Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 44, Number 2, 2018, Pages 238–249

ON THE APPROXIMATION OF FUNCTIONS BY SOME SINGULAR INTEGRALS

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Abstract. In this paper, we define a new linear operator with the help of convolution singular integral of Fejer's type, consider its convergence properties and obtain the degree of approximation in terms of higher order characteristics.

1. Introduction

Approximation of functions by singular integrals has long history and is an important topic in approximation theory. In this paper we study questions about the approximation of locally integrable functions by singular integrals. The rate of approximation in terms of various metric characteristics describing the structural properties of a given function is estimated. We define a new linear operator with the help of convolution singular integral of Fejer's type, consider its convergence properties and obtain the degree of approximation in terms of higher order characteristics.

Various aspects of the question of the approximation of a function by singular integrals have been investigated by many authors and quantitative estimates for approximation have been presented in a large literature (see, for example, [1], [3]-[9], [15], [16] and the literature cited therein).

2. Preliminaries

Let $\mathbb{R}^n$ be n-dimensional Euclidean space of points $x = (x_1, x_2, ..., x_n)$, $B(a, r) := \{x \in \mathbb{R}^n\colon\ |x - a| \le r\}$ –be a closed ball in $\mathbb{R}^n$ of radius $r > 0$ centered at the point $a \in \mathbb{R}^n$, N be the set of all natural numbers, $\nu = (\nu_1, \nu_2, ..., \nu_n)$, $x^\nu = x_1^{\nu_1} \cdot x_2^{\nu_2} \cdots x_n^{\nu_n}$, $|\nu| = \nu_1 + \nu_2 + ... + \nu_n$, where $\nu_1, \nu_2, ..., \nu_n$ are non-negative integers. Denote by $L_{loc}(\mathbb{R}^n)$ the union of all functions locally summable in $\mathbb{R}^n$.

2010 *Mathematics Subject Classification.* 41A17, 41A25, 41A35.
Key words and phrases. Approximation, singular integrals, mean oscillation.

Let $f \in L_{loc}(\mathbb{R}^n)$, $k \in \mathrm{N} \bigcup \{0\}$. Let us consider the polynomial (see [2], [10])

$$P_{k,B(a.r)} f(x) := \sum_{|\nu| \le k} \left(\frac{1}{|B(a,r)|} \int_{B(a,r)} f(t) \varphi_\nu \left(\frac{t-a}{r} \right) dt \right) \varphi_\nu \left(\frac{x-a}{r} \right),$$

where $|B(a,r)|$ denotes the volume of the ball $B(a,r)$ and $\{\varphi_\nu\}$, $|\nu| \le k$, is an orthonormal system obtained from applications of orthogonalization process with respect to the scalar product

$$(f,g) := \frac{1}{|B(0,1)|} \int_{B(0,1)} f(t) g(t) dt$$

to the system of power functions $\{x^\nu\}$, $|\nu| \le k$, located in partially lexicographic order (this means that x^ν precedes x^μ if either $|\nu| < |\mu|$, or $|\nu| = |\mu|$ but the first nonzero difference $\nu_i - \mu_i$ is negative) (see [11], [12]).

$P_{k,\, B(a,r)} f$ is a polynomial of degree at most k. We denote the union of all polynomials in $\mathbb{R}^n$ of degree at most k by P_k. Thus $P_{k,\, B(a,r)} f \in P_k$.

We determine the local modulus of the k-th order mean oscillation $(k \in N)$ of locally summable function f by the equality

$$m_f^k(x_0; \delta) := \sup \{ \Omega_k(f, B(x_0, r)) \ : \ 0 < \ r \le \delta \} \quad (x_0 \in \mathbb{R}^n, \quad \delta > 0),$$

where

$$\Omega_k(f, B(x,r)) := \frac{1}{|B(x,r)|} \int_{B(x,r)} \left| f(t) - P_{k-1, B(x,r)} f(t) \right| dt \ (x \in \mathbb{R}^n, \ r > 0).$$

$\Omega_k(f, B(a,r))$ is said to be the k-th order mean oscillation of the function f in the ball $B(a,r)$ in the metric L^1.

Modulus of the k-th order mean oscillation $(k \in N)$ of the locally summable function f is determined by the equality

$$M_f^k(\delta) := \sup \left\{ m_f^k(x;\delta) \ : \ x \in \mathbb{R}^n \right\}, \ \delta > 0.$$

