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BIFURCATION OF SOLUTIONS OF NONLINEARIZABLE DIRAC PROBLEMS WITH SPECTRAL PARAMETER IN THE BOUNDARY CONDITION

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Abstract. In this paper we study bifurcation from zero of the nonlinearizable one-dimensional Dirac problem with spectral parameter in boundary condition. We prove the existence of unbounded continua of nontrivial solutions of this problem which bifurcating from intervals of the line of trivial solutions and having oscillation properties of eigenvectors of the linear problem.

1. Introduction

We consider the following nonlinear Dirac equation

$$\ell w(x) \equiv Bw'(x) - P(x)w(x) = \lambda w(x) + f(x, w(x), \lambda),\ 0 < x < \pi, \tag{1.1}$$

subject to the boundary conditions $U(\lambda, w) = \begin{pmatrix} U_1(w) \\ U_2(\lambda, w) \end{pmatrix}$ given by

$$U_1(w) := (\sin\alpha,\ \cos\alpha)\, w(0) = v(0)\cos\alpha + u(0)\sin\alpha = 0, \tag{1.2}$$

$$\begin{aligned} U_2(\lambda, w) &:= (\lambda\sin\beta + b_1,\ \lambda\cos\beta + a_1)\, w(\pi) = \\ &(\lambda\cos\beta + a_1)\, v(\pi) + (\lambda\sin\beta + b_1)\, u(\pi) = 0, \end{aligned} \tag{1.3}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ P(x) = \begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix},\ w(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix},$$

$\lambda \in \mathbb{R}$ is a spectral parameter, $p(x),\ r(x) \in C([0,\pi];\mathbb{R}),\ \alpha, \beta, a_1, b_1 \in [0, \pi)$ and satisfies the condition

$$\sigma = a_1\sin\beta - b_1\cos\beta > 0. \tag{1.4}$$

The function $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in C([0,\pi] \times \mathbb{R}^2 \times \mathbb{R}; \mathbb{R}^2)$ and satisfies the condition: there are positive constants M, K and small positive number $\varkappa$ such that

$$\begin{aligned} |f_1(x, w, \lambda)| \le K|w|,\ |f_2(x, w, \lambda)| \le M|w|,\ x \in [0, \pi], \\ 0 < |w| \le \varkappa,\ \lambda \in \mathbb{R}. \end{aligned} \tag{1.5}$$

Bifurcation of nonlinear Sturm-Liouville problems of second and fourth order have been considered by Rabinowitz [15], Berestycki [10], Schmitt and Smith [17],

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Chiappinelli [12], Rynne [16], Binding, Browne, Watson [11], Aliyev [1, 2], Aliyev and Mamedova [3]. These authors prove the existence of unbounded continua of nontrivial solutions bifurcating from points and intervals of the line of trivial solutions and having usual nodal properties.

Only Schmitt and Smith [17] considered problem (1.1)-(1.3), in the case of $a_1 = b_1 = 0$ and $K + M < \frac{1}{2}$. But the authors of the paper [17] did not completely study the structure and behavior of continua of solutions emanating from bifurcation points and intervals of the line of trivial solutions.

In [6, 7] Aliev and Rzayeva completely investigated the oscillatory properties of eigenvector-functions of the one dimensional linear Dirac eigenvalue problem (1.1)-(1.3) with $h \equiv 0$ and $a_1 = b_1 = 0$. In [8], using these oscillating properties of eigenvector functions and refined asymptotic formulas for the eigenvalues of the linear problem (1.1)-(1.3) with $h = 0$ and $a_1 = b_1 = 0$, was investigated global bifurcation of the solutions of the non-linear problem (1.1)-(1.3) with $a_1 = b_1 = 0$.

In recent papers [4] and [5] a linear eigenvalue problem was obtained from (1.1)-(1.3) for $h \equiv 0$, where it was proved that the eigenvalues of this problem are real, algebraically simple and values range from $-\infty$ to $+\infty$ and can be numbered in ascending order. In addition, the oscillatory properties of the eigenvector functions are studied and asymptotic formulas for the eigenvalues and eigenvectors are obtained. In [5] the nonlinear eigenvalue problem (1.1)-(1.3) was also studied in the more general case.

