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SOME NEW CLASSES OF HARMONIC CONVEX FUNCTIONS AND INEQUALITIES

MUHAMMAD ASLAM NOOR, KHALIDA INAYAT NOOR, SABAH IFTIKHAR, AND MUHAMMAD UZAIR AWAN

Abstract. In this paper, we introduce and study some new classes of convex functions which are called generalized harmonic convex and generalized harmonic log-convex functions. These new classes of harmonic functions unify serval new and previous known classes of harmonic convex functions. We derive some estimates for the class of functions whose certain powers of the absolute value are generalized harmonic convex function. Results obtained in this paper continue to hold for several classes of harmonic convex functions, which can be viewed as special cases. The ideas and techniques of this may inspire research in this field.

1. Introduction

Convexity plays an important role in pure and applied mathematics. Inequalities present an attractive and active field of research. In recent years, various inequalities for convex functions and their variant forms are being developed using innovative techniques. The concept of convexity has been extended and generalized in several directions to tackle a wide class of problems which arise in pure and applied sciences. Tunc et. al. [34] considered the class of tgs-convex functions. A significant generalization of convex functions is that of harmonic convex functions which was introduced by Anderson et al. [1] and Iscan [7]. Noor and Noor [13] proved that the minimum of the differentiable harmonic convex functions can be characterized by a class of variational inequalities, called harmonic variational inequalities. For the applications and numerical methods of the harmonic variational inequalities, see Noor and Noor [13, 14, 19] and the references therein. In recent years, several extensions and generalizations for harmonic convex functions have been considered and analyzed using novel and innovative techniques, see [1, 7, 8, 9, 12, 15, 16, 17, 18, 20, 21, 22, 23, 24, 25, 26, 27, 28, 30, 31, 32, 33, 34, 35].

Inspired and motivated by ongoing research, we introduce and investigate some new classes of convex functions, called generalized harmonic convex functions and generalized harmonic log-convex functions. These classes are quite different from each other. We obtain new Hadamard type inequalities for these classes. It is

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shown that these new classes of harmonic convex functions contain several new and known classes of harmonic convex functions such as harmonic P-functions, harmonic tgs-convex functions and harmonic Godunova-Levin tgs-convex functions as special cases. Results obtained in this paper can be viewed as significant refinement and improvement of the known results.

2. Preliminaries

In this section, we discuss some new and known results.

Definition 2.1. [1, 7]. A set $I = [a, b] \subseteq \mathbb{R} \setminus \{0\}$ is said to be a harmonic convex set, if

$$\frac{xy}{tx + (1-t)y} \in I, \qquad \forall x, y \in I, t \in [0, 1].$$

Definition 2.2. [1, 7]. A function $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonic convex function, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq (1-t)f(x) + tf(y), \quad \forall x, y \in I, t \in [0, 1].$$

We now introduce a new class of harmonic convex functions.

Definition 2.3. A function $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be generalized harmonic convex function, where $s \in [-1, 1]$, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq t^s(1-t)^s[f(x) + f(y)], \quad \forall x, y \in I, t \in (0, 1). \tag{2.1}$$

The function f is said to be generalized harmonic concave function, if -f is generalized harmonic convex function.
We say that f is generalized harmonic mid convex, if

$$f\left(\frac{2xy}{x + y}\right) \leq \frac{f(x) + f(y)}{4^s}. \tag{2.2}$$

We now discuss some special cases of generalized harmonic convex functions, which appears to be new one.

I). If $s = 0$, then Definition 2.3 reduces to:

Definition 2.4. [18]. A function $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonic P-function, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq f(x) + f(y), \quad \forall x, y \in I, t \in [0, 1].$$

II). If $s = 1$, then Definition 2.3 reduces to:

Definition 2.5. A function $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonic tgs-convex, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq t(1-t)[f(x) + f(y)], \quad \forall x, y \in I, t \in [0, 1].$$

III). If $s = -1$, then Definition 2.3 reduces to:

Definition 2.6. A function $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonic Godunova-Levin tgs-convex, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq \frac{1}{t(1-t)}[f(x)+f(y)], \quad \forall x, y \in I, t \in (0,1).$$

We now introduce a new class of harmonic log-convex functions.

Definition 2.7. A function $f : I = [a, b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be generalized harmonic log-convex function on I, where $s \in [-1, 1]$, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq \big[f(x)f(y)\big]^{t^s(1-t)^s}, \qquad \forall x, y \in I, t \in (0,1).$$

It follows that

$$\log f\left(\frac{xy}{tx+(1-t)y}\right) \leq t^s(1-t)^s[\log f(x)+\log f(y)], \qquad \forall x, y \in I, t \in (0,1).$$

From definition 2.7, we have

$$\begin{aligned} f\left(\frac{xy}{tx+(1-t)y}\right) &\leq \big[f(x)f(y)\big]^{t^s(1-t)^s} \\ &\leq t^s(1-t)^s[f(x)+f(y)]. \end{aligned}$$

This shows that, generalized harmonic log-convex function implies generalized harmonic convex function, but the converse is not true.
For appropriate and suitable choices of s, one can obtain several new and known classes of harmonic convex functions from Definition 2.3 and Definition 2.7. This shows that generalized harmonic convex and generalized harmonic log-convex functions are quite general and unifying ones.

Definition 2.8. [9]. A function $f : [a, b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonic symmetric with respect to $\frac{2ab}{a+b}$, if

$$f(x) = f\left(\frac{abx}{(a+b)x-ab}\right) \quad \forall x \in [a, b].$$

We recall the following special functions which are known as Beta function and hypergeometric function respectively.

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t, \quad x, y > 0,$$

$${}_2F_1[a, b; c, z] = \frac{1}{\beta(b, c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\mathrm{d}t, \quad c > b > 0, |z| < 1.$$

We also need the following well-known result, which plays a crucial part in the derivation of our results.