Let $\Phi(x)$, $x \in \mathbb{R}^n$, be a function summable in $\mathbb{R}^n$ (i.e. $\Phi \in L^1(\mathbb{R}^n)$) such that $\Phi(x) \ge 0$ $(x \in \mathbb{R}^n)$, and let

$$\Phi_r(x) := r^{-n} \Phi \left(\frac{x}{r} \right), \ x \in \mathbb{R}^n, \ r > 0,$$

$$\Omega_{k,\Phi}(f, B(x,r)) := \int_{\mathbb{R}^n} \Phi_r(x-t) \left| f(t) - P_{k-1,\, B(x,r)} f(t) \right| dt,$$

where $f \in L_{loc}(\mathbb{R}^n)$, $k \in \mathrm{N}$. Further, we assume

$$h_f^{k,\Phi}(x;\delta) := \sup \{ \Omega_{k,\Phi}(f, B(x,r)) : \ 0 < r \le \delta \}, \ x \in \mathbb{R}^n, \ \delta > 0,$$

$$H_f^{k,\Phi}(\delta) := \sup \left\{ h_f^{k,\Phi}(x;\delta) : \quad x \in \mathbb{R}^n \right\}, \quad \delta > 0.$$

We call the quantity $\Omega_{k,\Phi}(f, B(x,r))$ the k-th order Φ-oscillation of the function f in the ball $B(a,r)$ in the metric L^1 [13].

3. On the quantity $P_{k-1,B(x_0.r)}f(x_0)$

From the definition of the polynomial $P_{k,B(a.r)}f(x)$ it follows that if $x_0 \in \mathbb{R}^n$ a fixed point, and $k \in N$, then

$$P_{k-1,B(x_0.r)}f(x_0) = \sum_{|\nu|\le k-1} \left(\frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} f(t)\,\varphi_\nu\left(\frac{t-x_0}{r}\right) dt \right) \varphi_\nu(0) =$$

$$= \int_{B(x_0,r)} \left\{ \sum_{|\nu|\le k-1} \frac{1}{|B(x_0,r)|} \varphi_\nu\left(\frac{t-x_0}{r}\right) \cdot \varphi_\nu(0) \right\} f(t)\,dt.$$

Denote

$$K(t) := \frac{1}{|B(0,1)|} X_{B(0,1)}(t) \sum_{|\nu|\le k-1} \varphi_\nu(-t)\,\varphi_\nu(0).$$

Then we have $P_{k-1,B(x_0,r)}f(x_0) = \frac{1}{r^n}\int_{\mathbb{R}^n} K\left(\frac{x_0-t}{r}\right) f(t)\,dt$.

By the property of the system $\{\varphi_\nu\}$, $|\nu| \le k-1$, we get

$$\int_{\mathbb{R}^n} K(t)\,dt = \frac{1}{|B(0,1)|} \int_{B(0,1)} \sum_{|\nu|\le k-1} \varphi_\nu(-t)\,\varphi_\nu(0)\,dt =$$

$$= \sum_{|\nu|\le k-1} \left(\frac{1}{|B(0,1)|} \int_{B(0,1)} \varphi_\nu(-t)\,dt \right) \varphi_\nu(0) = \varphi_0^2(0) = 1.$$

Thus the quantity $P_{k-1,B(x_0.r)}f(x_0)$ is expressed by a convolution type singular integral.

Denote $\lim\limits_{r\to 0} P_{k-1,B(x_0,r)}f(x_0) =: s_{k,f}(x_0)$.

Let $x_0 \in \mathbb{R}^n$ and $f \in L_{loc}(\mathbb{R}^n)$. Then we have

$$\left|P_{k-1,B(x_0,r)}f(x_0) - f(x_0)\right| = \left|P_{k-1,B(x_0,r)}(f - f(x_0))(x_0)\right| \le$$

$$\le \left\|P_{k-1,B(x_0,r)}(f - f(x_0))\right\|_{L^\infty(B(x_0,r))} \le c \cdot \frac{1}{|B(x_0;r)|} \int_{B(x_0,r)} |f(t) - f(x_0)|\,dt,$$

where the constant $c > 0$ depends only on the system $\{\varphi_\nu\}$, $|\nu| \le k-1$. Hence it follows that if $x_0 \in \mathbb{R}^n$ is the Lebesgue point of the function f, then

$$\lim_{r\to 0} P_{k-1,B(x_0,r)}f(x_0) = f(x_0).$$

Thus, almost everywhere in $\mathbb{R}^n$ there exists the limit $s_{k,f}(x)$ and almost everywhere the equality $s_{k,f}(x) = f(x)$ is fulfilled.

Lemma 3.1. *For any polynomial $\pi \in P_{k-1}$ the identity*

$$P_{k-1,B(x_0.r)}\pi(x) \equiv \pi(x), \quad x \in \mathbb{R}^n$$

is valid.