The purpose of this paper is to study the structure and behavior of global continua of nontrivial solutions of problem (1.1)-(1.3) bifurcating from intervals of the line of trivial solutions. Despite the fact that the problem (1.1)-(1.3) considered is a particular case of the problem considered in [5], but this problem has its own peculiarities. We shall prove that if condition (1.3) holds for all $w \in \mathbb{R}^2$, then unbounded continua of solutions are contained in the bands corresponding to the bifurcation intervals.

2. Preliminaries

In the case $f \equiv 0$ from (1.1)-(1.3) we get the following linear one-dimensional Dirac system

$$\begin{aligned} &\ell w(x) = \lambda w(x),\ 0 < x < \pi, \\ &U(\lambda, w) = 0. \end{aligned} \tag{2.1}$$

By [4, Lemma 2.1 and Theorem 3.2] the eigenvalues of the boundary value problem (2.1) are real, algebraically simple and the values range from $-\infty$ to $+\infty$, and can be numerated in increasing order.

In order to study the bifurcation of solutions of nonlinear problem (1.1)-(1.3) we consider a more general linear problem

$$\begin{aligned} &\tilde{\ell} w(x) \equiv Bw'(x) - \tilde{P}(x) w(x) = \lambda w(x),\ 0 < x < \pi, \\ &U(\lambda, w) = 0, \end{aligned} \tag{2.2}$$

where $\tilde{P}(x) = \begin{pmatrix} p(x) & q(x) \\ s(x) & r(x) \end{pmatrix}$ and $q(x), s(x)$ are real-valued continuous functions on the interval $[0, \pi]$.

One can readily show that there exists a unique solution $w(x,\lambda)=\begin{pmatrix} u(x,\lambda) \\ v(x,\lambda) \end{pmatrix}$ of Dirac equation

$$\tilde{\ell}w(x)=\lambda w(x),\ 0<x<\pi,$$

satisfying the initial condition

$$u(0,\lambda)=\cos\alpha,\quad v(0,\lambda)=-\sin\alpha\,; \tag{2.3}$$

moreover, for each fixed $x\in[0,\,\pi]$ the functions $u(x,\lambda)$ and $v(x,\lambda)$ are entire functions of the argument λ. The proof of this assertion reproduces that of Theorem 1.1 from [14, Ch. 1, § 1] with obvious modifications.

To study the oscillatory properties of eigenvector-functions of problem (1.1)-(1.3), we introduce the Prüfer angular variable

$$\theta(x,\lambda)=\cot^{-1}(u(x,\lambda)/v(x,\lambda)) \tag{2.4}$$

(see [9, Ch. 8, § 3]), or more precisely,

$$\theta(x,\lambda)=\arg\{u(x,\lambda)+iv(x,\lambda)\}. \tag{2.5}$$

We recall that u,v have fixed initial values for $x=0$, and all λ, given by (2.3). We define initially

$$\theta(0,\lambda)=-\,\alpha, \tag{2.6}$$

in view (2.3). For other x and λ, $\theta(x,\,\lambda)$ is given by (2.7) (or (2.6)) except for an arbitrary multiple of 2π, since u and v cannot vanish simultaneously. This multiple of 2π is to be fixed so that $\theta(x,\,\lambda)$ satisfies (2.7) and is continuous in x and λ. Since the (x,λ)-region, namely, $0\le x\le\pi$, $-\infty<\lambda<+\infty$, is simply-connected, this defines $\theta(x,\,\lambda)$ uniquely.

It is obvious that (1.3) may be rewritten in the form

$$\cot(1,\lambda)=-\frac{\lambda\cos\beta+a_1}{\lambda\sin\beta+b_1}. \tag{2.7}$$

To discuss our counterparts (2.7), let $\rho(\lambda)$ be a continuous function defined as follows:

$$\cot\rho(\lambda)=-\frac{\lambda\cos\beta+a_1}{\lambda\sin\beta+b_1},\ \rho\left(-\frac{b_1}{\sin\beta}\right)=-\pi.$$

For problem (1.1)-(1.3) we have the following result which proved in [5] (see also [4]).

Theorem 2.1 [5, Theorem 2.1]. *The eigenvalues λ_k, $k\in\mathbb{Z}$, of the problem* (2.2) *are real and simple, and can be numbered in ascending order on the real axis so that the corresponding angular function $\theta(x,\lambda_k)$ at $x=\pi$ satisfy the condition*

$$\theta(\pi,\lambda_k)=\rho(\lambda_k)+k\pi. \tag{2.8}$$

Let $E=C\left([0,\pi];\mathbb{R}^2\right)\cap\{w\,:\,U_1(w)=0\}$ be a Banach space with the usual norm $||w||=\max\limits_{x\in[0,\pi]}|u(x)|+\max\limits_{x\in[0,\pi]}|v(x)|$. Let S be the subset of E given by

$$S=\{w\in E:\ |u(x)+|v(x)|\ >0,\,\forall x\in[0,\pi]\}$$

with metric inherited from E.

