, we find

$$\left|\sum_{m_1'\le i_1<M_1'} \cdots \sum_{m_N'\le i_N<M_N'} \bar{c}_{i_1\ldots i_N}(qc_1,...,qc_N)\right|^2 \le$$

$$\le (M_1'-m_1')\cdots(M_N'-m_N') \sum_{m_1'\le i_1<M_1'} \cdots \sum_{m_N'\le i_N<M_N'} \left|\bar{c}_{i_1\ldots i_N}(qc_1,...,qc_N)\right|^2. \quad (3.3)$$

We denote:

$$M = (M_1'-m_1')\cdots(M_N'-m_N') \int_{m_1'}^{M_1'} \cdots \int_{m_N'}^{M_N'} |g(u_1,...,u_N)|^2 du_1\cdots du_N.$$

We have to prove that the series

$$\sum_{q=1}^{\infty} q^{-1-N\delta} \sum_{|c_1|<q^{r_1+\delta}} \cdots \sum_{|c_N|<q^{r_N+\delta}} |\bar{c}_{i_1\ldots i_N}(qc_1,...,qc_N)|^2 \quad (3.4)$$

is convergent. To do that we note that for every fixed $q$ the inner multiple sum is taken over all corteges of indexes components of which are divisible by $q$. Then this sum doesn't exceed the following one:

$$\sum_{|c_1|<q^{r_1+\delta}} \cdots \sum_{|c_N|<q^{r_N+\delta}} |\bar{c}_{i_1\ldots i_N}(qc_1,...,qc_N)|^2 \le$$

$$\le \sum_{m_1=-\infty, m_1 \vdots q}^{\infty} \cdots \sum_{m_N=-\infty, m_N \vdots q}^{\infty} |\bar{c}_{i_1\ldots i_N}(m_1,...,m_N)|^2.$$

Therefore, taking the sum over all corteges $(m_1,...,m_N)$, and using Parseval's formula, for the sum (3.4) we find the estimation:

$$\le \sum_{q=1}^{\infty} q^{-1-N\delta} \sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_N=-\infty}^{\infty} |\bar{c}_{i_1\ldots i_N}(m_1,...,m_N)|^2 \le$$

$$\le \int_{i_1}^{i_1+1} \cdots \int_{i_N}^{i_N+1} |g(u_1,...,u_N)|^2 du_1\cdots du_N \sum_{q=1}^{\infty} q^{-1-N\delta} < +\infty.$$

So, the series (3.4) converges. Denoting the last sum over $q$ by $c_0$ we have:

$$\sum_{q=1}^{\infty} \mu_q \leq Mc_0.$$

Then by the lemma 1 and remarks made above we find that the set of points $\bar{x} \in \Omega$ for which $\bar{x} \in A_q$ for infinitely many natural $q$ has a zero measure. Since the $\delta$ is arbitrarily small, then the considered manifold is extremal. Proof of the theorem 3.1 is completed. □

Consider now the other question on the extremality. We will show that the algebraic variety $\Gamma = (\gamma_1(\bar{x}), ..., \gamma_N(\bar{x}))$ where $\gamma_j(\bar{x}) = x_1^{k_{1j}} x_2^{k_{2j}} \cdots x_m^{k_{mj}}$ are monomials being not a constant, and $0 < k_{1j}, k_{2j}, ..., k_{mj} \leq n$, $k_{ij} \geq 0$, is extremal. These monomials is a monomials of a polynomial of a view

$$\sum_{n_1=0}^{n} \cdots \sum_{n_k=0}^{n} \alpha_{n_1,...,n_k} x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m}.$$

**Theorem 3.2.** *The algebraic variety* $\Gamma = (\gamma_1(\bar{x}), ..., \gamma_N(\bar{x}))$ *for which the conditions above are satisfied is extremal.*

*Proof.* Take in the lemma 2.2 $f_j(\bar{x}) = q\gamma_j(\bar{x}) = qx_1^{k_{1j}} x_2^{k_{2j}} \cdots x_m^{k_{mj}}$, $r_j = 1/N + \delta$, ($\delta$ is arbitrarily small) for all $j = 1, ..., N$. We can apply the reasonings above to get the bound for the measure $\mu(q)$ . Note that the expression at the right side of (3.4) is the integral

$$\int_{m_1}^{M_1} \cdots \int_{m_n}^{M_N} |g(u_1, ..., u_N)|^2 \, du_1 \cdots du_N,$$

multiplied by a constant. This integral is equal to the special integral

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left| \int_{\Omega} e^{2\pi i(\alpha_1 f_1(\bar{x}) + \cdots + \alpha_N f_N(\bar{x}))} d\bar{x} \right|^{2h} d\alpha_1 \cdots d\alpha_N,$$

of multidimensional Tarry's problem, in consent with the relation from [2, 5].

In the theorem 9 of the work ([1] p. 50) an estimation for the trigonometric integrals with the polynomials of a special form on the exponent was obtained. Every polynomial can be written in the same form also. Hence, for every polynomial the special integral has an exponent $\gamma$ of convergence. When $2h > \gamma$ the inner integral is convergent. So, we get that the series

$$\sum_{q=1}^{\infty} \mu(q)^h$$

is convergent. Then the set $E \subset \Omega$ for the points of which the conditions

$$\|q\gamma_j(\bar{x})\| < q^{-1/N-\delta} \, (1 \leq j \leq n)$$

are satisfied for infinitely many natural $q$, in consent with the lemma 1, has zero measure. Since $\delta > 0$ is arbitrarily small, then the proof of the theorem 3.2 is completed. □

The advantage of this theorem consists of the fact that the map introduced above may have degenerating Jacoby matrix, and there is not need in assumption on existence of a number $h$ as in the theorem 1. This fact shows that the class of extremal manifolds is wide. We can await that the manifold $\Gamma = (f_1(\bar{x}), ..., f_N(\bar{x}))$ with any differentiable functions $f_1(\bar{x}), ..., f_N(\bar{x})$, being not a constant, is extremal.

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