Remark 2.1. Let $I = [a, b] \subseteq \mathbb{R} \setminus \{0\}$ and consider the function $g : \left[\frac{1}{b}, \frac{1}{a}\right] \to \mathbb{R}$ defined by $g(s) = f\left(\frac{1}{s}\right)$. Then f is generalized harmonic convex function on $[a, b]$, if and only if, g is generalized convex function on $\left[\frac{1}{b}, \frac{1}{a}\right]$.

3. Main results

In this section, we derive Hermite-Hadamard inequalities for generalized harmonic convex and generalized harmonic log-convex functions.

Theorem 3.1. *Let $f : I = [a,b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be generalized harmonic convex function. If $f \in L[a,b]$, then*

$$2^{2s-1} f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,\mathrm{d}x \leq [f(a)+f(b)]\beta(s+1,s+1). \tag{3.1}$$

Proof. Let f be a generalized harmonic convex function. Then, taking $x = \frac{ab}{ta+(1-t)b}$ and $y = \frac{ab}{(1-t)a+tb}$ in (2.2), we have

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\leq \frac{1}{4^s}\left[f\left(\frac{ab}{ta+(1-t)b}\right) + f\left(\frac{ab}{(1-t)a+tb}\right)\right] \\ &= \frac{1}{4^s}\left[\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t + \int_0^1 f\left(\frac{ab}{(1-t)a+tb}\right)\mathrm{d}t\right] \\ &= \frac{ab}{2^{2s-1}(b-a)}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \\ &= \frac{1}{2^{2s-1}}\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t \\ &\leq \frac{f(a)+f(b)}{2^{2s-1}}\int_0^1 t^s(1-t)^s\mathrm{d}t \\ &= \frac{f(a)+f(b)}{2^{2s-1}}\beta(s+1,s+1), \end{aligned}$$

the required result. □

Theorem 3.2. *Let $f, g : I = [a,b] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be generalized harmonic convex functions. If $f, g \in L[a,b]$, then*

$$\begin{aligned} &\frac{ab}{b-a}\int_a^b \frac{f(x)g\left(\frac{abx}{(a+b)x-ab}\right)}{x^2}\mathrm{d}x \leq [M(a,b)+N(a,b)]\beta(2s+1,2s+1) \\ &\leq \frac{\left[f(a)+f(b)\right]^2 + \left[g(a)+g(b)\right]^2}{2}\beta(2s+1,2s+1), \end{aligned}$$

where

$$M(a,b) = f(a)g(a) + f(b)g(b) \tag{3.2}$$

$$N(a,b) = f(a)g(b) + f(b)g(a) \tag{3.3}$$

Proof. Let f, g be generalized harmonic convex functions. Then

$$\begin{aligned} &\frac{ab}{b-a}\int_a^b \frac{f(x)g\left(\frac{abx}{(a+b)x-ab}\right)}{x^2}\mathrm{d}x \\ &= \int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{(1-t)a+tb}\right)\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^1 [t^s(1-t)^s[f(a)+f(b)]][t^s(1-t)^s[g(a)+g(b)]\mathrm{d}t \\
&= [f(a)g(a)+f(b)g(b)+f(a)g(b)+f(b)g(a)]\int_0^1 t^{2s}(1-t)^{2s}\mathrm{d}t \\
&= [M(a,b)+N(a,b)]\beta(2s+1,2s+1) \\
&\leq \int_0^1 \left\{\frac{t^{2s}(1-t)^{2s}[\big[f(a)+f(b)\big]^2+\big[g(a)+g(b)\big]^2]}{2}\right\}\mathrm{d}t \\
&= \frac{\big[f(a)+f(b)\big]^2+\big[g(a)+g(b)\big]^2}{2}\int_0^1 t^{2s}(1-t)^{2s}\mathrm{d}t \\
&= \frac{\big[f(a)+f(b)\big]^2+\big[g(a)+g(b)\big]^2}{2}\beta(2s+1,2s+1),
\end{aligned}
$$

the required result. □

If $f=g$ in Theorem 3.2, then it reduces to the following result.

Corollary 3.1. *Let $f: I=[a,b]\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be generalized harmonic convex functions. If $f\in L[a,b]$, then*

$$
\frac{ab}{b-a}\int_a^b \frac{f(x)f\big(\frac{abx}{(a+b)x-ab}\big)}{x^2}\mathrm{d}x \quad\leq\quad [f(a)+f(b)]^2\beta(2s+1,2s+1).
$$

Theorem 3.3. *Let $f,g: I=[a,b]\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be generalized harmonic convex functions. If $fg\in L[a,b]$, then*

$$
2^{4s-1}f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right)-\frac{ab}{b-a}\int_a^b\frac{f(x)g(x)}{x^2}\mathrm{d}x \leq [M(a,b)+N(a,b)]\beta(2s+1,2s+1).
$$

where $M(a,b)$ and $N(a,b)$ are given by (3.2) *and* (3.3) *respectively.*

Proof. Let f,g be generalized harmonic convex functions. Then, taking $x=\frac{ab}{ta+(1-t)b}$ and $y=\frac{ab}{(1-t)a+tb}$ in (2.2), we have

$$
\begin{aligned}
f\left(\frac{2ab}{a+b}\right) &\leq \frac{1}{4^s}\left[f\left(\frac{ab}{ta+(1-t)b}\right)+f\left(\frac{ab}{(1-t)a+tb}\right)\right]. \\
g\left(\frac{2ab}{a+b}\right) &\leq \frac{1}{4^s}\left[g\left(\frac{ab}{ta+(1-t)b}\right)+g\left(\frac{ab}{(1-t)a+tb}\right)\right].
\end{aligned}
$$