For each $w = \begin{pmatrix} u \\ v \end{pmatrix} \in S$ we define $\theta(w, \cdot)$ to be continuous function on $[0, \pi]$ satisfying

$$\cot(w, x) = \frac{u(x)}{v(x)}, \;\; \theta(w, 0) = -\alpha$$

(see, e.g. [8. 11]). It is apparent that $\theta : S \times [0, \pi] \to \mathbb{R}$ is continuous. From (2.6) and (2.8) we have

$$\theta(w_k, 0) = -\alpha, \;\; \theta(w_k, \pi) = \rho(\lambda_k) + k\pi, \;\; k \in \mathbb{Z}, \tag{2.9}$$

where $w_k(x)$ is an eigenfunction corresponding to the eigenvalue λ_k of problem (2.2).

For each $k \in \mathbb{Z}$ and each $\lambda \in \mathbb{R}$ let $S^+_{k,\lambda}$ be the set of functions $w = \begin{pmatrix} u \\ v \end{pmatrix} \in S$ satisfying the following conditions:

(i) $\theta(w, \pi) = \varrho(\lambda) + k\pi$;

(ii) the function $u(x)$ is positive in a neighborhood of $x = 0$;

(iii) if $\beta = 0$ and $k > 0$ or $k = 0$, $\varrho(\lambda) \geq -\alpha$; or $\beta \neq 0$ and $k > 1$ or $k = 0, 1$, $\varrho(\lambda) \geq -\alpha$, then for fixed w, as x increases from 0 to π, the function θ cannot tend to a multiple of $\pi/2$ from above, and as x decreases, the function θ cannot tend to a multiple of $\pi/2$ from below; if $\beta = 0$ and $k < 0$ or $k = 0$, $\varrho(\lambda) \leq -\alpha$; or $\beta \neq 0$ and $k < 0$ or $k = 0, 1$, $\varrho(\lambda) \leq -\alpha$, then for fixed w, as x increases, the function θ cannot tend to a multiple of $\pi/2$ from below, and as x decreases, the function θ cannot tend to a multiple of $\pi/2$ from above.

Let $S^-_{k,\lambda} = -S^+_{k,\lambda}$ and $S_{k,\lambda} = S^+_{k,\lambda} \cup S^-_{k,\lambda}$. It follows from (2.9) and statement (ii) of Lemma 3 from [7] that for each $\lambda \in \mathbb{R}$ the sets $S^+_{k,\lambda}$, $S^-_{k,\lambda}$ and $S_{k,\lambda}$, $k \in \mathbb{Z}$, are nonempty.

From now on ν will denote an element of $\{+, -\}$ that is, either $\nu = +$ or $\nu = -$.

It follows directly from the definition of the sets $S^\nu_{k,\lambda}$, $k \in \mathbb{Z}$, that for each $\lambda \in \mathbb{R}$ they are disjoint and open in E. Furthermore, if $w \in \partial S^\nu_{k,\lambda}$, then there exists a point $t \in [0, \pi]$ such that $|w(t)| = 0$, i.e. $u(t) = v(t) = 0$ (see [8]).

Now we define the sets S_k and S^ν_k, $k \in \mathbb{Z}$, as follows:

$$S_k = \bigcup_{\lambda \in R} S_{k,\lambda} \;\text{ and }\; S^\nu_k = \bigcup_{\lambda \in R} S^\nu_{k,\lambda}.$$

It is obvious that sets S_k and S^ν_k, $k \in \mathbb{Z}$, are are disjoint and open in E. Moreover, if $w \in \partial S^\nu_k$, $k \in \mathbb{Z}$, then there exists a point $t \in [0, \pi]$ such that $|w(t)| = 0$, i.e. $u(t) = v(t) = 0$.

