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THE SWEEP ALGORITHM FOR SOLVING THE SYSTEM OF HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS DESCRIBING THE MOTION IN OIL PRODUCTION

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Abstract. The initial problem for the system of partial differential equations of hyperbolic type, arising in modeling of the oil production by the gas lift method, is considered. The sweep algorithm for its solution is suggested, requiring to solve two differential equations, one of which corresponds to the classical quasi-linear partial differential equations, and the second is a linear ordinary differential equation of the first order, whose coefficients depend on the solution of the first equation with appropriate initial conditions. Covering the domain of definition of the solution of the system of quasi-linear equations by the uniform grid, two types of templates are used which define the solution in the all nodal points of the considered domain. This allows to solve for each time layer the linear ordinary differential equation with the help of fundamental solutions. Then, using the corresponding difference schemes the solution (the volume of gas or gas-liquid mixture, depending on the coordinates and pressure) is constructed for the original system of hyperbolic partial differential equations. On a simple example, when the initial data are constant, it is shown that the solutions coincide with the known ones.

1. Introduction

It is known [2–6, 11, 13, 14, 16, 17] that for finding of the volume of gas-liquid mixture (GLM) and the pressure at any point of the lift in the oil production by gas lift method (after changing the role of arguments x and t) the following system of partial differential equations of hyperbolic type of the first order should be investigated

$$\begin{aligned} \frac{\partial P(x,t)}{\partial x} &= -\frac{c}{F}\cdot\frac{\partial Q(x,t)}{\partial t}\ , \\ \frac{\partial Q(x,t)}{\partial x} &= -F\frac{\partial P(x,t)}{\partial t} - 2aQ(x,t),\ -\infty < x < \infty,\ \ t>0, \end{aligned} \tag{1.1}$$

with initial conditions

$$P(x,0) = P_o(x),\ \ Q\left(x,0\right) = Q_0\left(x\right),\ \ \infty < x < \infty\,, \tag{1.2}$$

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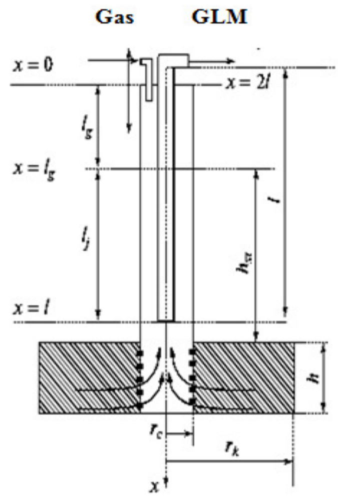

FIGURE 1

where a, F, c are constants, which have the concrete physical values [12], and $P_0(x)$, $Q_0(x)$ are given continuous and differentiable functions on x.

From the linearity of the equation (1.1), analogically to [6, 7, 13, 15], we can seek the pressure $P(x,t)$, as a linear function of the volume of GLM - $Q(x,t)$ in the following form [8–10]

$$P(x,t) = S(x,t) \cdot Q(x,t) + \alpha(t)R(x), \tag{1.3}$$

where $S(x,t)$, $R(x)$ will be determined, but in regard of a scalar function $\alpha(t)$, it is any function satisfying the following conditions [4]

$$\alpha(0) = 0, \quad \int_0^\infty \alpha(t)dt = 1, \tag{1.4}$$

i.e., in a special case, $\alpha(t)$ can be chosen as $\alpha(t) = te^{-t}$.

2. Sweep method

As proved in [4], for determining the coefficient $S(x,t)$ from (1.3) we have the following quasi-linear partial differential equation

$$\frac{\partial S(x,t)}{\partial x} + FS(x,t)\frac{\partial S(x,t)}{\partial t} - 2aS(x,t) = 0, \tag{2.1}$$

with initial condition [1, 19]

$$S(x,0) = r(x) = \frac{P_0(x)}{Q_0(x)}. \tag{2.2}$$

$R(x)$ is found from the linear ordinary differential equation

$$R'(x) - F\left(\int_0^\infty S(x,t)\alpha'(t)dt\right) R(x) + \left(FS^2(x,0) - \frac{c}{F}\right) Q(x,0) = 0, \tag{2.3}$$

with condition

$$\lim_{x\to\infty} R(x) = 0. \tag{2.4}$$

Note that this condition arises from the fact that the pressure and the gas volume (or GLM) at infinity is damped, i.e. from (1.3) we have the relation (2.2).

