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THE INVERSE SPECTRAL PROBLEM FOR THE PERTURBED HARMONIC OSCILLATOR ON THE ENTIRE AXIS

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Abstract. Transformation operators with a condition at infinity for the perturbed harmonic oscillator are constructed. The inverse spectral problem for perturbed harmonic oscillators on the whole axis with the same spectrum is investigated. The main equations of the inverse problem are obtained. The unique solvability of the main equations is proved.

1. Introduction

The problem of a quantum oscillator was an essential problem solved by Heisenberg in the framework of matrix mechanics and by Schrodinger in the language of wave mechanics. The problem of describing the oscillatory motions of atoms in molecules and crystals reduces to solving precisely this problem (see [2]). A "quantized" electromagnetic field is equivalent to a system of oscillators. McKean and Trubowitz [14] considered the problem of reconstruction for perturbed oscillator on the real line

$$T = \hat{T} + q(x), \ \widehat{T} = -\frac{d^2}{dx^2} + x^2.$$

They gave an algorithm for the reconstruction of $q(x)$ from norming constants for the class of real infinitely differentiable potentials, vanishing rapidly at $\pm\infty$, for fixed eigenvalues $\lambda_n(q) = \lambda_n(0)$ for all n and "norming constants" $\to 0$ rapidly as $n \to \infty$. Later on, Levitan [12] reproved some results of [14] without an exact definition of the class of potentials. It was also noted there that the perturbation potentials may be constructed by the standard procedure of the method of transformation operator. However, the rationale for some heuristic considerations of [14] will require the construction of a transformation operator with a condition at infinity.

We consider the perturbed oscillator T, generated on $L_2(-\infty,\infty)$ by the anharmonic equation

$$-y'' + x^2 y + q(x)\, y = \lambda y, \ -\infty < x < \infty, \quad \lambda \in C, \tag{1.1}$$

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where the real potential $q(x)$ satisfies the conditions

$$q(x) \in C^{(1)}(-\infty, \infty), \int_{-\infty}^{\infty} \left|x^j q(x)\right| dx < \infty, \; j = 0, 1, 2. \tag{1.2}$$

In present paper we construct transformation operators with a condition at infinity for the perturbed oscillator T. Furthermore, the method transformation operators used to solve the inverse spectral problem for the perturbed harmonic oscillator T having the same spectrum as the harmonic oscillator $\hat{T}$.

We note that the method transformation operators were used in [9,11,18] to solve the inverse spectral problem for the Schrodinger operators.

There are only few papers about the inverse problem for the perturbed harmonic oscillator. Some uniqueness theorems were proved in paper [3]. In the work [4] obtained the characterization and described the isospectral set for the case on the real line for $q \in H = \{q \in L_2(-\infty, \infty) : q', xq \in L_2(-\infty, \infty)\}$. A similar problem for the perturbed harmonic oscillator on the half-line with a Dirichlet boundary condition was investigated in [5]. In the work [10] considered a special kind of perturbations with exact asymptotics at $\pm\infty$.

Many papers have appeared devoted to various problems of spectral analysis of a perturbed harmonic oscillator (see [15-17], and references quoted therein).

2. The transformation operator

We consider the unperturbed equation

$$-y'' + x^2 y = \lambda y, \; 0 < x < \infty, \quad \lambda \in C. \tag{2.1}$$

It has [1] two solutions $f_0^{\pm}(x, \lambda) = D_{\frac{\lambda}{2} - \frac{1}{2}}\left(\pm\sqrt{2}x\right)$, where we use a standard notation for the Weber function $D_\nu(x)$ (or the parabolic cylinder function). It is well known (see [1,3]) that for each x the functions $f_0^{\pm}(x, \lambda)$ are entire and the following asymptotics are fulfilled:

$$f_0^{\pm}(x, \lambda) = \left(\sqrt{2}x\right)^{\frac{\lambda-1}{2}} e^{-\frac{x^2}{2}} \left(1 + O\left(x^{-2}\right)\right), \quad x \to \pm\infty,$$

uniformly with respect to λ on bounded domains.