Consider

$$
\begin{aligned}
&f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right) \\
&\leq \frac{1}{16^s}\left[f\left(\frac{ab}{ta+(1-t)b}\right)+f\left(\frac{ab}{(1-t)a+tb}\right)\right] \\
&\qquad \left[g\left(\frac{ab}{ta+(1-t)b}\right)+g\left(\frac{ab}{(1-t)a+tb}\right)\right] \\
&= \frac{1}{16^s}\left[f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)\right. \\
&\quad + f\left(\frac{ab}{(1-t)a+tb}\right)g\left(\frac{ab}{(1-t)a+tb}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{(1-t)a+tb}\right)\\
&+ f\left(\frac{ab}{(1-t)a+tb}\right)g\left(\frac{ab}{ta+(1-t)b}\right)\Bigg]\\
\leq &\frac{1}{16^s}\left[f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)\right.\\
&+ f\left(\frac{ab}{(1-t)a+tb}\right)g\left(\frac{ab}{(1-t)a+tb}\right)\\
&\left.+ 2t^{2s}(1-t)^{2s}[f(a)+f(b)][g(a)+g(b)]\right]\\
= &\frac{1}{16^s}\left[\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t\right.\\
&+ \int_0^1 f\left(\frac{ab}{(1-t)a+tb}\right)g\left(\frac{ab}{(1-t)a+tb}\right)\mathrm{d}t\\
&\left.+ 2[f(a)g(a)+f(b)g(b)+f(a)g(b)+f(b)g(a)]\int_0^1 t^{2s}(1-t)^{2s}\mathrm{d}t\right]\\
= &\frac{1}{16^s}\left[\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t\right.\\
&+ \int_0^1 f\left(\frac{ab}{(1-t)a+tb}\right)g\left(\frac{ab}{(1-t)a+tb}\right)\mathrm{d}t\\
&\left.+ 2[M(a,b)+N(a,b)]\beta(2s+1,2s+1)\right]\\
= &\frac{1}{2^{4s-1}}\left[\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}t + [M(a,b)+N(a,b)]\beta(2s+1,2s+1)\right].
\end{aligned}
$$

This completes the proof. □

Theorem 3.4. *Let* $f, g : I = [a, b] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ *be generalized harmonic convex functions. If* $fg \in L[a, b]$, *then*

$$
\begin{aligned}
&\frac{ab}{b-a}\int_a^b \mu(x)\frac{f(a)g(x)+f(b)g(x)}{x^2}\mathrm{d}x + \frac{ab}{b-a}\int_a^b \mu(x)\frac{g(a)f(x)+g(b)f(x)}{x^2}\mathrm{d}x\\
&\leq [M(a,b)+N(a,b)]\beta(2s+1,2s+1) + \frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x, \qquad (3.4)
\end{aligned}
$$

where $M(a, b)$ *and* $N(a, b)$ *are given by* (3.2) *and* (3.3) *respectively and*

$$
\mu(x) = \left(\frac{ab(x-a)(b-x)}{x^2(b-a)^2}\right)^s. \qquad (3.5)
$$

Proof. Let f, g be generalized harmonic convex functions. Then

$$
\begin{aligned}
f\left(\frac{ab}{ta+(1-t)b}\right) &\leq t^s(1-t)^s[f(a)+f(b)].\\
g\left(\frac{ab}{ta+(1-t)b}\right) &\leq t^s(1-t)^s[g(a)+g(b)].
\end{aligned}
$$

Now, using $\langle x_1 - x_2, x_3 - x_4\rangle \geq 0, (x_1, x_2, x_3, x_4 \in \mathbb{R})$ and $x_1 < x_2$, $x_3 < x_4$, we have

$$\begin{aligned}
&f\left(\frac{ab}{ta+(1-t)b}\right)t^s(1-t)^s[g(a)+g(b)] + g\left(\frac{ab}{ta+(1-t)b}\right)t^s(1-t)^s[f(a)+f(b)]\\
&\leq t^{2s}(1-t)^{2s}[f(a)+f(b)][g(a)+g(b)]\\
&+ f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)\\
&= t^{2s}(1-t)^{2s}[f(a)g(a)+f(b)g(b)+f(a)g(b)+f(b)g(a)]\\
&\quad + f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)
\end{aligned}$$

Integrating over $[0, 1]$, we have

$$\begin{aligned}
&[g(a)+g(b)]\int_0^1 t^s(1-t)^s f\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t\\
&\quad + [f(a)+f(b)]\int_0^1 t^s(1-t)^s g\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t\\
&\leq [f(a)g(a)+f(b)g(b)+f(a)g(b)+f(b)g(a)]\int_0^1 t^{2s}(1-t)^{2s}\mathrm{d}t\\
&\quad + \int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t
\end{aligned}$$

This implies

$$\begin{aligned}
&\frac{ab}{b-a}\int_a^b \mu(x)\frac{f(a)g(x)+f(b)g(x)}{x^2}\mathrm{d}x\\
&+\frac{ab}{b-a}\int_a^b \mu(x)\frac{g(a)f(x)+g(b)f(x)}{x^2}\mathrm{d}x\\
&\leq [M(a,b)+N(a,b)]\beta(2s+1, 2s+1) + \frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x,
\end{aligned}$$

the required result. □

Theorem 3.5. *Let $f, g : I = [a, b] \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be generalized harmonic convex functions. If $fg \in L[a, b]$, then*

$$\begin{aligned}
&f\left(\frac{2ab}{a+b}\right)\frac{ab}{b-a}\int_a^b \frac{g(x)}{x^2}\mathrm{d}x + g\left(\frac{2ab}{a+b}\right)\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\\
&\leq \frac{1}{4^s}\left[\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x + [M(a,b)+N(a,b)]\beta(2s+1, 2s+1)\right]\\
&\quad + 2^{2s-1}f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right),
\end{aligned}$$

where $M(a, b)$ and $N(a, b)$ are given by (3.2) and (3.3) respectively.