To find a solution of equation (2.1), (1.4), (2.4) with the conditions (2.2) and (2.3), respectively, one can use different numerical methods (due to their quasi-linearity). Now we focus on the solution of the problem (2.1), (2.2).

3. The numerical algorithm for solving of the equation (2.1) with condition (2.2)

First, we focus on the construction of a numerical algorithm [18] for the quasi-linear equation (2.1) with the initial condition (2.2), at $(x,t) \in D \subset R^2$, where $D = \{(x,t) : x \in (0,L),\ t \in (0,T)\}$. Consider a grid with step $h = \frac{L}{N}$ in the direction of the axis x, with step $\tau = \frac{T}{M}$ in the direction of the axis t, i.e.

$$x_i = ih,\ \ i = \overline{0,N},\ \ hN = L,$$
$$t_j = j\tau,\ \ j = \overline{0,M},\ \ \tau M = T.$$

Then, taking designations

$$S(x_i,t_j) = S(ih,j\tau) = S_{ij},$$

by replacing the derivatives $\frac{\partial S(x,t)}{\partial x}$ and $\frac{\partial S(x,t)}{\partial t}$ by

$$\frac{\partial S(x,t)}{\partial x} \approx \frac{S_{i+1j} - S_{i,j}}{h},\ \ \frac{\partial S(x,t)}{\partial t} \approx \frac{S_{ij+1} - S_{i,j}}{\tau},$$

the equations (2.1) are discretized as follows

$$\frac{S_{i+1j} - S_{i,j}}{h} + FS_{ij}\frac{S_{i,j+1} - S_{i,j}}{\tau} - 2aS_{ij} = 0, i = \overline{0,N-1}; j = \overline{0,M-1},$$

which after corresponding changes turns to

$$\tau S_{i+1j} - \tau S_{ij} + FhS_{ij}S_{i,j+1} - FhS_{ij}^2 - 2ah\tau S_{ij} = 0,$$

where from the latter for S_{ij+1} we find the expression

$$S_{ij+1} = S_{ij} + \frac{\tau}{Fh}(1+2ah) - \frac{\tau}{Fh}\frac{S_{i+1j}}{S_{ij}},\ \ i = \overline{0,N-1}; j = \overline{0,M-1}. \tag{3.1}$$

Now, getting back to equation (2.1), taking another discretization (significantly different from the previous)

$$\frac{\partial S(x,t)}{\partial x} \approx \frac{S_{ij} - S_{i-1j}}{h},\ \ \frac{\partial S(x,t)}{\partial t} \approx \frac{S_{ij+1} - S_{i,j}}{\tau},$$

and substituting them into (2.1), we have:

$$\frac{S_{ij} - S_{i-1j}}{h} + FS_{ij}\frac{S_{i,j+1} - S_{i,j}}{\tau} - 2aS_{ij} = 0, i = \overline{1,\ N}; j = \overline{0,M-1}.$$

After simple transformations the last takes the form

$$\tau S_{ij} - \tau S_{i-1j} + FhS_{ij}S_{i,j+1} - FhS_{ij}^2 - 2ah\tau S_{ij} = 0,$$

which allows to determine S_{ij+1} by the following relation:

$$S_{ij+1} = S_{ij} + (2ah-1)\frac{\tau}{Fh} + \frac{\tau}{Fh}\frac{S_{i-1j}}{S_{ij}}, \quad i = \overline{1,N}; j = \overline{0,M-1}. \tag{3.2}$$

The approximate formulas (3.1) and (3.2) alone can not define S_{ij} at the all nodes of the defined above grid. Therefore, we try to combine (3.1) and (3.2) so as to completely embrace all nodes of the specified grid.