We now consider the perturbed equation (1.1). As is shown in [3,16], the equation (1.1) under condition (1.2) has two solutions $f^{\pm}(x, \lambda)$ with asymptotics $f^{\pm}(x, \lambda) = f_0^{\pm}(x, \lambda)(1 + o(1))$, $x \to \pm\infty$. We set

$$\sigma^{\pm}(x) = \pm \int_x^{\pm\infty} |q(t)|\, dt, \sigma_1^{\pm}(x) = \pm \int_x^{\pm\infty} \sigma^{\pm}(t)\, dt\,.$$

In the next theorem, by means of the transformation operator, representations of the solutions $f^{\pm}(x, \lambda)$ are obtained.

Theorem 2.1. *If $q(x)$satisfies condition (1.2) for $j = 1$, then for all λ equation (1.1) has two solution $f^{\pm}(x, \lambda)$, representable in the form*

$$f^{\pm}(x, \lambda) = f_0^{\pm}(x, \lambda) \pm \int_x^{\pm\infty} K^{\pm}(x, t) \; f_0^{\pm}(t, \lambda)\, dt, \tag{2.2}$$

where the kernels $K^{\pm}(x, t)$ are a continuous function and satisfy the relations

$$\left|K^{\pm}(x,t)\right| \leq \frac{1}{2}\sigma^{\pm}\left(\frac{x+t}{2}\right)e^{\sigma_1^{\pm}(x)}, \tag{2.3}$$

$$K^{\pm}(x,x) = \pm\frac{1}{2}\int_x^{\pm\infty} q\,(t)\;dt. \tag{2.4}$$

Proof. Without loss of generality, consider the case of "+" and assume that $x \geq 0$.Substituting the representation (2.2) into equation (1.1), we find that function (2.2) satisfies equation (1.1), if only the kernel $K^{\pm}(x,t)$ satisfies a hyperbolic equation of the second order

$$\frac{\partial K^{+}(x,t)}{\partial x^2} - \frac{\partial K^{+}(x,t)}{\partial t^2} - \left(x^2 - t^2 + q\,(x)\right)\;K^{+}(x,t) = 0,\; 0 < x < t, \tag{2.5}$$

and the conditions

$$K^{+}(x,x) = \frac{1}{2}\int_x^{\infty} q\,(t)\;dt,\;\lim_{x+t\to\infty} K^{+}(x,t) = 0. \tag{2.6}$$

Reduce problem (2.5)-(2.6) to the integral equation. To this end we reduce equation (2.5) to the canonical form. Assuming $U(\xi,\eta) = U\left(\frac{t+x}{2},\frac{t-x}{2}\right) = K^{+}(x,t) = K^{+}(\xi-\eta,\xi+\eta)$, for this function we find following

$$L\,[U] \equiv \frac{\partial^2 U(\xi,\eta)}{\partial\xi\partial\eta} - 4\xi\eta U(\xi,\eta) = -U(\xi,\eta)\,q\,(\xi-\eta) \tag{2.7}$$

with boundary conditions

$$U(\xi,0) = \frac{1}{2}\int_{\xi}^{\infty} q\,(\alpha)\,d\alpha,\;\lim_{\xi\to\infty} U(\xi,\eta) = 0,\;\eta > 0. \tag{2.8}$$

Introduce the Riemann function $R(\xi,\eta;\xi_0,\eta_0)$ of the equation $L\,[U] = \psi(\xi,\eta)$, where $\psi(\xi,\eta) = -U(\xi,\eta)q(\xi-\eta)$, i.e., the function satisfying the equation

$$L^*(R) \equiv \frac{\partial^2 R}{\partial\xi\,\partial\eta} - 4\xi\,\eta\,R = 0,\;\; 0 < \eta < \eta_0,\;\;\; \xi_0 < \xi < \infty\,, 0 < \eta < \xi$$

and the conditions on the characteristics

$$R(\xi,\eta;\xi_0,\eta_0)\,|_{\xi=\xi_0} = 1, 0 \leq \eta \leq \eta_0, R(\xi,\eta;\xi_0,\eta_0)\,|_{\eta=\eta_0} = 1,\;\xi_0 \leq \xi < \infty.$$