Proof. Obviously, the polynomial $\pi \in P_{k-1}$ may be represented in the form

$$\pi(x) = \sum_{|\nu|\le k-1} c_\nu(x_0,r) \cdot \varphi_\nu\left(\frac{x-x_0}{r}\right),$$

where $c_\nu(x_0,r)$ are the coefficients of representation. Then we have

$$P_{k-1,B(x_0,r)}\pi(x) =$$

$$= \sum_{|\mu|\le k-1}\left(\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)}\pi(t)\,\varphi_\mu\left(\frac{t-x_0}{r}\right)dt\right)\varphi_\mu\left(\frac{x-x_0}{r}\right) =$$

$$= \sum_{|\mu|\le k-1}\left(\sum_{|\nu|\le k-1} c_\nu(x_0,r)\cdot\frac{1}{|B(x_0,r)|}\times\right.$$

$$\left.\times\int_{B(x_0,r)}\varphi_\nu\left(\frac{t-x_0}{r}\right)\varphi_\mu\left(\frac{t-x_0}{r}\right)dt\right)\varphi_\mu\left(\frac{x-x_0}{r}\right) =$$

$$= \sum_{|\mu|\le k-1} c_\mu(x_0,r)\left(\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)}\varphi_\mu^2\left(\frac{t-x_0}{r}\right)dt\right)\varphi_\mu\left(\frac{x-x_0}{r}\right) =$$

$$= \sum_{|\mu|\le k-1} c_\mu(x_0,r)\cdot\varphi_\mu\left(\frac{x-x_0}{r}\right) = \pi(x),$$

i.e. $P_{k-1,B(x_0.r)}\pi(x)\equiv\pi(x)$, $x\in\mathbb{R}^n$. The lemma is proved. □

Now we prove a lemma that will be used in the sequel.

Lemma 3.2. *If $f\in L_{loc}(\mathbb{R}^n)$, $x_0\in\mathbb{R}^n$, $k\in N$, $0<\eta<\xi<\infty$, then the following inequality is valid*

$$\left|P_{k-1,B(x_0,\xi)}f(x_0)-P_{k-1,B(x_0,\eta)}f(x_0)\right|\le c\left(m_f^k(x_0;\xi)+\int_\eta^\xi\frac{m_f^k(x_0;t)}{t}dt\right), \tag{3.1}$$

where the constant $c>0$ is independent of η, ξ, f and x_0.

Proof. Taking into account the definition of functions φ_ν, from the expression for the quantity $P_{k-1,B(x_0.r)}f(x_0)$ we easily get

$$\left|P_{k-1,B(x_0.r)}f(x_0)\right|\le c_1\cdot\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)}|f(y)|\,dy, \tag{3.2}$$

where $c_1>0$ depends only on the system $\{\varphi_\nu\}$, $|\nu|\le k-1$. Let $2^{-m-1}\cdot\xi<\eta\le 2^{-m}\cdot\xi$, where m is an entire non-negative number. Denote $B_i=B\left(x_0,\frac{\xi}{2^i}\right)$, $i=0,1,...,m$. Then $|B_i|=\gamma_n\cdot\left(2^{-i}\xi\right)^n$, where $\gamma_n=\pi^{\frac{n}{2}}\cdot\left[\Gamma\left(\frac{n}{2}+1\right)\right]^{-1}$. Therefore $|B_{i+1}|=\frac{1}{2^n}|B_i|$ $(i=0,1,...,m-1)$ and $|B(x_0,\eta)|>\left|B\left(x_0,\frac{\xi}{2^{m+1}}\right)\right|=\frac{1}{2^n}|B_m|$. Further we have

$$\left|P_{k-1,B(x_0,\xi)}f(x_0)-P_{k-1,B(x_0,\eta)}f(x_0)\right|\le\sum_{i=0}^{m-1}\left|P_{k-1,B_i}f(x_0)-P_{k-1,B_{i+1}}f(x_0)\right|+$$

$$+\left|P_{k-1,B_m}f(x_0)-P_{k-1,B(x_0,\eta)}f(x_0)\right|. \tag{3.3}$$

By inequality (3.2) for $i=0,1,...,m-1$ we get

$$\left|P_{k-1,B_i}f(x_0)-P_{k-1,B_{i+1}}f(x_0)\right|=\left|P_{k-1,B_{i+1}}\left(f-P_{k-1,B_i}f\right)(x_0)\right|\le$$

$$\leq c_1 \cdot \frac{1}{|B_{i+1}|}\int_{B_{i+1}} |(t) - P_{k-1,B_i} f(t)|dt \leq c_1 \cdot 2^n \cdot \frac{1}{|B_i|}\int_{B_i} |f(t) - P_{k-1,B_i} f(t)|\, dt \leq$$

$$\leq c_1 \cdot 2^n \cdot m_f^k\left(x_0; \frac{\xi}{2^i}\right). \tag{3.4}$$

Furthermore,

$$\left|P_{k-1,B\left(x_0,\frac{\xi}{2^m}\right)} f(x_0) - P_{k-1,B(x_0,\eta)} f(x_0)\right| = \left|P_{k-1,B(x_0,\eta)}\left(f - P_{k-1,B_m} f\right)(x_0)\right| \leq$$

$$\leq c_1 \cdot \frac{1}{|B(x_0,\eta)|}\int_{B(x_0,\eta)} |f(t) - P_{k-1,B_m} f(t)|\, dt \leq$$

$$\leq c_1 \cdot 2^n \cdot \frac{1}{|B_m|}\int_{B_m} |f(t) - P_{k-1,B_m} f(t)|\, dt \leq c_1 \cdot 2^n \cdot m_f^k\left(x_0; \frac{\xi}{2^m}\right). \tag{3.5}$$