Lemma 2.1. [8, Lemma 2.8] *If $(\lambda, w) \in \mathbb{R} \times E$ is a solution of problem* (1.1)-(1.3) *and $w \in \partial S^\nu_k$, then $w \equiv 0$ (more precisely, $u \equiv 0$ and $v \equiv 0$).*

Let $\hat{E} = E \oplus \mathbb{R}$ be the Banach space with the norm $||\hat{w}|| = \left\| \begin{pmatrix} w \\ \eta \end{pmatrix} \right\| = ||w|| + |\eta|$.

Let us define the operator L

$$L(\hat{w}) = L\begin{pmatrix} w \\ \eta \end{pmatrix} = \begin{pmatrix} \ell(w) \\ a_1 v(\pi) - b_1 u(\pi) \end{pmatrix}$$

with the domain

$$D(L)=\left\{\hat{w}=\left(\begin{array}{c} w \\ \eta \end{array}\right)\in\hat{E}\ :\ w\in C^1\left([0,\pi];R^2\right),\ \eta=u\left(\pi\right)\sin\beta-v(\pi)\cos\beta\right\}.$$

Obviously, the operator L is well defined in $\hat{E}$. Then linear problem (2.1) takes the form

$$L\hat{w}=\lambda\hat{w}, \tag{2.10}$$

i.e., the eigenvalues λ_k, $k\in\mathbb{Z}$, of problem (2.1) and the operator L coincide, and between the eigenvector-functions, there is a one-to-one correspondence

$$w_k=\left(\begin{array}{c} u_k \\ v_k \end{array}\right)\leftrightarrow\hat{w}_k=\left(\begin{array}{c} w_k \\ \eta_k \end{array}\right)=\left(\begin{array}{c} u_k \\ v_k \\ \eta_k \end{array}\right),\ \eta_k=u_k(\pi)\,\sin\beta-v_k(\pi)\,\cos\beta.$$

It is obvious that L is a closed (generally, nonself-adjoint) operator in $\hat{E}$ with compact resolvent.

Define the operator $F:R\times\hat{E}\to\hat{E}$ as follows:

$$F(\lambda,\hat{w})=F\left(\lambda,\left(\begin{array}{c} w \\ \eta \end{array}\right)\right)=\left(\begin{array}{c} f(x,w,\lambda) \\ 0 \end{array}\right)=\left(\begin{array}{c} f_1(x,w,\lambda) \\ f_2(x,w,\lambda) \\ 0 \end{array}\right)$$

where $\eta=u\left(\pi\right)\sin\beta-v(\pi)\cos\beta$. Then problem (1.1)-(1.3) reduces to the nonlinear problem

$$L\hat{w}=\lambda\hat{w}+F(\lambda,\hat{w}). \tag{2.11}$$

i.e., between the solutions of these problems, there is a correspondence

$$(\lambda,w)\leftrightarrow(\lambda,\hat{w}).$$

If $\lambda=0$ is not an eigenvalue of the linear problem (2.1), then L^{-1} exists and $L^{-1}:\hat{E}\to\hat{E}$. Define the operators $\hat{L}:\hat{E}\to\hat{E}$ and $\hat{F}:R\times\hat{E}\to\hat{E}$ as follows:

$$\hat{L}=L^{-1},\ \text{and}\ \ \hat{F}=L^{-1}F.$$

Then problem (2.11) (or (1.1)-(1.3)) can be written in the following equivalent form

$$\hat{w}=\lambda\hat{L}\hat{w}+\hat{F}(\lambda,\hat{w}). \tag{2.12}$$

Since L has a compact resolvent in $\hat{E}$, we can be regarded $\hat{L}$ as a completely continuous operator in $\hat{E}$. Hence $\hat{F}:R\times\hat{E}\to\hat{E}$ is completely continuous.

Let

$$\hat{S}_k^\nu=\{\hat{w}\in\hat{E}:w\in S_k^\nu\},\ \hat{S}_k=\{\hat{w}\in\hat{E}:w\in S_k\},\ k\in\mathbb{Z}.$$

3. Global bifurcation of solutions of problem (1.1)-(1.3)

We denote $\hat{\mathcal{C}}$ the closure in $\mathbb{R}\times\hat{E}$ of the set of nontrivial solutions of problem (2.12). Then it obvious that $\mathcal{C}=\{(\lambda,w)\in\mathbb{R}\times E:(\lambda,\hat{w})\in\hat{\mathcal{C}}\}$ is the closure in $\mathbb{R}\times E$ of the set of nontrivial solutions of problem (1.1)-(1.3).