Let $R(\xi,\eta,\xi_0,\eta_0) = J_0(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{z}{2}\right)^{2n}$,$z = 2\sqrt{\left(\xi^2-\xi_0^2\right)\left(\eta_0^2-\eta^2\right)}$, where $J_n(z)$ is the Bessel function of the first kind. It is easy to verify that this function satisfies the last three relations. In other words, $R(\xi,\eta,\,\xi_0,\eta_0)$ is the Riemann function of the equation (2.7) and has the symmetric property

$$R(\xi,\eta,\xi_0,\eta_0) = R(\xi_0,\eta_0,\xi,\eta)\,.$$

Using the well-known properties of the Bessel function, we find that the following relations hold

$$\begin{aligned} &\tfrac{\partial R}{\partial\xi} = O(\xi)\,,\;\tfrac{\partial R}{\partial\eta} = O(\xi)\,,\;\tfrac{\partial^2 R}{\partial\xi\partial\eta} = O(\xi)\,,\xi\to\infty,\\ &\tfrac{\partial^2 R}{\partial\xi^2} = O\left(\xi^2\right),\;\tfrac{\partial^2 R}{\partial\eta^2} = O\left(\xi^2\right),\;\xi\to\infty. \end{aligned} \tag{2.9}$$

Now, apply the Riemann method (see [6]) to the equation (2.7). Then we obtain the following integral equation for $U(\xi_0,\eta_0)$

$$U(\xi_0,\eta_0)=\frac{1}{2}\int_{\xi_0}^{\infty}R(\xi,0;\xi_0,\eta_0)q(\xi)d\xi$$
$$-\int_{\xi_0}^{\infty}d\xi\int_0^{\eta_0}U(\xi,\eta)q(\xi-\eta)R(\xi,\eta;\xi_0,\eta_0)d\eta. \tag{2.10}$$

Thus, for solving problem (2.7) - (2.8) it is enough to solve integral equation (2.10) with respect to $U(\xi_0,\eta_0)$.

Now we proceed to the investigation of the integral equation (2.10). By the method of successive approximation, let

$$U_0(\xi_0,\eta_0)=\frac{1}{2}\int_{\xi_0}^{\infty}R(\xi,0;\xi_0,\eta_0)q(\xi)d\xi,$$
$$U_n(\xi_0,\eta_0)=-\int_{\xi_0}^{\infty}d\xi\int_0^{\eta_0}U_{n-1}(\xi,\eta)q(\xi-\eta)R(\xi,\eta;\xi_0,\eta_0)d\eta.$$

From the relation $|R|\le 1$, we have

$$|U_0(\xi_0,\eta_0)|\le\frac{1}{2}\int_{\xi_0}^{\infty}|\,R(\xi,0;\xi_0,\eta_0)\,|\;|q(\xi)|\,d\xi\le\frac{1}{2}\int_{\xi_0}^{\infty}|q\left(\xi\right)|\,d\xi,$$

i.e.