Thus, by means of inequalities (3.4) and (3.5) from inequality (3.3) we get

$$\left|P_{k-1,B(x_0,\xi)} f(x_0) - P_{k-1,B(x_0,\eta)} f(x_0)\right| \leq$$

$$\leq c_1 \cdot 2^n \cdot \left(m_f^k(x_0;\xi) + \sum_{i=1}^{m} m_f^k\left(x_0; \frac{\xi}{2^i}\right)\right), \tag{3.6}$$

moreover, if $m = 0$, then the second term in brackets on the right hand side of the inequality, is absent.

For $m \geq 1$ we have

$$\int_{\xi/2^m}^{\xi} \frac{m_f^k(x_0;t)}{t} dt = \sum_{i=1}^{m}\int_{\xi/2^i}^{\xi/2^{i-1}} \frac{m_f^k(x_0;t)}{t} dt \geq \sum_{i=1}^{m} m_f^k\left(x_0; \frac{\xi}{2^i}\right) \cdot \ln 2. \tag{3.7}$$

Since

$$\int_{\eta}^{\xi} \frac{m_f^k(x_0;t)}{t} dt \geq \int_{\xi/2^m}^{\xi} \frac{m_f^k(x_0;t)}{t} dt,$$

then from inequalities (3.6) and (3.7) we get

$$\left|P_{k-1,B(x_0,\xi)} f(x_0) - P_{k-1,B(x_0,\eta)} f(x_0)\right| \leq c_1 \cdot \frac{2^n}{\ln 2} \cdot \left(m_f^k(x_0;\xi) + \int_{\eta}^{\xi} \frac{m_f^k(x_0;t)}{t} dt\right).$$

The lemma is proved. □

Theorem 3.1. *Let $f \in L_{loc}(\mathbb{R}^n)$, $x_0 \in \mathbb{R}^n$, $k \in N$, and*

$$\int_0^1 \frac{m_f^k(x_0;t)}{t} dt < +\infty. \tag{3.8}$$

Then there exists finite limit $s_{k,f}(x_0) = \lim_{r \to 0} P_{k-1,B(x_0,r)} f(x_0)$ and the following inequality is valid

$$\left|P_{k-1,B(x_0,r)} f(x_0) - s_{k,f}(x_0)\right| \leq c\left(m_f^k(x_0;r) + \int_0^r \frac{m_f^k(x_0;t)}{t} dt\right), \tag{3.9}$$

where the constant $c > 0$ is independent of r, f and x_0.

Proof. The existence of the finite limit $s_{k,f}(x_0)$ follows from condition (3.8) and inequality (3.1). If we pass to limit as $\eta \to 0$ in inequality (3.1), then we get the desired inequality (3.9). The theorem is proved. □

4. On approximation of functions by singular integrals

Introduce the following singular integral

$$S_{k,r}(f)(x) = S_{k,r}(f;K)(x) =$$
$$= \int_{\mathbb{R}^n} K_r(x-t)\left[f(t) - P_{k-1,B(x,r)}f(t)\right]dt + P_{k-1,B(x,r)}f(x), \qquad (4.1)$$

where $K \in L^1(\mathbb{R}^n)$, $K_r(x) := r^{-n}K\left(\frac{x}{r}\right)$, $r>0$, $k \in N$, $x \in \mathbb{R}^n$. The function $K(x)$ with the indicated properties is called a kernel.

For approximation of the locally summable function f by the singular integrals in the terms of the characteristics $m_f^k(x_0;\delta)$ it is convenient to choose a singular integral in the form (4.1).

Remark 4.1. Introduce the denotation

$$K_r f(x) = K_r * f(x) = \int_{\mathbb{R}^n} K_r(x-t) f(t)\,dt.$$

Let is satisfied the condition

$$\forall \pi \in P_{k-1} : K_r\pi(x) \equiv \pi(x)\,(x \in \mathbb{R}^n, \ \ r>0). \qquad (4.2)$$

Then for $f \in L_{loc}(\mathbb{R}^n)$ we have

$$K_r f(x) = K_r * f(x) = \int_{\mathbb{R}^n} K_r(x-t) f(t)\,dt =$$
$$= \int_{\mathbb{R}^n} K_r(x-t)\left[f(t) - P_{k-1,B(x,r)}f(t)\right]dt + \int_{\mathbb{R}^n} K_r(x-t)\,P_{k-1,B(x,r)}f(t)\,dt =$$
$$= \int_{\mathbb{R}^n} K_r(x-t)\left[f(t) - P_{k-1,B(x,r)}f(t)\right]dt + \left(K_r * P_{k-1,B(x,r)}f\right)(x) =$$
$$= \int_{\mathbb{R}^n} K_r(x-t)\left[f(t) - P_{k-1,B(x,r)}f(t)\right]dt + P_{k-1,B(x,r)}f(x) = S_{k,r}f(x).$$

Thus, if condition (4.2) is fulfilled, then we can represent the convolution type singular integral $K_r f(x)$ in the form of the singular integral $S_{k,r}f(x)$.