Proof. Let f, g be generalized harmonic convex function. Then, taking $x = \frac{ab}{ta+(1-t)b}$ and $y = \frac{ab}{(1-t)a+tb}$ in (2.2), we have

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{1}{4^s}\left[f\left(\frac{ab}{ta+(1-t)b}\right) + f\left(\frac{ab}{(1-t)a+tb}\right)\right].$$

$$g\left(\frac{2ab}{a+b}\right) \leq \frac{1}{4^s}\left[g\left(\frac{ab}{ta+(1-t)b}\right) + g\left(\frac{ab}{(1-t)a+tb}\right)\right].$$

Now, using $\langle x_1 - x_2, x_3 - x_4\rangle \geq 0, (x_1, x_2, x_3, x_4 \in \mathbb{R})$ and $x_1 < x_2$, $x_3 < x_4$, we have

$$\begin{aligned}
&\frac{1}{4^s} f\left(\frac{2ab}{a+b}\right)\left[g\left(\frac{ab}{ta+(1-t)b}\right) + g\left(\frac{ab}{(1-t)a+tb}\right)\right] \\
&\quad + \frac{1}{4^s} g\left(\frac{2ab}{a+b}\right)\left[f\left(\frac{ab}{ta+(1-t)b}\right) + f\left(\frac{ab}{(1-t)a+tb}\right)\right] \\
&\leq \frac{1}{16^s}\left[f\left(\frac{ab}{ta+(1-t)b}\right) + f\left(\frac{ab}{(1-t)a+tb}\right)\right] \\
&\qquad \left[g\left(\frac{ab}{ta+(1-t)b}\right) + g\left(\frac{ab}{(1-t)a+tb}\right)\right] + f\left(\frac{2ab}{a+b}\right) g\left(\frac{2ab}{a+b}\right) \\
&= \frac{1}{16^s}\left[f\left(\frac{ab}{ta+(1-t)b}\right) g\left(\frac{ab}{ta+(1-t)b}\right) + f\left(\frac{ab}{(1-t)a+tb}\right) g\left(\frac{ab}{(1-t)a+b}\right)\right. \\
&\quad \left. + f\left(\frac{ab}{ta+(1-t)b}\right) g\left(\frac{ab}{(1-t)a+tb}\right) + f\left(\frac{ab}{(1-t)a+tb}\right) g\left(\frac{ab}{ta+(1-t)b}\right)\right] \\
&\quad + f\left(\frac{2ab}{a+b}\right) g\left(\frac{2ab}{a+b}\right) \\
&\leq \frac{1}{16^s}\left[f\left(\frac{ab}{ta+(1-t)b}\right) g\left(\frac{ab}{ta+(1-t)b}\right) + f\left(\frac{ab}{(1-t)a+tb}\right) g\left(\frac{ab}{(1-t)a+b}\right)\right. \\
&\quad \left. + 2t^{2s}(1-t)^{2s}[f(a) + f(b)][g(a) + g(b)]\right] + f\left(\frac{2ab}{a+b}\right) g\left(\frac{2ab}{a+b}\right)
\end{aligned}$$

Integrating over [0, 1], we have

$$\begin{aligned}
&\frac{1}{4^s} f\left(\frac{2ab}{a+b}\right) \int_0^1 \left[g\left(\frac{ab}{ta+(1-t)b}\right) + g\left(\frac{ab}{(1-t)a+tb}\right)\right] \mathrm{d}t \\
&\quad + \frac{1}{4^s} g\left(\frac{2ab}{a+b}\right) \int_0^1 \left[f\left(\frac{ab}{ta+(1-t)b}\right) + f\left(\frac{ab}{(1-t)a+tb}\right)\right] \mathrm{d}t \\
&\leq \frac{1}{16^s}\left[\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right) g\left(\frac{ab}{ta+(1-t)b}\right) \mathrm{d}t\right. \\
&\quad + \int_0^1 f\left(\frac{ab}{(1-t)a+tb}\right) g\left(\frac{ab}{(1-t)a+b}\right) \mathrm{d}t \\
&\quad + 2[f(a)g(a) + f(b)g(b) + f(a)g(b) + f(b)g(a)] \int_0^1 t^{2s}(1-t)^{2s} \mathrm{d}t \\
&\quad \left. + \int_0^1 f\left(\frac{2ab}{a+b}\right) g\left(\frac{2ab}{a+b}\right)\right] \mathrm{d}t \\
&= \frac{1}{16^s}\left[\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right) g\left(\frac{ab}{ta+(1-t)b}\right) \mathrm{d}t\right. \\
&\quad + \int_0^1 f\left(\frac{ab}{(1-t)a+tb}\right) g\left(\frac{ab}{(1-t)a+b}\right) \mathrm{d}t
\end{aligned}$$

$$+2[M(a,b)+N(a,b)]\beta(s+1,s+1)\Big]+f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right).$$

From the above inequality, it follows that

$$\begin{aligned}
&f\left(\frac{2ab}{a+b}\right)\frac{ab}{b-a}\int_a^b\frac{g(x)}{x^2}\mathrm{d}x+g\left(\frac{2ab}{a+b}\right)\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x\\
&\leq\frac{1}{4^s}\left[\frac{ab}{b-a}\int_a^b\frac{f(x)g(x)}{x^2}\mathrm{d}x+[M(a,b)+N(a,b)]\beta(s+1,s+1)\right]\\
&\quad+2^{2s-1}f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right),
\end{aligned}$$

which is the requires result. □

Theorem 3.6. *Let $f:I=[a,b]\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be increasing generalized harmonic* log-*convex function. If $f\in L[a,b]$, then*