$$|U_0\left(\xi_0,\eta_0\right)|\le\frac{1}{2}\sigma^+\left(\xi_0\right).$$

Further, we find that

$$\begin{aligned}
&|U_1\left(\xi_0,\eta_0\right)|\le\textstyle\int_{\xi_0}^{\infty}d\xi\int_0^{\eta_0}|U_0\left(\xi,\eta\right)|\cdot|q\left(\xi-\eta\right)\cdot R\left(\xi,\eta;\,\xi_0,\eta_0\right)|\,d\eta\le\\
&\le\textstyle\frac{1}{2}\int_{\xi_0}^{\infty}d\xi\int_0^{\eta_0}\sigma^+\left(\xi\right)\;|q\left(\xi-\eta\right)|\,d\eta\le\frac{1}{2}\int_{\xi_0}^{\infty}\sigma^+\left(\xi\right)d\xi\int_0^{\eta_0}|q\left(\xi-\eta\right)|\;d\eta\le\\
&\le\textstyle\frac{\sigma^+(\xi_0)}{2}\int_{\xi_0}^{\infty}d\xi\int_0^{\eta_0}|q\left(\xi-\eta\right)|\;d\eta=\frac{\sigma^+(\xi_0)}{2}\int_{\xi_0}^{\infty}d\xi\int_{\xi-\eta_0}^{\xi}|q\left(\alpha\right)|\;d\alpha\le\\
&\le\textstyle\frac{\sigma^+(\xi_0)}{2}\int_{\xi_0}^{\infty}\int_{\xi-\eta_0}^{\infty}|q\left(\alpha\right)|\,d\alpha d\xi\;\le\frac{\sigma^+(\xi_0)}{2}\sigma_1^+\left(\xi_0-\eta_0\right).
\end{aligned}$$

Now let

$$|U_{n-1}(\xi_0,\eta_0)|\le\frac{1}{2}\,\sigma^+\left(\xi_0\right)\frac{\left(\sigma_1^+\left(\xi_0-\eta_0\right)\right)^{n-1}}{(n-1)!}.$$

Using last relation we get

$$|U_n(\xi_0,\eta_0)|\le\int_{\xi_0}^{\infty}d\xi\int_0^{\eta_0}|q(\xi-\eta)R(\xi,\eta,\,\xi_0,\eta_0)U_{n-1}(\xi,\eta)|\,d\eta\;\le$$
$$\frac{1}{2}\,\sigma^+\left(\xi_0\right)\int_{\xi_0}^{\infty}\frac{\left(\sigma_1^+\left(\xi-\eta\right)\right)^{n-1}}{(n-1)!}\int_{\xi-\eta_0}^{\xi}|q\left(\alpha\right)|\,d\alpha d\xi$$
$$\le\;\frac{1}{2}\,\sigma^+\left(\xi_0\right)\int_{\xi_0}^{\infty}\frac{\left(\sigma_1^+\left(\xi-\eta\right)\right)^{n-1}}{(n-1)!}\int_{\xi-\eta_0}^{\infty}|q\left(\alpha\right)|\,d\alpha d\xi$$
$$=-\frac{1}{2}\,\sigma^+\left(\xi_0\right)\int_{\xi_0}^{\infty}\frac{\left(\sigma_1^+\left(\xi-\eta_0\right)\right)^{n-1}}{(n-1)!}d\sigma_1^+\left(\xi-\eta_0\right)=\frac{1}{2}\,\sigma^+\left(\xi_0\right)\frac{\left(\sigma_1^+\left(\xi_0-\eta_0\right)\right)^{n}}{n!}.$$

Thus, the series $U(\xi_0, \eta_0) = \sum_{n=0}^{\infty} U_n(\xi_0, \eta_0)$ converges absolutely and uniformly and its sum is a solution of equation (2.10). Moreover, $U(\xi_0, \eta_0)$ satisfies the inequality

$$|U(\xi_0, \eta_0)| \leq \frac{1}{2}\sigma^+(\xi_0)\, e^{\sigma_1^+(\xi_0-\eta_0)}. \tag{2.11}$$

Differentiating equation (2.10) directly and using relations (2.9), we find that the function $U(\xi_0, \eta_0)$ and thus the function $K^+(x,t) = U\left(\frac{t+x}{2}, \frac{t-x}{2}\right)$ are twice continuously differentiable. Moreover, for each fixed x we have the relations

$$\frac{\partial K^+(x,t)}{\partial x} = O\left(t^2\right), \ \frac{\partial K^+(x,t)}{\partial t} = O\left(t^2\right),$$
$$\frac{\partial^2 K^+(x,t)}{\partial x^2} = O\left(t^4\right), \ \frac{\partial^2 K^+(x,t)}{\partial t^2} = O\left(t^4\right), t \to \infty.$$