Suppose that in (2.2) the function $S(x,0)$ is known. Let us write it in the form

$$S(ih,0) = S_{i0} = \frac{P(ih)}{Q(ih)} = \frac{P_{i0}}{Q_{i0}}, \quad i = \overline{0,N}. \tag{3.3}$$

In order to cover all grid nodes, we proceed as follows
1_1) from (3.1) at $i=0,\ j=0$ we have

$$S_{01} = S_{00} + \frac{\tau}{Fh}(1+2ah) - \frac{\tau}{Fh}\frac{S_{10}}{S_{00}}, \tag{3.4}$$

1_2) from (3.2) at $i=1,\ j=0$ we obtain

$$S_{11} = S_{10} + (2ah-1)\frac{\tau}{Fh} + \frac{\tau}{Fh}\frac{S_{00}}{S_{10}}. \tag{3.5}$$

Continuing this process, we will define S_{02} and S_{12} as follows
2_1) from (3.1) at $i=0,\ j=1$ we have:

$$S_{02} = S_{01} + \tfrac{\tau}{Fh}(1+2ah) - \tfrac{\tau}{Fh}\tfrac{S_{11}}{S_{01}} = S_{00} + \tfrac{2\tau}{Fh}(1+2ah) - \\ - \tfrac{\tau}{Fh}\tfrac{S_{10}}{S_{00}} - \tfrac{\tau}{Fh}\frac{S_{10}+(2ah-1)\frac{\tau}{Fh}+\frac{\tau}{Fh}\frac{S_{00}}{S_{10}}}{S_{00}+\frac{\tau}{Fh}(2ah+1)-\frac{\tau}{Fh}\frac{S_{10}}{S_{00}}}. \tag{3.6}$$

2_2) from (3.2) at $i=1,\ j=1$ we will define S_{12} in the form:

$$S_{12} = S_{11} + (2ah-1)\tfrac{\tau}{Fh} + \tfrac{\tau}{Fh}\tfrac{S_{01}}{S_{11}} = S_{10} + (-1+2ah)\tfrac{2\tau}{Fh} + \\ + \tfrac{\tau}{Fh}\tfrac{S_{00}}{S_{10}} + \tfrac{\tau}{Fh}\frac{S_{00}+\frac{\tau}{Fh}(1+2ah)-\frac{\tau}{Fh}\frac{S_{10}}{S_{00}}}{S_{10}+(2ah-1)\frac{\tau}{Fh}+\frac{\tau}{Fh}\frac{S_{00}}{S_{10}}}, \tag{3.7}$$

here we defined S_{02} and S_{12} through S_{00}, S_{10}. Continuing this process, we define all S_{0j} and S_{1j} on the vertical strips at $j = \overline{1,M}$. Indeed, if there were determined all S_{0j} and S_{1j} at $j = \overline{1,M-1}$, then
M_1) from (3.1) at $i=0$, $j=M-1$ we have:

$$S_{0M} = S_{0M-1} + \frac{\tau}{Fh}(1+2ah) - \frac{\tau}{Fh}\frac{S_{1M-1}}{S_{0M-1}}, \tag{3.8}$$

where S_{0M-1} and S_{1M-1} are known.
M_2) from (3.2) at $i=1$, $j=M-1$ we obtain:

$$S_{1M} = S_{1M-1} + (2ah-1)\frac{\tau}{Fh} + \frac{\tau}{Fh}\frac{S_{0M-1}}{S_{1M-1}}, \tag{3.9}$$

Thus, from formulas (3.4) - (3.9), we define all S_{0j} and S_{1j}at $j = \overline{1,M}$ with the initial condition (3.3).

Further, on the basis of (3.2) at $i = 2$, $j = \overline{0, M-1}$, we get all S_{2j} at $j = \overline{1, M}$. Continuing this process (based only on the formula (3.10)), we get all

$$S_{ij}, i = \overline{0, N}; j = \overline{1, M}. \tag{3.10}$$

Indeed, the last column S_{Nj}, is obtained from (3.2) at $i = N$, $j = \overline{0, M-1}$.

Note 3.1. As is seen from the above scheme, in order to compute S_{ij} at all node points there have been used two types of templates; one of them replaces derivatives on x by the step forward and the other discretizes the derivatives on x by the step back. This proves the following theorem.