$$\int_a^b\frac{f(x)}{x^2}\mathrm{d}x\mu_1(t,a;b)\leq\frac{ab}{8(b-a)}\int_a^b\frac{f^4(x)}{x^2}\mathrm{d}x+\frac{\mu_2(t,a;b)}{8}+1,$$

where

$$\mu_1(t,a;b)=\int_0^1[f(a)f(b)]^{t^s(1-t)^s}\mathrm{d}t,$$

and

$$\mu_2(t,a;b)=\int_0^1[f(a)f(b)]^{4t^s(1-t)^s}\mathrm{d}t.$$

Proof. Let f be generalized harmonic log-convex function, then

$$f\left(\frac{ab}{ta+(1-t)b}\right)\leq[f(a)f(b)]^{t^s(1-t)^s}.$$

Now using the identity

$$8xy\leq x^4+y^4+8,\qquad x,y\in\mathbb{R},$$

we have

$$\begin{aligned}
&8\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)[f(a)f(b)]^{t^s(1-t)^s}\mathrm{d}t\\
&\leq\int_0^1 f^4\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t+\int_0^1[f(a)f(b)]^{4t^s(1-t)^s}\mathrm{d}t+8.
\end{aligned}$$

Since f is an increasing function, we have

$$\begin{aligned}
&8\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t\int_0^1[f(a)f(b)]^{t^s(1-t)^s}\mathrm{d}t\\
&\leq\int_0^1 f^4\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t+\int_0^1[f(a)f(b)]^{4t^s(1-t)^s}\mathrm{d}t+8.
\end{aligned}$$

From the above inequality it is easy to observe that

$$\int_a^b\frac{f(x)}{x^2}\mathrm{d}x\mu_1(t,a;b)\leq\frac{ab}{8(b-a)}\int_a^b\frac{f^4(x)}{x^2}\mathrm{d}x+\frac{\mu_2(t,a;b)}{8}+1,$$

the required result. □

Theorem 3.7. *Let $f, g : I = [a,b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be harmonic* log-*convex functions. If $fg \in L[a,b]$, then*

$$\frac{ab}{b-a}\int_a^b \left(\frac{f(x)}{x^2}\right)[g(a)g(b)]^{\mu(x)}\mathrm{d}x + \frac{ab}{b-a}\int_a^b \left(\frac{g(x)}{x^2}\right)[f(a)f(b)]^{\mu(x)}\mathrm{d}x$$
$$\leq \frac{ab}{b-a}\int_a^b \left(\frac{f(x)g(x)}{x^2}\right)\mathrm{d}x + \mu_3(t,a;b).$$

where $\mu(x)$ is given by (3.5) *and*

$$\mu_3(t,a;b) = \int_0^1 [f(a)f(b)g(a)g(b)]^{t^s(1-t)^s}\mathrm{d}t.$$

Proof. Let f and g be generalized harmonic log-convex functions. Then

$$f\left(\frac{ab}{ta+(1-t)b}\right) \leq [f(a)f(b)]^{t^s(1-t)^s},$$
$$g\left(\frac{ab}{ta+(1-t)b}\right) \leq [g(a)g(b)]^{t^s(1-t)^s}.$$

Now, using $\langle x_1 - x_2, x_3 - x_4\rangle \geq 0$, $(x_1, x_2, x_3, x_4 \in \mathbb{R})$ and $x_1 < x_2$, $x_3 < x_4$, we have

$$f\left(\frac{ab}{ta+(1-t)b}\right)[g(a)g(b)]^{t^s(1-t)^s} + g\left(\frac{ab}{ta+(1-t)b}\right)[f(a)f(b)]^{t^s(1-t)^s}$$
$$\leq f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right) + [f(a)f(b)g(a)g(b)]^{t^s(1-t)^s}.$$

Integrating over $[0,1]$, we have

$$\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)[g(a)g(b)]^{\mu(x)}\mathrm{d}t$$
$$+ \int_0^1 g\left(\frac{ab}{ta+(1-t)b}\right)[f(a)f(b)]^{\mu(x)}\mathrm{d}t$$
$$\leq \int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t + \int_0^1 [f(a)f(b)g(a)g(b)]^{t^s(1-t)^s}\mathrm{d}t.$$

From the above inequality it follows that

$$\frac{ab}{b-a}\int_a^b \left(\frac{f(x)}{x^2}\right)[g(a)g(b)]^{\mu(x)}\mathrm{d}x + \frac{ab}{b-a}\int_a^b \left(\frac{g(x)}{x^2}\right)[f(a)f(b)]^{\mu(x)}\mathrm{d}x$$
$$\leq \frac{ab}{b-a}\int_a^b \left(\frac{f(x)g(x)}{x^2}\right)\mathrm{d}x + \mu_3(t,a;b),$$

the required result. □

4. Integral Inequalities

We need the following result in order to obtain our results for generalized harmonic convex functions, which is proved using the technique of Liu[10].

Lemma 4.1. [29]. *If $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is a function such that $f \in L[a, b]$, then the following equality holds for some fixed $p, q > 0$.*

$$\int_a^b (x-a)^p (b-x)^q f(x)\mathrm{d}x = a^{p+1} b^{q+1} (b-a)^{p+q+1} \int_0^1 \frac{t^p (1-t)^q}{A_t^{p+q+2}} f\left(\frac{ab}{A_t}\right)\mathrm{d}t,$$

where $A_t = ta + (1-t)b$.