Remark 4.2. Show that

$$\forall \pi \in P_{k-1} : S_{k,r}\pi(x) \equiv \pi(x)\,(x \in \mathbb{R}^n, \ \ r>0). \qquad (4.3)$$

Indeed, let $\pi \in P_{k-1}$ be any polynomial. Then taking into account that for any polynomial $\pi \in P_{k-1}$ the identity

$$P_{k-1,B(x.r)}\pi(t) \equiv \pi(t), t \in \mathbb{R}^n,$$

is valid, we have

$$S_{k,r}\pi(x) = \int_{\mathbb{R}^n} K_r(x-t)\left[\pi(t) - P_{k-1,B(x,r)}\pi(t)\right]dt + P_{k-1,B(x,r)}\pi(x) =$$
$$= \int_{\mathbb{R}^n} K_r(x-t)\left[\pi(t) - \pi(t)\right]dt + P_{k-1,B(x,r)}\pi(x) \equiv \pi(x).$$

The least decreasing radial majorant of the function $K(x)$ measurable in $\mathbb{R}^n$ is determined by the equality $\psi(x) := esssup\{|K(y)| : \ \ |y| \geq |x|\}$. In future, we will use the denotation $\varphi(|x|) := \psi(x)$. Let $k \in N$. By $\Lambda_k$we will denote the class of all functions $K(x)$ measurable in $\mathbb{R}^n$ such that $\psi \in L^1(B(0,1))$, $|x|^{k-1} \cdot \psi \in L^1(\mathbb{R}^n \backslash B(0,1))$. It is easy to see that $\Lambda_k \subset L^1(\mathbb{R}^n)$.

Theorem 4.1. *Let $K \in \Lambda_k$, $k \in N$, $x_0 \in \mathbb{R}^n$, $f \in L_{loc}(\mathbb{R}^n)$. Then under convergence of the integrals in the right hand side, the following inequality is valid*

$$\left|S_{k,r}(f;K)(x_0) - P_{k-1,B(x_0,r)}f(x_0)\right| \le c(n,\psi,k)\left(m_f^k(x_0;r) + \right.$$

$$+\int_0^\infty x^{n-1}\varphi(x)m_f^k(x_0;4rx)\,dx + \int_0^r \frac{m_f^k(x_0;t)}{t}\left(\int_0^{t/r} x^{n-1}\varphi(x)dx\right)dt +$$

$$\left. + r^{k-1}\int_r^\infty \frac{m_f^k(x_0;t)}{t^k}\left(\int_{t/r}^\infty x^{n+k-2}\varphi(x)dx\right)dt\right), \tag{4.4}$$

where $c(n,\psi,k)$ is a positive constant dependent only on n, ψ and k.

Proof. From the definition of the singular integral $S_{k,r}(f;K)(x)$ it follows that

$$\left|S_{k,r}(f;K)(x_0) - P_{k-1,B(x_0,r)}f(x_0)\right| =$$

$$= \left|\int_{\mathbb{R}^n} K_r(x_0-t)\left[f(t) - P_{k-1,B(x_0,r)}f(t)\right]dt\right| \le$$

$$\le \int_{\mathbb{R}^n} |K_r(x_0-t)|\left|f(t) - P_{k-1,B(x_0,r)}f(t)\right|dt =$$

$$= \frac{1}{r^n}\int_{\mathbb{R}^n}\left|K\left(\frac{x_0-t}{r}\right)\right|\left|f(t) - P_{k-1,B(x_0,r)}f(t)\right|dt =$$

$$= \int_{\mathbb{R}^n} |K|_r(x_0-t)\left|f(t) - P_{k-1,B(x_0,r)}f(t)\right|dt = \Omega_{k,|K|}(f,B(x_0,r)).$$

Applying Theorem 1 from [14] we find

$$\left|S_{k,r}(f;K)(x_0) - P_{k-1,B(x_0,r)}f(x_0)\right| \le \Omega_{k,|K|}(f,B(x_0,r)) \le$$

$$\le c(n,\psi,k)\left(m_f^k(x_0;r) + \int_0^\infty x^{n-1}\varphi(x)m_f^k(x_0;4rx)\,dx + \right.$$

$$+\int_0^r \frac{m_f^k(x_0;t)}{t}\left(\int_0^{t/r} x^{n-1}\varphi(x)dx\right)dt +$$

$$\left. + r^{k-1}\int_r^\infty \frac{m_f^k(x_0;t)}{t^k}\left(\int_{t/r}^\infty x^{n+k-2}\varphi(x)dx\right)dt\right),$$

where $c(n,\psi,k)$ is a positive constant dependent only on n, ψ and k. The theorem is proved. □

The following statement is obtained from theorems 3.1 and 4.1.