From this and (2.11) it follows that the function $K^+(x,t) = U\left(\frac{t+x}{2}, \frac{t-x}{2}\right)$ satisfies the problem (2.5), (2.6). This completes the proof of the theorem. □

3. Inverse problem

Consider the quantum-mechanical harmonic oscillator

$$\widehat{T} = -\frac{d^2}{dx^2} + x^2$$

on $L_2(-\infty, \infty)$. It is well known that the spectrum of $\widehat{T}$ is purely discrete and consists of the simple eigenvalues $\lambda_n = 2n+1$, $n = 0, 1, \ldots$. The corresponding normalized eigenfunctions $\{\psi_0(x, \lambda_n)\}_{n=0}^{\infty}$ have the form $\psi_0(x, \lambda_n) = \left(\alpha_n^0\right)^{-1} f_0^{\pm}(x, \lambda_n) = \left(\alpha_n^0\right)^{-1} 2^{-\frac{n}{2}} e^{-\frac{x^2}{2}} H_n(\pm x)$, where $H_n(x)$ is the Hermite polynomial and

$$\left(\alpha_n^0\right)^2 = \int_{-\infty}^{\infty} \left|f_0^{\pm}(x, \lambda_n)\right|^2 dx = n!\sqrt{\pi}. \tag{3.1}$$

Further, as in the papers [12,14], we assume that the perturbed oscillators T have the same spectrum as the harmonic oscillator $\hat{T}$. The functions $f^+(x, \lambda_n)$ and $f^-(x, \lambda_n)$ are eigenfunctions of the perturbed oscillators T and there exists a sequence C_n such that

$$f^+(x, \lambda_n) = C_n f^-(x, \lambda_n), \ C_n \neq 0. \tag{3.2}$$

Define the weight numbers $\alpha_n^{\pm}$ by the formulas

$$\alpha_n^{\pm} = \sqrt{\int_{-\infty}^{\infty} \left|f^{\pm}(x, \lambda_n)\right|^2 dx}\,, \ n = 0, 1, 2, \ldots. \tag{3.3}$$

The numbers $\{\lambda_n = 2n+1;\ \alpha_n^+ > 0\}_{n=0}^{\infty}$ or $\{\lambda_n = 2n+1;\ \alpha_n^- > 0\}_{n=0}^{\infty}$ are called spectral data of perturbed harmonic oscillator T. The inverse scattering problem for perturbed harmonic oscillator T is recovering the potential $q(x)$ by the spectral data.

Below we need some properties of the weight numbers.

Lemma 3.1. *For $n \to \infty$ the following asymptotic formulae hold*

$$C_n = (-1)^n \left[1 + O\left(n^{-\frac{1}{2}}\right)\right], \tag{3.4}$$

$$\left(\alpha_n^{\pm}\right)^2 = \left(\alpha_n^0\right)^2 \left[1 + O\left(n^{-\frac{1}{2}}\right)\right]. \tag{3.5}$$

Proof. We use the notation$\{u, v\} = uv' - u'v$ for the Wronskian of u, v. Let $f(x, \lambda)$ be some solution of (1.1). The standard identity $f^2 = \left\{\dot{f}, f\right\}'$ yields

$$\left(\alpha_n^{\pm}\right)^2 = -C_n^{\pm 1} \dot{\Delta}(\lambda_n), \tag{3.6}$$

where $\Delta(\lambda) \equiv \left\{f_0^+(x, \lambda), f_0^-(x, \lambda)\right\} = -2e^{-\left(\frac{c}{2} + \ln 2\right)\lambda} \prod_{n=0}^{\infty}\left(1 - \frac{\lambda}{\lambda_n}\right) e^{\frac{\lambda}{\lambda_n}}$, c is the Euler constant and $\dot{f} = \frac{\partial f}{\partial \lambda}$(see [1,5]).