Theorem 4.1. *Let $f : I = [a, b] \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a function such that $f \in L[a, b]$. If f is generalized harmonic convex function, where $s \in [-1, 1]$ and $p, q > 0$, then*

$$\int_a^b (x-a)^p (b-x)^q f(x)\mathrm{d}x \leq a^{p+1} b^{q+1} (b-a)^{p+q+1} [f(a) + f(b)]\psi_1(t, a; b).$$

where

$$\begin{aligned} \psi_1(t, a; b) &= \int_0^1 \frac{t^{p+s}(1-t)^{q+s}}{A_t^{p+q+2}} \\ &= \frac{\beta(p+s+1, q+s+1)}{b^{p+q+2}} \\ &\quad {}_2F_1[p+q+2, p+s+1; p+q+2s+2; 1-\frac{a}{b}]. \end{aligned} \tag{4.1}$$

Proof. Using Lemma 4.1 and generalized harmonic convexity of f, we have

$$\begin{aligned} &\int_a^b (x-a)^p (b-x)^q f(x)\mathrm{d}x \\ &= a^{p+1} b^{q+1} (b-a)^{p+q+1} \int_0^1 \frac{t^p (1-t)^q}{A_t^{p+q+2}} f\left(\frac{ab}{A_t}\right)\mathrm{d}t \\ &\leq a^{p+1} b^{q+1} (b-a)^{p+q+1} \int_0^1 \frac{t^p (1-t)^q}{A_t^{p+q+2}} [t^s (1-t)^s][f(a) + f(b)]\mathrm{d}t \\ &= a^{p+1} b^{q+1} (b-a)^{p+q+1} [f(a) + f(b)] \int_0^1 \frac{t^{p+s}(1-t)^{q+s}}{A_t^{p+q+2}}\mathrm{d}t \\ &= a^{p+1} b^{q+1} (b-a)^{p+q+1} [f(a) + f(b)]\psi_1(t, a; b). \end{aligned}$$

the required result. □

Corollary 4.1. *Under the assumptions of Theorem 4.1 with $s = 0$, we have*

$$\begin{aligned} \int_a^b (x-a)^p (b-x)^q f(x)\mathrm{d}x &\leq \left(\frac{a}{b}\right)^{p+1} (b-a)^{p+q+1} [f(a) + f(b)]\beta(p+2, q+2) \\ &\quad {}_2F_1[p+q+2, p+1; p+q+2; 1-\frac{a}{b}]. \end{aligned}$$

Corollary 4.2. *Under the assumptions of Theorem 4.1 with $s = 1$, we have*

$$\begin{aligned} \int_a^b (x-a)^p (b-x)^q f(x)\mathrm{d}x &\leq \left(\frac{a}{b}\right)^{p+1} (b-a)^{p+q+1} [f(a) + f(b)]\beta(p+2, q+2) \\ &\quad {}_2F_1[p+q+2, p+2; p+q+4; 1-\frac{a}{b}]. \end{aligned}$$

Corollary 4.3. *Under the assumptions of Theorem 4.1 with $s=-1$, we have*

$$\int_a^b (x-a)^p(b-x)^q f(x)\mathrm{d}x \leq \left(\frac{a}{b}\right)^{p+1}(b-a)^{p+q+1}[f(a)+f(b)]\beta(p,q)$$
$$\quad {}_2F_1[p+q+2,p;p+q;1-\frac{a}{b}].$$

Theorem 4.2. *Let $f: I=[a,b]\subseteq \mathbb{R}\backslash\{0\}\to\mathbb{R}$ be a function such that $f\in L[a,b]$. If $|f|^\lambda$ is generalized harmonic convex function, where $s\in[-1,1]$ and $p,q>0$, $\lambda\geq 1$, then*

$$\int_a^b (x-a)^p(b-x)^q f(x)\mathrm{d}x \leq a^{p+1}b^{q+1}(b-a)^{p+q+1}\big(\psi_2(t,a;b)\big)^{1-\frac{1}{\lambda}}$$
$$\left([|f(a)|^\lambda+|f(b)|^\lambda]\psi_1(t,a;b)\right)^{\frac{1}{\lambda}},$$

where $\psi_1(t,a;b)$ is given by (4.1) *and*

$$\psi_2(t,a;b) = \int_0^1 \frac{t^p(1-t)^q}{A_t^{p+q+2}}\mathrm{d}t$$
$$= \frac{\beta(p+1,q+1)}{b^{p+q+2}}{}_2F_1[p+q+2,p+1;p+q+2;1-\frac{a}{b}].$$

Proof. Using Lemma 4.1, generalized harmonic convexity of $|f|^\lambda$ and the power mean inequality, we have

$$\int_a^b (x-a)^p(b-x)^q f(x)\mathrm{d}x$$
$$= a^{p+1}b^{q+1}(b-a)^{p+q+1}\int_0^1 \frac{t^p(1-t)^q}{A_t^{p+q+2}}\left|f\left(\frac{ab}{A_t}\right)\right|\mathrm{d}t$$
$$\leq a^{p+1}b^{q+1}(b-a)^{p+q+1}\left(\int_0^1 \frac{t^p(1-t)^q}{A_t^{p+q+2}}\mathrm{d}t\right)^{1-\frac{1}{\lambda}}\left(\int_0^1 \frac{t^p(1-t)^q}{A_t^{p+q+2}}\left|f\left(\frac{ab}{A_t}\right)\right|^\lambda\mathrm{d}t\right)^{\frac{1}{\lambda}}$$
$$\leq a^{p+1}b^{q+1}(b-a)^{p+q+1}\left(\int_0^1 \frac{t^p(1-t)^q}{A_t^{p+q+2}}\mathrm{d}t\right)^{1-\frac{1}{\lambda}}$$
$$\times\left(\int_0^1 \frac{t^p(1-t)^q}{A_t^{p+q+2}}[t^s(1-t)^s][|f(a)|^\lambda+|f(b)|^\lambda]\mathrm{d}t\right)^{\frac{1}{\lambda}}$$
$$= a^{p+1}b^{q+1}(b-a)^{p+q+1}\left(\int_0^1 \frac{t^p(1-t)^q}{A_t^{p+q+2}}\mathrm{d}t\right)^{1-\frac{1}{\lambda}}$$
$$\times\left([|f(a)|^\lambda+|f(b)|^\lambda]\int_0^1 \frac{t^{p+s}(1-t)^{q+s}}{A_t^{p+q+2}}\mathrm{d}t\right)^{\frac{1}{\lambda}}$$
$$= a^{p+1}b^{q+1}(b-a)^{p+q+1}\big(\psi_2(t,a;b)\big)^{1-\frac{1}{\lambda}}\left([|f(a)|^\lambda+|f(b)|^\lambda]\psi_1(t,a;b)\right)^{\frac{1}{\lambda}},$$