We say that $(\lambda,0)$ is a bifurcation point of problem (1.1)-(1.3) with respect to the set $\mathbb{R}\times S_k^\nu,\ k\in\mathbb{Z}$, if each small neighborhood of this point there is a solution of problem (1.1)-(1.3) which is contained in $\mathbb{R}\times S_k^\nu$ (see [3]).

Alongside (1.1)-(1.3) we will consider the following approximation problem

$$\begin{aligned} &\ell w(x) = \lambda w(x) + f(x, |w(x)|^{\varepsilon} w(x), \lambda),\ 0 < x < \pi, \\ &U(\lambda, w) = 0, \end{aligned} \tag{3.1}$$

where $\varepsilon \in (0, 1]$.

By (2.12) problem (3.1) can be expressed in the following equivalent form:

$$\hat{w} = \lambda \hat{L} \hat{w} + \hat{F}(\lambda, ||\hat{w}||^{\varepsilon} \hat{w}). \tag{3.2}$$

By virtue of (1.5) for any $\varepsilon \in (0, 1]$ we have

$$f(x, |w(x)|^{\varepsilon} w(x), \lambda) = o(|w|) \ \text{ as } \ |w| \to 0,$$

uniformly in $x \in [0, \pi]$ and $\lambda \in \Lambda$. Hence we have

$$\hat{F}(\lambda, ||\hat{w}||^{\varepsilon} \hat{w}) = o(||\hat{w}||) \ \text{ as } \ ||\hat{w}|| \to 0,$$

uniformly in $\lambda \in \Lambda$. Then following the appropriate reasoning, carried out in the proof of Theorem 2.3 of [15], we can prove that for each $k \in \mathbb{Z}$ and each ν, there exists an unbounded continuum $\hat{\mathcal{C}}^{\nu}_{k,\varepsilon}$ of solutions of (3.2) such that

$$\left(\lambda_k, \hat{0}\right) \in \hat{\mathcal{C}}^{\nu}_{k,\varepsilon} \subset \left(\mathbb{R} \times \hat{S}^{\nu}_k\right) \cup \left\{\left(\lambda_k, \hat{0}\right)\right\},$$

where $\hat{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Consequently, for each $k \in \mathbb{Z}$ and each ν the continuum $\mathcal{C}^{\nu}_{k,\varepsilon} = \{(\lambda, w) \in \mathbb{R} \times E : (\lambda, \hat{w}) \in \hat{\mathcal{C}}^{\nu}_{k,\varepsilon}\}$ of solutions of problem (3.1) is unbounded in $\mathbb{R} \times E$ and satisfies the condition

$$(\lambda_k, 0) \in \mathcal{C}^{\nu}_{k,\varepsilon} \subset (\mathbb{R} \times S^{\nu}_k) \cup \{(\lambda_k, 0)\}.$$

Lemma 3.1 *For each $k \in \mathbb{Z}$ and each ν, and for sufficiently small $0 < \tau < \varkappa$ there exists a solution $(\lambda_{\tau,k}, w_{\tau,k})$ of the nonlinear problem* (1.1)-(1.3) *such that $w_{\tau,k} \in S^{\nu}_k$, $\lambda_{\tau,k} \in I_k$ and $||w_{\tau,k}|| = \tau$, where $I_k = [\lambda_k - \tilde{c}_k, \lambda_k + \tilde{c}_k]$, $\tilde{c}_k = \frac{1}{2}(M + K) + 1 + c_k$, $c_k = O\left(\frac{1}{k}\right)$.*

Proof. It follows from the above that for each $k \in \mathbb{Z}$ and for any $\varepsilon \in (0, 1]$ there exists a solution $(\lambda_{\tau,\varepsilon}, w_{\tau,\varepsilon}) = \left(\lambda_{\tau,k,\varepsilon}, \begin{pmatrix} u_{\tau,k,\varepsilon} \\ v_{\tau,k\,\varepsilon} \end{pmatrix}\right)$ of problem (3.1) such that $w_{\tau,k,\varepsilon} \in S^{\nu}_k$ and $||w_{\tau,k,\varepsilon}|| = \tau$, where $\tau < \varkappa$.