Taking into account the relation $f_0^+(x, \lambda_n) = (-1)^n f_0^-(x, \lambda_n)$, by virtue of (3.6) we get

$$\left(\alpha_n^0\right)^2 = (-1)^{n+1} \dot{\Delta}(\lambda_n). \tag{3.7}$$

As is shown in the work [16], the behavior of $f^{\pm}(x, \lambda)$ as $\lambda \to \infty$ and fixed x is determined by the expansion

$$f^{\pm}(x, \lambda) = \pi^{-\frac{1}{2}} 2^{\frac{\lambda - 1}{4}} \Gamma\left(\frac{\lambda + 1}{4}\right) \left\{\cos\left[\pi \frac{\lambda - 1}{4} \mp x\sqrt{\lambda}\right] + O\left(\lambda^{-\frac{1}{2}}\right)\right\}. \tag{3.8}$$

Using (3.3), (3.8) and taking into account that $\lambda_n = 2n + 1$, we obtain (3.4). Further, using (3.6), (3.7) we get (3.5).

We note that relations (3.6) allows one to establish connections between the spectral data $\{\lambda_n = 2n + 1;\ \alpha_n^+ > 0\}_{n=0}^{\infty}$ and$\{\lambda_n = 2n + 1;\ \alpha_n^- > 0\}_{n=0}^{\infty}$. More precisely, from the given data $\{\lambda_n = 2n + 1;\ \alpha_n^+ > 0\}_{n=0}^{\infty}$ one can uniquely reconstruct $\{\lambda_n = 2n + 1;\ \alpha_n^- > 0\}_{n=0}^{\infty}$ (and vice versa).

Denote

$$F^{\pm}(x, y) = \sum_{n=0}^{\infty} \left\{\left(\alpha_n^{\pm}\right)^{-2} - \left(\alpha_n^0\right)^{-2}\right\} f_0^{\pm}(x, \lambda_n) f_0^{\pm}(y, \lambda_n). \tag{3.9}$$

Since $\left(\alpha_n^0\right)^2 = n!\sqrt{\pi} = \sqrt{\pi}\Gamma(n + 1)$, by virtue of the well-known relations for the Gamma function it follows from (3.8) that for each fixed x the relation $\frac{f_0^{\pm}(x, \lambda_n)}{\alpha_n^0} = O\left(n^{-\frac{1}{4}}\right)$, $n \to \infty$ holds. From this and (3.9) it follows that for each fixed x the series (3.9) converges in the metric of $L_2(-\infty, \infty)$. Hence, for each fixed x the function $F(x, y)$ belongs to $L_2(-\infty, \infty)$ as a function of y. □

The central role for constructing the solution of the inverse spectral problem is played by the so-called main equation which is a linear integral equation of Fredholm type.

Theorem 3.1. *For each fixed $x, x \in (-\infty, \infty)$, the functions $K^{\pm}(x, y)$, defined in representation (2.2) satisfy the linear integral equations*

$$F^{\pm}(x,y)+K^{\pm}(x,y)\pm\int_{x}^{\pm\infty}K^{\pm}(x,t)\,F^{\pm}(t,y)\,dt=0,\ \ \pm y>\pm x. \tag{3.10}$$

Equations (3.10) are called the main equations or Gelfand-Levitan-Marchenko equations.

Proof. The functions $\left\{\frac{f_0^{\pm}(x,\lambda_n)}{\alpha_n^0}\right\}_{n=0}^{\infty}$ and $\left\{\frac{f^{\pm}(x,\lambda_n)}{\alpha_n}\right\}_{n=0}^{\infty}$ are normalized eigenfunctions of $\hat{T}$ and T respectively. Consequently,

$$\sum_{n=0}^{\infty}\frac{f_0^{\pm}(x,\lambda_n)}{\alpha_n^0}\frac{f_0^{\pm}(y,\lambda_n)}{\alpha_n^0}=\delta(x-y)\,,\sum_{n=0}^{\infty}\frac{f^{\pm}(x,\lambda_n)}{\alpha_n^{\pm}}\frac{f^{\pm}(y,\lambda_n)}{\alpha_n^{\pm}}=\delta(x-y)\ , \tag{3.11}$$