Theorem 3.1. *Let c, F and a be given real constants, $P_0(x)$ and $Q_0(x)$ be given continuous functions, $\alpha(t)$ satisfies the condition* (1.4)*, $R(\infty) = 0$, then S_{ij} at $i = \overline{0, N}$, $j = \overline{0, M}$ are determined at all node points with the help of two types of columns* (3.1) *and* (3.2)*, respectively.*

Thus, the following algorithm can be proposed for finding the solution of equation (2.1).

Algorithm 3.1.

1. The parameters c, F, a included to the equation (1.1) are given, P_{i0}, Q_{i0} from (3.6) and steps of partitioning h on coordinates x, τ of time t in the form

$$x_i = ih, \ i = \overline{0, N}; \ \ t_j = j\tau, \ j = \overline{0, M}.$$

2. All nodes S_{0j} and S_{1j} are filled from the formulas (3.1) and (3.2), respectively, by columns $i = 0$, $j = \overline{1, M}$ and $i = 1$, $j = \overline{1, M}$ with alternation.

3. Other nodes S_{ij} at $i = \overline{2, N}$, $j = \overline{1, M}$ are defined by using of the formulas (3.2).

4. After appropriate interpolations S_{ij}, the obtained polynomial $\tilde{S}(x, t)$ is substituted into (2.1), and the discrepancy is determined. If it satisfies the required accuracy, the calculation is terminated, otherwise, by reducing the steps h and τ, the above steps are repeated.

4. The computational algorithm for finding $R(x)$ from (2.3)

Now we consider the calculation of $R(x)$ from the equations (2.3) with the condition (2.4). For this first on the basis of (2.4) we give the formula for finding $R(0)$ to make up a computational algorithm for finding $R(x)$. Since the equation (2.3) is a linear inhomogeneous system of ordinary differential equations, then its general solution can be represented as follows [20]

$$R(x) = e^{F\int\limits_0^x \left(\int\limits_0^\infty S(\xi,t)\alpha'(t)dt\right)d\xi} R(0) -$$

$$\int\limits_0^x e^{F\int\limits_\eta^x \left(\int\limits_0^\infty S(\xi,t)\alpha'(t)dt\right)d\xi} \left(FS^2(\eta, 0) - \frac{c}{F}\right) Q(\eta, 0)\, d\eta. \tag{4.1}$$

Considering the conditions (2.4) in (4.1) for the $R(0)$ after simple transformations we obtain the following expression

$$R(0) = \int\limits_0^\infty e^{-F\int\limits_0^\eta \left(\int\limits_0^\infty S(\xi,t)\alpha'(t)dt\right)d\xi} \left(FS^2(\eta, 0) - \frac{c}{F}\right) Q(\eta, 0)\, d\eta, \tag{4.2}$$

i.e., from the relation (4.2) $R(0)$ can be calculated through the corresponding difference scheme, in which instead of $S(x,t)$ it is necessary to consider S_{ij} from (3.10). Thus, with the initial condition $R(0)$ in the form (4.2) from (2.3) according to the following difference scheme $R(x_i) = R_i$ can be determined in the form

$$R_{i+1} = \left(1 + Fh\tau \sum_{j=0}^{\infty} S_{ij}\,(1-\tau j)\; e^{-\tau j}\right) R_i - h\left(FS_{i0}^2 - \frac{c}{F}\right) Q_{i\,0}. \tag{4.3}$$

Here $R(0)$ from (4.2) is computing according to the following approximate formula

$$R_0 = h\sum_{j=0}^{\infty} e^{-h\tau F \sum\limits_{k=0}^{j}\sum\limits_{m=0}^{\infty} S_{km}(1-\tau m)e^{-\tau m}} \left(FS_{j0}^2 - \frac{c}{F}\right) Q_{j0}. \tag{4.4}$$

Thus, we have proved the following theorem.

Theorem 4.1. *Under the conditions of Theorem 3.1, R_i is determined at all nodal points $i = \overline{1,N}$ by formulas (4.3), (4.4).*

Thus for the approximate calculation of $R(x)$ we have following

Algorithm 4.1.