Theorem 4.2. *Let $K \in \Lambda_k$, $k \in N$, $x_0 \in \mathbb{R}^n$, $f \in L_{loc}(\mathbb{R}^n)$. Then under the convergence of the integrals in the right hand side, there exists finite limit $s_{k,f}(x_0) = \lim\limits_{r\to 0} P_{k-1,B(x_0,r)}f(x_0)$ and the following inequality is valid*

$$|S_{k,r}(f;K)(x_0) - s_{k,f}(x_0)| \le$$

$$\le c(n,\psi,k)\left(m_f^k(x_0;r) + \int_0^r \frac{m_f^k(x_0;t)}{t}dt + \int_0^\infty x^{n-1}\varphi(x) m_f^k(x_0;4rx)\,dx + \right.$$

$$+\int_0^r \frac{m_f^k(x_0;t)}{t}\left(\int_0^{t/r} x^{n-1}\varphi(x)dx\right)dt +$$

$$\left. + r^{k-1}\int_r^\infty \frac{m_f^k(x_0;t)}{t^k}\left(\int_{t/r}^\infty x^{n+k-2}\varphi(x)dx\right)dt\right),$$

where $c(n,\psi,k)$ is a positive constant dependent only on n, ψ and k.

Proposition 4.1. *Let $K \in L^1(\mathbb{R}^n)$, $K_r(x) := r^{-n}K\left(\frac{x}{r}\right)$, $r > 0$, $x \in \mathbb{R}^n$, $k \in N$. Then the following conditions are equivalent:*

A) $\int_{\mathbb{R}^n} K(-t)\varphi_\nu(t)\,dt = \varphi_\nu(0)$, where $|\nu| \le k-1$;

B) for any polynomial $\pi \in P_{k-1}$ and for any number $r > 0$ it holds the identity $(K_r * \pi)(x) \equiv \pi(x)$, $x \in \mathbb{R}^n$.

Proof. Let condition A) be fulfilled. Then for $f \in L_{loc}(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} K_r(x-t)\,P_{k-1,B(x,r)}f(t)\,dt =$$

$$= \int_{\mathbb{R}^n} K_r(x-t)\sum_{|\nu|\le k-1}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\varphi_\nu\left(\frac{y-x}{r}\right)dy\right)\varphi_\nu\left(\frac{t-x}{r}\right)dt =$$

$$= \sum_{|\nu|\le k-1}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\varphi_\nu\left(\frac{y-x}{r}\right)dy\right)\int_{\mathbb{R}^n} K_r(x-t)\varphi_\nu\left(\frac{t-x}{r}\right)dt =$$

$$= \sum_{|\nu|\le k-1}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\varphi_\nu\left(\frac{y-x}{r}\right)dy\right)\frac{1}{r^n}\times$$

$$\times\int_{\mathbb{R}^n} K\left(\frac{x-t}{r}\right)\varphi_\nu\left(\frac{t-x}{r}\right)dt,\ x \in \mathbb{R}^n.$$

Making change of variables by the formulas $t - x = ru$, $dt = r^n du$ in the last integral we get

$$\int_{\mathbb{R}^n} K_r(x-t)\,P_{k-1,B(x,r)}f(t)\,dt =$$

$$= \sum_{|\nu|\le k-1}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\varphi_\nu\left(\frac{y-x}{r}\right)dy\right)\int_{\mathbb{R}^n} K(-u)\varphi_\nu(u)\,du,\ x \in \mathbb{R}^n. \tag{4.5}$$

Furthermore, from the definition of the polynomial $P_{k-1,B(x,r)}f$ it follows that

$$P_{k-1,B(x,r)}f(x)=\sum_{|\nu|\leq k-1}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}f(y)\varphi_\nu\left(\frac{y-x}{r}\right)dy\right)\varphi_\nu(0)\,,\ x\in\mathbb{R}^n. \tag{4.6}$$

From equalities (4.5) and (4.6) by condition A) we get

$$\int_{\mathbb{R}^n}K_r(x-t)\,P_{k-1,B(x,r)}f(t)\,dt=P_{k-1,B(x,r)}f(x),\ x\in\mathbb{R}^n. \tag{4.7}$$

Therefore, if $f\in L_{loc}(\mathbb{R}^n)$, then

$$S_{k,r}(f)(x)=\int_{\mathbb{R}^n}K_r(x-t)\left[f(t)-P_{k-1,B(x,r)}f(t)\right]dt+P_{k-1,B(x,r)}f(x)=$$

$$=\int_{\mathbb{R}^n}K_r(x-t)\,f(t)\,dt-\int_{\mathbb{R}^n}K_r(x-t)\,P_{k-1,B(x,r)}f(t)\,dt+P_{k-1,B(x,r)}f(x)=$$

$$=(K_r*f)(x)-P_{k-1,B(x,r)}f(x)+P_{k-1,B(x,r)}f(x)=(K_r*f)(x),\ x\in\mathbb{R}^n. \tag{4.8}$$

We showed above that for any polynomial $\pi\in P_{k-1}$ and for any number $r>0$ the identity $S_{k,r}\pi(x)\equiv\pi(x)$, $x\in\mathbb{R}^n$, is valid. Therefore, as it follows from equality (4.8) the relation

$$\forall\pi\in P_{k-1}:(K_r*\pi)(x)\equiv\pi(x),\ x\in\mathbb{R}^n,\ r>0,$$

will be also fulfilled, i.e. condition B) holds.