where $\delta(x)$ is the Dirac delta-function. On the other hand, one can consider the relation (2.2) as a Volterra integral equation with respect to $f_0^{\pm}(x,\lambda)$. Solving this equation we obtain

$$f_0^{\pm}(y,\lambda)=f^{\pm}(y,\lambda)\pm\int_{y}^{\pm\infty}\tilde{K}^{\pm}(y,t)f^{\pm}(t,\lambda)dt. \tag{3.12}$$

Moreover, from the well-known properties of the transformation operators[19] it follows that the kernel $\tilde{K}^{\pm}(y,t)$ satisfies an inequality analogous to (2.3). Since $\tilde{K}^{\pm}(y,x)=0$, $\pm x>\pm y$, then from (3.11), (3.12), we have

$$\sum_{n=0}^{\infty}\frac{f^{\pm}(x,\lambda_n)}{\alpha_n^{\pm}}\frac{f_0^{\pm}(y,\lambda_n)}{\alpha_n^{\pm}}=$$

$$=\sum_{n=0}^{\infty}\frac{f^{\pm}(x,\lambda_n)}{\alpha_n^{\pm}}\frac{f^{\pm}(y,\lambda_n)}{\alpha_n^{\pm}}\pm\int_{y}^{\pm\infty}\tilde{K}^{\pm}(y,t)\left\{\sum_{n=0}^{\infty}\frac{f^{\pm}(x,\lambda_n)}{\alpha_n^{\pm}}\frac{f^{\pm}(t,\lambda_n)}{\alpha_n^{\pm}}\right\}dt=$$

$$=\delta(x-y)\pm\int_{y}^{\pm\infty}\tilde{K}^{\pm}(y,t)\,\delta(x-t)\,dt=\delta(x-y)\pm\tilde{K}(y,x)=\delta(x-y)\,,$$

and hence with the help of (2.2),

$$\sum_{n=0}^{\infty}\frac{f^{\pm}(x,\lambda_n)}{\alpha_n^{\pm}}\frac{f_0^{\pm}(y,\lambda_n)}{\alpha_n^{\pm}}=$$

$$=\sum_{n=0}^{\infty}\frac{f_0^{\pm}(x,\lambda_n)}{\alpha_n^{\pm}}\frac{f_0^{\pm}(y,\lambda_n)}{\alpha_n^{\pm}}\pm\int_{x}^{\pm\infty}K^{\pm}(x,t)\left\{\sum_{n=0}^{\infty}\frac{f_0^{\pm}(t,\lambda_n)}{\alpha_n^{\pm}}\frac{f_0^{\pm}(y,\lambda_n)}{\alpha_n^{\pm}}\right\}dt=$$

$$=\delta(x-y)+F^{\pm}(x,y)+K^{\pm}(x,y)\pm\int_{x}^{\pm\infty}K^{\pm}(x,t)\,F^{\pm}(t,y)\,dt.$$

Comparing the last two equations, we arrive at (3.10).

If $q(x)$ satisfies condition (1.2) for $j=2$, then as is shown in [8](see Lemma 6.3) the kernel $F^{\pm}(x,y)$ of the main equation (3.10) for each fixed $a$satisfies the inequality

$$\left|F^{\pm}(x,y)\right|\leq C^{\pm}(a)\,\sigma^{\pm}\left(\frac{x+y}{2}\right),\ \pm x>a,\ \pm y>a. \tag{3.13}$$

In addition, function $F^{\pm}(x,y)$ is symmetric with respect to x,$y$and is continuous in the set of arguments. It follows from (3.13) that

$$\pm\int_a^{\pm\infty}\sup_{\pm(x-a)>0}\left|F^{\pm}(x,y)\right|dy<\infty. \tag{3.14}$$