(1) By using S_{ij}, computed in Algorithm 3.1, R_0 is found from the formula (4.4).
(2) With help of the formulas (4.4) and (3.10), from the recurrence relation (4.3) $R_i,\ i = \overline{1,N}$ are defined.
(3) Similarly to claim 4. of Algorithm 3.1, upon receipt of the R_i the discrepancy of the equation (2.3) is checked, where the specification of steps h and τ is consistent with the results of Algorithm 3.1.

5. The difference scheme for calculating $P(x,t)$ and $Q(x,t)$

From the difference equations (3.1), (3.2) at initial conditions (3.3) we are defining the functions $S_{i,j}$ and are supplying them into (4.3), by the initial condition (4.4) we are determining R_i. Then, substituting the found functions $S(i,j)$ and $R(i)$ into discrete analogue (1.3)

$$P_{ij} = S_{ij}Q_{ij} + \tau j e^{-\tau j} R_i, \tag{5.1}$$

we find P_{ij} using Q_{ij}.

By substituting the found relations for the function P_{ij} into the discrete analogue of the system (1.1)

$$\frac{P_{i+1j} - P_{ij}}{h} = -\frac{c}{F}\frac{Q_{ij+1} - Q_{ij}}{\tau}, \quad \frac{Q_{i+1j} - Q_{ij}}{h} = -F\frac{P_{ij+1} - P_{ij}}{\tau} - 2aQ_{ij}, \tag{5.2}$$

from the first difference equation (5.2) we have

$$hcQ_{ij+1} + F\tau S_{i+1j}Q_{i+1j} - F\tau S_{ij}Q_{ij} + F\tau^2 j e^{-\tau j} R_{i+1} - F\tau^2 j e^{-\tau j} R_i - hcQ_{ij} = 0,$$
$$i = \overline{0, N-1},\ j = \overline{0, M-1}, \tag{5.3}$$

with initial condition

$$Q_{i0} = Q_i. \tag{5.4}$$

Considering (5.4) in (5.3) at $j=0$ we have

$$Q_{i1}=\frac{1}{hc}\left[-F\tau S_{i+1,0}Q_{i+1}+(F\tau S_{i0}+hc)Q_i\right].$$

Continuing this process, for Q_{ij} we obtain the following approximate expression in the form

$$\begin{array}{l} Q_{ij}=\frac{1}{hc}[-F\tau S_{i+1,j-1}Q_{i+1,j-1}+ \\ (F\tau S_{ij-1}-hc)Q_{ij-1}-F\tau^2(j-1)e^{-\tau(j-1)}(R_{i+1}-R_i)], \\ (i=0,\,N-1,\ j=1,\,M), \end{array} \tag{5.5}$$

what proves the following theorem.

Theorem 5.1. *Under the conditions of the theorem 4.1, Q_{ij} and P_{ij} are defined using the (5.5) and (4.3), respectively.*

Thus, at the each marked i we determine Q_{ij} at all nodes of the domain under consideration and we compute P_{ij} from (5.1) with (5.5), which allows to offer the next **Algorithm 5.1.**

(1) Considering $S_{i,j}$ and R_i from the algorithms 3.1, 4.1 from (5.5) we determine the Q_{ij} with the initial condition (5.4).
(2) Substituting the defined Q_{ij}, $S_{i,j}$, R_i into (5.1) we find P_{ij}.
(3) Similar to the previous algorithms on the base of finding the discrepancy of the system of equations (1.1). the steps h and τ are specified by the given accuracy.

Let us illustrate the above results on the following example, when Q_{i0}, P_{i0} are constant.