which the the required result. □

Theorem 4.3. *Let $f: I=[a,b]\subseteq \mathbb{R}\backslash\{0\}\to\mathbb{R}$ be a function such that $f\in L[a,b]$. If $|f|^\lambda$ is generalized harmonic convex function, where $s\in[-1,1]$ and $p,q>0$,* *then*

$$\int_a^b (x-a)^p(b-x)^q f(x)\mathrm{d}x$$
$$\leq \; a^{p+1}b^{q+1}(b-a)^{p+q+1}\big(\psi_3(t,a;b)\big)^{\frac{1}{\mu}}\left[\{|f(a)|^\lambda+|f(b)|^\lambda\}\beta(s+1,s+1)\right]^{\frac{1}{\lambda}},$$

where

$$\begin{aligned}\frac{1}{\lambda}+\frac{1}{\mu} &= 1,\\ \psi_3(t,a;b) &= \int_0^1 \frac{t^{p\mu}(1-t)^{q\mu}}{A_t^{(p+q+2)\mu}}\mathrm{d}t\\ &= \frac{\beta(p+1,q+1)}{b^{(p+q+2)\mu}}\,{}_2F_1[(p+q+2)\mu, p\mu+1;(p+q)\mu+1;1-\frac{a}{b}].\end{aligned}$$

Proof. Using Lemma 4.1, generalized harmonic convexity of $|f|^\lambda$ and the Holder integral inequality, we have

$$\begin{aligned}&\int_a^b (x-a)^p(b-x)^q f(x)\mathrm{d}x\\ &= a^{p+1}b^{q+1}(b-a)^{p+q+1}\int_0^1 \frac{t^p(1-t)^q}{A_t^{p+q+2}}\left|f\left(\frac{ab}{A_t}\right)\right|\mathrm{d}t\\ &\leq a^{p+1}b^{q+1}(b-a)^{p+q+1}\left(\int_0^1 \frac{t^{p\mu}(1-t)^{q\mu}}{A_t^{(p+q+2)\mu}}\mathrm{d}t\right)^{\frac{1}{\mu}}\left(\int_0^1 \left|f\left(\frac{ab}{A_t}\right)\right|^\lambda\mathrm{d}t\right)^{\frac{1}{\lambda}}\\ &\leq a^{p+1}b^{q+1}(b-a)^{p+q+1}\left(\int_0^1 \frac{t^{p\mu}(1-t)^{q\mu}}{A_t^{(p+q+2)\mu}}\mathrm{d}t\right)^{\frac{1}{\mu}}\\ &\quad\times\left(\int_0^1 [t^s(1-t)^s][|f(a)|^\lambda+|f(b)|^\lambda]\mathrm{d}t\right)^{\frac{1}{\lambda}}\\ &= a^{p+1}b^{q+1}(b-a)^{p+q+1}\left(\int_0^1 \frac{t^{p\mu}(1-t)^{q\mu}}{A_t^{(p+q+2)\mu}}\mathrm{d}t\right)^{\frac{1}{\mu}}\\ &\quad\times\left([|f(a)|^\lambda+|f(b)|^\lambda]\int_0^1 [t^s(1-t)^s]\mathrm{d}t\right)^{\frac{1}{\lambda}}\\ &= a^{p+1}b^{q+1}(b-a)^{p+q+1}\big(\psi_3(t,a;b)\big)^{\frac{1}{\mu}}\left[\{|f(a)|^\lambda+|f(b)|^\lambda\}\beta(s+1,s+1)\right]^{\frac{1}{\lambda}},\end{aligned}$$

the required result. □

Theorem 4.4. *Let $f: I=[a,b]\subseteq\mathbb{R}\backslash\{0\}\to\mathbb{R}$ be a function such that $f\in L[a,b]$. If $|f|^\lambda$ is generalized harmonic convex function, where $s\in[-1,1]$ and $p,q>0$, then*

$$\begin{aligned}\int_a^b (x-a)^p(b-x)^q f(x)\mathrm{d}x &\leq a^{p+1}b^{q+1}(b-a)^{p+q+1}\beta^{\frac{1}{\mu}}(p\mu+1,q\mu+1)\\ &\qquad\left([|f(a)|^\lambda+|f(b)|^\lambda\psi_4(t,a;b)\right)^{\frac{1}{\lambda}},\end{aligned}$$

where $\frac{1}{\lambda}+\frac{1}{\mu}=1$ *and*

$$\begin{aligned}\psi_4(t,a;b) &= \int_0^1 \frac{t^s(1-t)^s}{A_t^{(p+q+2)\lambda}}\mathrm{d}t \\ &= \frac{\beta(s+1,s+1)}{b^{(p+q+2)\lambda}} \\ &\quad {}_2F_1[(p+q+2)\lambda, s+1; 2s+2; 1-\frac{a}{b}].\end{aligned}$$