□

Theorem 3.2. *If function $F^{\pm}(x,y)$ satisfies conditions (3.14), (3.14), then for each fixed $x, x\in(-\infty,\infty)$ equation (3.10) has a unique solution $K^{\pm}(x,y)$ in $L_2(x,\pm\infty)$.*

Proof. For definiteness we consider equation (3.10) in the case "+". For the case "-" the arguments are the same. It is easy to check that for each fixed x, the operator

$$\Omega_x f(y)=\int_x^{\infty}F^{+}(y,t)\,f(t)\,dt$$

is compact in $L_2(x,\infty)$.

Indeed, we have

$$\int_x^{\infty}dt\int_x^{\infty}\left|F^{+}(t,y)\right|^2dy\le\int_x^{\infty}\sup_{y\ge x}\left|F^{+}(t,y)\right|dt\int_x^{\infty}\left|F^{+}(t,y)\right|dy\le$$
$$\le\int_x^{\infty}\sup_{y\ge x}\left|F^{+}(t,y)\right|dt\int_x^{\infty}\sup_{t\ge x}\left|F^{+}(t,y)\right|dy<\infty$$

Hence, the operator Ω_x is a Hilbert-Schmidt type operator.

Since (3.10) is a Fredholm equation it is sufficient to prove that the homogeneous equation

$$h(y)+\int_x^{\infty}F^{+}(t,y)\,h(t)\,dt=0 \tag{3.15}$$

has only the trivial solution $h(y)=0$. Let$h(y)$ be a solution of (3.15). Then

$$\int_x^{\infty}h^2(y)\,dy+\int_x^{\infty}\int_x^{\infty}F^{+}(t,y)\,h(t)\,h(y)\,dtdy=0,$$

or

$$\int_x^{\infty}h^2(y)\,dy+\sum_{n=0}^{\infty}\left(\alpha_n^{+}\right)^{-2}\left(\int_x^{\infty}h(y)\,f_0^{+}(y,\lambda_n)\,dy\right)^2-$$
$$-\sum_{n=0}^{\infty}\left(\alpha_n^{0}\right)^{-2}\left(\int_x^{\infty}h(y)\,f_0^{+}(y,\lambda_n)\,dy\right)^2=0.$$

Using Parsevals equality $\int_x^{\infty}h^2(y)\,dy=\sum_{n=0}^{\infty}\left(\alpha_n^{0}\right)^{-2}\left(\int_x^{\infty}h(y)\,f_0^{+}(y,\lambda_n)\,dy\right)^2$for the function $h(y)$, extended by zero for $y<x$, we obtain

$$\sum_{n=0}^{\infty}\left(\alpha_n^{+}\right)^{-2}\left(\int_x^{\infty}h(y)\,f_0^{+}(y,\lambda_n)\,dy\right)^2=0.$$

Since $(\alpha_n^+)^{-2} > 0$, then $\int_x^\infty h(y) f_0^+(y, \lambda_n)\, dy = 0, n \geq 0$. The system of functions $\{f_0^+(y, \lambda_n)\}_0^\infty$ is orthogonal basis in $L_2(-\infty, \infty)$. This yields $h(y) = 0$. □

Remark 3.1. The solution of the inverse scattering problem can be constructed by the following algorithm. Calculate the function $F^\pm(x, y)$ by the spectral data $\{\lambda_n, \alpha_n^\pm > 0\}_{n=0}^\infty$ and (3.9). Find $K(x, y)$ by solving the main equation (3.10). Construct $q(x)$ by (2.4). In this case, following the corresponding arguments in the paper [18], in a narrower class of potentials, one can achieve a complete solution of the inverse problem.

Remark 3.2. The obtained results also extend to the case when the spectra of perturbed harmonic oscillators are different. In this case we will have to use the asymptotic formula (see [4, 16]) $\lambda_n = 2n + 1 + O\left(n^{-\frac{1}{2}}\right)$, $n \to \infty$.

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