Example 5.1. Let us consider the particular case, when S_{i0} doesn't depend on i, i.e. $S_{i0}=S=const$, $S=\frac{P}{Q}$. Then from (3.4) and (3.5) we have:

$$S_{01}=S+\frac{\tau}{Fh}(1+2ah)-\frac{\tau}{Fh}\frac{S}{S}=S+\frac{\tau}{Fh}+2a\frac{\tau}{F}-\frac{\tau}{Fh}=S+\frac{2a\tau}{F},$$

$$S_{11}=S+(2ah-1)\frac{\tau}{Fh}+\frac{\tau}{Fh}\frac{S}{S}=S+\frac{2a\tau}{Fh}-\frac{\tau}{Fh}+\frac{\tau}{Fh}=S+\frac{2a\tau}{F}.$$

Similarly, from (3.6) and (3.7) we have:

$$\begin{array}{l} S_{02}=S_{01}+\frac{\tau}{Fh}(1+2ah)-\frac{\tau}{Fh}\frac{S_{11}}{S_{01}}=S+\frac{2a\tau}{F}+ \\ +\frac{\tau}{Fh}+\frac{2a\tau}{F}-\frac{\tau}{Fh}\frac{S+\frac{2a\tau}{F}}{S+\frac{2a\tau}{F}}=S+\frac{4a\tau}{F}=\frac{P}{Q}+\frac{2a}{F}(2\tau), \end{array}$$

$$\begin{array}{l} S_{12}=S_{11}+(2ah-1)\frac{\tau}{Fh}+\frac{\tau}{Fh}\frac{S_{01}}{S_{11}}=S+\frac{2a\tau}{F}+\frac{2a\tau}{F}-\frac{\tau}{Fh}+ \\ +\frac{\tau}{Fh}+\frac{S+\frac{2a\tau}{F}}{S+\frac{2a\tau}{F}}=S+\frac{4a\tau}{F}=\frac{P}{Q}+\frac{2a}{F}(2\tau). \end{array}$$

Thus,taking

$$S_{ij}=S+\frac{2a}{F}j\tau$$

from (3.2) we can easily calculate

$$S_{ij+1}=S+\frac{2a}{F}(j+1)\tau. \tag{5.6}$$

If, $S^2=\frac{c}{F^2}$ substituting (5.6) into (4.3) and (4.4) for R_i we have:

$$R_i=0,\ \ i\geq 0. \tag{5.7}$$

Taking into account (5.6) and (5.7) for Q_{ij} from (5.5) we obtain

$$Q_{ij} = Q = const. \tag{5.8}$$

Finally, substituting (5.6), (5.7) and (5.8) into for P_{ij} we have the following formula

$$P_{ij} = P + \frac{2a}{F}(j\tau)Q.$$

Let us note that from (5.6)-(5.8) at $h \to 0$ and $\tau \to 0$ (in other words at $i \to \infty$, $j \to \infty$) for the $Q(x,t)$, $P(x,t)$, $S(x,t)$, $R(x)$ we have $Q(x,t) = Q$, $P(x,t) = P + \frac{2a}{F}tQ$, $S(x,t) = \frac{P}{Q} + \frac{2a}{F}t$, $R(x) = 0$, which completely coincide with the results obtained in [3, 4, 8–10].

In the future, we will consider the continuous dependence of the solution of the problem on the initial data.

Now, we will illustrate the computer realization on the algorithms 3.1-5.1 with a comparison of analytical solutions $S^{an}(x,t)$, $P^{an}(x,t)$, $Q^{an}(x,t)$, $R^{an}(x)$.

Suppose that in (1.1) the parameters have the following concrete values

$a = 1.4896$, $c = 850m/\sec$, $g = 9.8m/\sec^2$, $F = 0.073\,m$, $\rho = 700kg/m^3$,
$l = 1485m$, $\tau = 0,\ 5$, $h = 0,\ 1$

then using the formulas (3.1), (3.2)

$$\|S(i,j) - S^{an}(i,j)\| = \max_{i,j} |S(i,j) - S^{an}(i,j)| =\sim 3.5 \cdot 10^{-8},$$

analogously,

$$\|P(i,j) - P^{an}(i,j)\| = \max_{i,j} |P(i,j) - P^{an}(i,j)| =\sim 2.7 \cdot 10^{-5},$$

$$\|Q(i,j) - Q^{an}(i,j)\| = \max_{i,j} |Q(i,j) - Q^{an}(i,j)| =\sim 2.3 \cdot 10^{-7},$$

which confirms that the numerical solution of the problem (1.1), (1.2) requires more accurate approach.