Proof. Using Lemma 4.1, generalized harmonic convexity of $|f|^\lambda$ and the Holder integral inequality, we have

$$\begin{aligned}&\int_a^b (x-a)^p(b-x)^q f(x)\mathrm{d}x \\ &= \int_0^1 \frac{t^p(1-t)^q}{A_t^{p+q+2}}\left|f\left(\frac{ab}{A_t}\right)\right|\mathrm{d}t \\ &\leq \left(\int_0^1 t^{p\mu}(1-t)^{q\mu}\mathrm{d}t\right)^{\frac{1}{\mu}}\left(\int_0^1 \frac{1}{A_t^{(p+q+2)\lambda}}\left|f\left(\frac{ab}{A_t}\right)\right|^\lambda \mathrm{d}t\right)^{\frac{1}{\lambda}} \\ &\leq a^{p+1}b^{q+1}(b-a)^{p+q+1}\left(\int_0^1 t^{p\mu}(1-t)^{q\mu}\mathrm{d}t\right)^{\frac{1}{\mu}} \\ &\quad \times \left(\int_0^1 \frac{1}{A_t^{(p+q+2)\lambda}}t^s(1-t)^s[|f(a)|^\lambda+|f(b)|^\lambda]\mathrm{d}t\right)^{\frac{1}{\lambda}} \\ &= a^{p+1}b^{q+1}(b-a)^{p+q+1}\left(\int_0^1 t^{p\mu}(1-t)^{q\mu}\mathrm{d}t\right)^{\frac{1}{\mu}} \\ &\quad \times \left([|f(a)|^\lambda+|f(b)|^\lambda \int_0^1 \frac{t^s(1-t)^s}{A_t^{(p+q+2)\lambda}}\mathrm{d}t\right)^{\frac{1}{\lambda}} \\ &= a^{p+1}b^{q+1}(b-a)^{p+q+1}\beta^{\frac{1}{\mu}}(p\mu+1,q\mu+1)\left([|f(a)|^\lambda+|f(b)|^\lambda\psi_4(t,a;b)\right)^{\frac{1}{\lambda}},\end{aligned}$$

the required result. □

Theorem 4.5. *Let* $f: I=[a,b]\subseteq\mathbb{R}\backslash\{0\}\to\mathbb{R}$ *be a function such that* $f\in L[a,b]$*. If* $|f|^\lambda$ *is generalized harmonic convex function and* $p,q>0$*, then*

$$\begin{aligned}&\int_a^b (x-a)^p(b-x)^q f(x)\mathrm{d}x \\ &\leq a^{p+1}b^{q+1}(b-a)^{p+q+1}\big(\psi_5(t,a;b)\big)^{\frac{1}{\mu}} \\ &\quad \times \left([|f(a)|^\lambda+|f(b)|^\lambda]\beta(p\lambda+s+1,q\lambda+s+1)\right)^{\frac{1}{\lambda}},\end{aligned}$$

where $\frac{1}{\lambda}+\frac{1}{\mu}=1$ *and*

$$\begin{aligned}\psi_5(t,a;b) &= \int_0^1 \frac{1}{A_t^{(p+q+2)\mu}}\mathrm{d}t \\ &= \frac{{}_2F_1[(p+q+2)\mu,1;2;1-\frac{a}{b}]}{b^{(p+q+2)\mu}}.\end{aligned}$$

Proof. Using Lemma 4.1, generalized harmonic convexity of $|f|^\lambda$ and the Holder integral inequality, we have

$$\begin{aligned}
&\int_a^b (x-a)^p(b-x)^q f(x)\mathrm{d}x \\
&= a^{p+1}b^{q+1}(b-a)^{p+q+1}\int_0^1 \frac{t^p(1-t)^q}{A_t^{p+q+2}}\left|f\left(\frac{ab}{A_t}\right)\right|\mathrm{d}t \\
&\leq a^{p+1}b^{q+1}(b-a)^{p+q+1}\left(\int_0^1 \frac{1}{A_t^{(p+q+2)\mu}}\mathrm{d}t\right)^{\frac{1}{\mu}}\left(\int_0^1 t^{p\lambda}(1-t)^{q\lambda}\left|f\left(\frac{ab}{A_t}\right)\right|^\lambda \mathrm{d}t\right)^{\frac{1}{\lambda}} \\
&\leq a^{p+1}b^{q+1}(b-a)^{p+q+1}\left(\int_0^1 \frac{1}{A_t^{(p+q+2)\mu}}\mathrm{d}t\right)^{\frac{1}{\mu}} \\
&\quad\times\left(\int_0^1 t^{p\lambda+s}(1-t)^{q\lambda+s}[|f(a)|^\lambda+|f(b)|^\lambda]\mathrm{d}t\right)^{\frac{1}{\lambda}} \\
&= a^{p+1}b^{q+1}(b-a)^{p+q+1}\big(\psi_5(t,a;b)\big)^{\frac{1}{\mu}} \\
&\quad\times\Big([|f(a)|^\lambda+|f(b)|^\lambda]\beta(p\lambda+s+1,q\lambda+s+1)\Big)^{\frac{1}{\lambda}},
\end{aligned}$$

the required result. □

Remark 4.1. For $s=0$, $s=1$ and $s=-1$, the classes of generalized harmonic convex and generalized harmonic convex function reduce to the class of harmonic P-functions, harmonic tgs-convex functions and harmonic Godunova-Levin tgs-convex functions. It is obvious that these classes of generalized harmonic convex and generalized harmonic log-convex functions are quite general and unifying ones. Results obtained in this paper continue to hold for these new and known classes of harmonic convex functions.

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Muhammad Aslam Noor
Department of Mathematics, COMSATS University Islamabad, Islamabad, Pakistan.
E-mail address: noormaslam@gmail.com

Khalida Inayat Noor
Department of Mathematics, COMSATS University Islamabad, Islamabad, Pakistan.
E-mail address: khalidan@gmail.com

Sabah Iftikhar
Department of Mathematics, COMSATS University Islamabad, Islamabad, Pakistan.
E-mail address: sabah.iftikhar22@gmail.com

Muhammad Uzair Awan
Department of Mathematics, GC University Faisalabad, Pakistan.
E-mail address: awan.uzair@gmail.com

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