Now, let condition B) be fulfilled. There in particular it follows that for each multi-index $\nu=(\nu_1,\nu_2,...,\nu_n)$ with the condition $|\nu|\leq k-1$ the identity

$$(K_r*\varphi_\nu)(x)\equiv\varphi_\nu(x),\ x\in\mathbb{R}^n,\ r>0$$

is valid. Then

$$(K_r*\varphi_\nu)(0)=\varphi_\nu(0)\,,\ r>0,\ |\nu|\leq k-1.$$

Finally, if $r=1$, $|\nu|\leq k-1$, then $(K*\varphi_\nu)(0)=\varphi_\nu(0)$, and this is condition A). The proposition is proved. □

Remark 4.3. In the course of the proof of proposition 4.1 we showed that if the kernel $K(x)$ satisfies condition A), then for any function $f\in L_{loc}(\mathbb{R}^n)$ equality (4.8) is fulfilled, i.e. singular integrals $S_{k,r}f(x)$ and $(K_r*f)(x)$ coincide between themselves. Therefore, from theorems 4.1 and 4.2 we get appropriate statements for the singular integral $(K_r*f)(x)$.

Theorem 4.3. *Let $K\in\Lambda_k$, $k\in N$, $x_0\in\mathbb{R}^n$, $f\in L_{loc}(\mathbb{R}^n)$ and let condition A) be fulfilled. Then under the convergence of the integrals in the right hand side, the following inequality is valid*

$$\left|(K_r*f)(x_0)-P_{k-1,B(x_0,r)}f(x_0)\right|\leq c(n,\psi,k)\Big(m_f^k(x_0;r)+$$

$$+\int_0^\infty x^{n-1}\varphi(x)m_f^k(x_0;4rx)\,dx+\int_0^r\frac{m_f^k(x_0;t)}{t}\left(\int_0^{t/r}x^{n-1}\varphi(x)dx\right)dt+$$

$$+r^{k-1}\int_r^\infty\frac{m_f^k(x_0;t)}{t^k}\left(\int_{t/r}^\infty x^{n+k-2}\varphi(x)dx\right)dt\Big)\,, \tag{4.9}$$

where $c(n,\psi,k)$ is a positive constant dependent only on n, ψ and k.

Theorem 4.4. *Let $K \in \Lambda_k$, $k \in N$, $x_0 \in \mathbb{R}^n$, $f \in L_{loc}(\mathbb{R}^n)$ and let condition A) be fulfilled. Then under the convergence of the integrals in the right hand side, there exists finite limit $s_{k,f}(x_0) = \lim_{r\to 0} P_{k-1,B(x_0,r)} f(x_0)$ and the following inequality is valid*

$$|(K_r * f)(x_0) - s_{k,f}(x_0)| \le c(n,\psi,k) \left(m_f^k(x_0;r) + \int_0^r \frac{m_f^k(x_0;t)}{t} dt + \right.$$

$$+ \int_0^\infty x^{n-1}\varphi(x) m_f^k(x_0;4rx)\,dx + \int_0^r \frac{m_f^k(x_0;t)}{t} \left(\int_0^{t/r} x^{n-1}\varphi(x)dx \right) dt +$$

$$\left. + r^{k-1} \int_r^\infty \frac{m_f^k(x_0;t)}{t^k} \left(\int_{t/r}^\infty x^{n+k-2}\varphi(x)dx \right) dt \right),$$

where $c(n,\psi,k)$ is a positive constant dependent only on n, ψ and k.

Theorem 4.5. *Let $K \in \Lambda_k$, $k \in N$, $f \in L_{loc}(\mathbb{R}^n)$. Then under the convergence of the integrals in the right hand side, the following inequality is valid*

$$\left| S_{k,r}(f;K)(x) - P_{k-1,B(x,r)} f(x) \right| \le c(n,\psi,k) \left(M_f^k(r) + \right.$$

$$+ \int_0^\infty y^{n-1}\varphi(y) M_f^k(4ry)\,dy + \int_0^r \frac{M_f^k(t)}{t} \left(\int_0^{t/r} y^{n-1}\varphi(y)dy \right) dt +$$

$$\left. + r^{k-1} \int_r^\infty \frac{M_f^k(t)}{t^k} \left(\int_{t/r}^\infty y^{n+k-2}\varphi(y)dy \right) dt \right), \quad x \in \mathbb{R}^n, \tag{4.10}$$

where $c(n,\psi,k)$ is a positive constant dependent only on n, ψ and k.