References

[1] S.A. Aisagaliev, I.V. Sevryugin, Controllability of process described by linear system of ordinary differential equations, *TWMS J. Pure Appl. Math.* **8** (2017), 170–185.

[2] F.A. Aliev, N.A. Aliev, A.P. Guliev, Time frequency method of solving one boundary value problem for a hyperbolic system and its application to the oil extraction, *Journal of Mathematical Physics, Analysis, Geometry*, (2016), **2**, 101-112.

[3] F.A. Aliev, K.K. Gasanov, A.P. Guliev, Sweep method for solving of system of hyperbolic equation describing the motion in oil production, *Proc. of IAM*, **3** (2014), 249-256.

[4] F.A. Aliev, N.A. Ismailov, Inverse problem to determine the hydraulic resistance coefficient in the gas lift process, *Appl. Comput. Math.* **12** (2013), 306–313.

[5] F.A. Aliev, N.A. Ismailov, L.N. Temirbekova, Extremal solution of the problem of the choice of optimum modes for gas-lift process, *Appl. Comput. Math.* **11** (2012), 348–357.

[6] F.A. Aliev, M.M. Mutallimov, N.A. Ismailov, M.F. Radzhabov, Algorithms for constructing optimal controllers for gaslift operation, *Automation and Remote Control*, **73** (2012), 1279–1289.

[7] F.A. Aliev, V.B. Larin, *Optimization of linear control system: Analytical methods and Computational Algorithms*, Gordon and Breach, Amsterdam, 1998, 272p.

[8] F.A. Aliev, M.A. Dzhamalbekov, M.Kh. Ilyasov, Mathematical simulation and control of gas lift, *International Journal of Computer and System Sciences*, **50**, (2011) 805–814.

[9] F.A.Aliev, N.A.Ismailov, H.Haciyev, M.F.Guliev, A method to determine the coefficient of hydraulic resistance in different areas of pump-compressor pipes, *TWMS J. Pure Appl. Math.* **7** (2016), 237–241.

[10] F.A. Aliev, N.A. Ismailov, Optimization problems with a periodic boundary-value condition and boundary control for gas-lifting wells, *Neliniini Kolyvannya (Nonlinear Oscillations)* **17**, (2014), 151–160.

[11] Fikret A. Aliev, N.A. Ismailov, A.A. Namazov, M.F. Rajabov, Algorithm for calculating the parameters of formation of gas-liquid mixture in the shoe of gas lift well, *Applied and Computational Mathematics*, **15**, (2016), 370-376.

[12] F.A. Aliev, N.A. Aliev, A.P. Guliev, R.M. Tagiyev, M.A. Dzhamlbekov, A method for solving a boundary value problem for a system of hyperbolic equations describing motion in a gas-lift process, *Applied Mathematics and Mechanics*, **82**, 512-519.

[13] A. Bryson, Ho, Yu-Shi, Applied theory of optimal control, Mir, Moscow, (1972), 544 p.

[14] I.A. Charniy, *Unstable motion of a real fluid in pipes,* Gostekhizdat, Moscow, (1951), 389p.

[15] S.K. Godunov, *Equations of mathematical physics*. Nauka, Moscow, 1972, 392p.

[16] M.A. Jamalbayov, N.A. Veliyev, The technique of early determination of reservoir drive of gas condensate and velotail oil deposits on the basis of new diagnosis indicators, *TWMS J. Pure Appl. Math.* **8** (2017), 236–250.

[17] A.X. Mirzadjanzadeh, et al., Technology and mechanics of oil production, Nauka, Moscow, (1986), 382p.

[18] M. Sarfraz, M. Z. Hussain, F. Hussain, Shape Preserving Convex Data Interpolation, *Appl. Comput. Math.* **16** (2017), 205–227.

[19] T.E. Simos, Ch. Tsitouras, I.Th. Famelis, Explicit Numerov type methods with constant coefficients: A review *Appl. Comput. Math.* **16** (2017), 89–113.

[20] V.V. Stepanov, *Course of differential equations*, 1950, 466p.

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