Theorem 4.6. *Let $K \in \Lambda_k$, $k \in N$, $f \in L_{loc}(\mathbb{R}^n)$. Then under the convergence of the integrals in the right hand side, at any point $x \in \mathbb{R}^n$ there exists finite limit $s_{k,f}(x) = \lim_{r\to 0} P_{k-1,B(x,r)} f(x)$ and the following inequality is valid*

$$|S_{k,r}(f;K)(x) - s_{k,f}(x)| \le c(n,\psi,k) \left(M_f^k(r) + \right.$$

$$+ \int_0^r \frac{M_f^k(t)}{t} dt + \int_0^\infty y^{n-1}\varphi(y) M_f^k(4ry)\,dy + \int_0^r \frac{M_f^k(t)}{t} \left(\int_0^{t/r} y^{n-1}\varphi(y)dy \right) dt +$$

$$\left. + r^{k-1} \int_r^\infty \frac{M_f^k(t)}{t^k} \left(\int_{t/r}^\infty y^{n+k-2}\varphi(y)dy \right) dt \right), \quad x \in \mathbb{R}^n,$$

where $c(n,\psi,k)$ is a positive constant dependent only on n, ψ and k.

Theorem 4.7. *Let $K \in \Lambda_k$, $k \in N$, $f \in L_{loc}(\mathbb{R}^n)$ and let condition A) be fulfilled. Then under the convergence of the integrals in the right hand side, the following inequality is valid*

$$\left| (K_r * f)(x) - P_{k-1,B(x,r)} f(x) \right| \le c(n,\psi,k) \left(M_f^k(r) + \right.$$

$$+ \int_0^\infty y^{n-1}\varphi(y) M_f^k(4ry)\,dy + \int_0^r \frac{M_f^k(t)}{t} \left(\int_0^{t/r} y^{n-1}\varphi(y)dy \right) dt +$$

$$+r^{k-1}\int_r^\infty \frac{M_f^k(t)}{t^k}\left(\int_{t/r}^\infty y^{n+k-2}\varphi(y)dy\right)dt\Bigg),\ x\in\mathbb{R}^n, \tag{4.11}$$

where $c(n,\psi,k)$ *is a positive constant dependent only on* n, ψ *and* k.

Theorem 4.8. *Let* $K\in\Lambda_k$, $k\in N$, $f\in L_{loc}(\mathbb{R}^n)$ *and let condition A) be fulfilled. Then under the convergence of the integrals in the right hand side, at any point* $x\in\mathbb{R}^n$ *there exists finite limit* $s_{k,f}(x)=\lim\limits_{r\to 0}P_{k-1,B(x,r)}f(x)$ *and the following inequality is valid*

$$|(K_r*f)(x)-s_{k,f}(x)|\le c(n,\psi,k)\Bigg(M_f^k(r)+\int_0^r\frac{M_f^k(t)}{t}dt+$$

$$+\int_0^\infty y^{n-1}\varphi(y)M_f^k(4ry)\,dy+\int_0^r\frac{M_f^k(t)}{t}\left(\int_0^{t/r}y^{n-1}\varphi(y)dy\right)dt+$$

$$+r^{k-1}\int_r^\infty\frac{M_f^k(t)}{t^k}\left(\int_{t/r}^\infty y^{n+k-2}\varphi(y)dy\right)dt\Bigg),x\in\mathbb{R}^n,$$

where $c(n,\psi,k)$ *is a positive constant dependent only on* n, ψ *and* k.

Theorem 4.9. *Let* $K\in\Lambda_k$, $k\in N$ *and let for any function* $f\in L_{loc}(\mathbb{R}^n)$, *for which the integrals in the right hand side of inequality* (4.11) *converge, this inequality is valid. Then condition A) is fulfilled.*

Proof. Let $f(x)\equiv\pi(x)$, where $\pi\in P_{k-1}$. Then $M_f^k(r)\equiv 0$, $r>0$. Therefore, for any polynomial $\pi\in P_{k-1}$ the integrals in the right hand side of inequality (4.11) converge, and by the theorem conditions this inequality is valid. Furthermore, $P_{k-1,B(x,r)}\pi(x)\equiv\pi(x)$, $x\in\mathbb{R}^n$. Therefore, from inequality (4.11) we get that for any polynomial $\pi\in P_{k-1}$ it holds the identity

$$(K_r*\pi)(x)\equiv\pi(x),\ x\in\mathbb{R}^n,\ r>0,$$

i.e. condition B) from proposition 4.1 is fulfilled. Hence, by this proposition it follows that condition A) is fulfilled. The theorem is proved. □

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Received: May 7, 2018; Accepted: September 27, 2018