Obviously, $w_{\tau,k,\varepsilon}$ is an eigenvector-function corresponding to the kth eigenvalue $\lambda_{\tau,k,\varepsilon}$ of the linear eigenvalue problem

$$\begin{aligned} &\ell w(x) = \lambda w(x) + P_{\tau,\varepsilon}(x) w(x),\ 0 < x < \pi, \\ &U(\lambda, w) = 0. \end{aligned} \tag{3.3}$$

where $P_\varepsilon(x)=\begin{pmatrix} \varphi_{\tau,\,\varepsilon}(x) & \psi_{\tau,\,\varepsilon}(x) \\ -\phi_{\tau,\,\varepsilon}(x) & -\,\omega_{\tau,\,\varepsilon}(x)\end{pmatrix}$ and the functions $\varphi_\varepsilon(x)$, $\psi_\varepsilon(x)$, $\phi_\varepsilon(x)$ and $\tau_\varepsilon(x)$ are determined as follows:

$$\begin{gathered}
\varphi_{\tau,\,\varepsilon}(x)=\frac{f_1(x,|w_{\tau,\,\varepsilon}(x)|^\varepsilon u_{\tau,\,\varepsilon}(x),|w_{\tau,\,\varepsilon}(x)|^\varepsilon v_{\tau,\,\varepsilon}(x),\lambda_{\tau,\,\varepsilon})\,u_{\tau,\,\varepsilon}(x)}{u^2_{\tau,\,\varepsilon}(x)+v^2_{\tau,\,\varepsilon}(x)},\\
\psi_{\tau,\,\varepsilon}(x)=\frac{f_1(x,|w_{\tau,\,\varepsilon}(x)|^\varepsilon u_{\tau,\,\varepsilon}(x),|w_{\tau,\,\varepsilon}(x)|^\varepsilon v_{\tau,\,\varepsilon}(x),\lambda_{\tau,\,\varepsilon})\,v_{\tau,\,\varepsilon}(x)}{u^2_{\tau,\,\varepsilon}(x)+v^2_{\tau,\,\varepsilon}(x)},\\
\phi_{\tau,\,\varepsilon}(x)=\frac{f_2(x,|w_{\tau,\,\varepsilon}(x)|^\varepsilon u_{\tau,\,\varepsilon}(x),|w_{\tau,\,\varepsilon}(x)|^\varepsilon v_{\tau,\,\varepsilon}(x),\lambda_{\tau,\,\varepsilon})\,u_{\tau,\,\varepsilon}(x)}{u^2_{\tau,\,\varepsilon}(x)+v^2_{\tau,\,\varepsilon}(x)},\\
\omega_{\tau,\,\varepsilon}(x)=\frac{f_2(x,|w_{\tau,\,\varepsilon}(x)|^\varepsilon u_{\tau,\,\varepsilon}(x),|w_{\tau,\,\varepsilon}(x)|^\varepsilon v_{\tau,\,\varepsilon}(x),\lambda_{\tau,\,\varepsilon})\,v_{\tau,\,\varepsilon}(x)}{u^2_{\tau,\,\varepsilon}(x)+v^2_{\tau,\,\varepsilon}(x)}.
\end{gathered}\tag{3.4}$$

By virtue of (1.5) it follows from (3.4) that

$$|\varphi_{\tau,\,\varepsilon}(x)|,\ |\psi_{\tau,\,\varepsilon}(x)|\le M,\ |\phi_{\tau,\,\varepsilon}(x)|,\ |\omega_{\tau,\,\varepsilon}(x)|\le K,\ \ x\in[0,\pi].\tag{3.5}$$

In view of (2.9) from [5] we have

$$\theta_k'(x)=\lambda+\frac{1}{2}\{p(x)+r(x)\}+\frac{1}{2}\{p(x)-r(x)\}\cos 2\,\theta_k(x),\tag{3.6}$$

$$\begin{gathered}
\theta'_{k,\,\tau,\,\varepsilon}(x)=\lambda+\{p(x)+r(x)+\varphi_{\tau,\varepsilon}(x)+\omega_{\tau,\varepsilon}(x)\}+\\
\tfrac{1}{2}\{p(x)+\varphi_{\tau,\varepsilon}(x)-r(x)-\omega_{\tau,\,\varepsilon}(x)\}\cos 2\,\theta_{\tau,\,\varepsilon}(x)+\\
\tfrac{1}{2}\{\psi_{\tau,\,\varepsilon}(x)+\phi_{\tau,\,\varepsilon}(x)\}\sin 2\,\theta_{\tau,\,\varepsilon},
\end{gathered}\tag{3.7}$$

where

$$\theta_k(x)=\theta(w_k,x),\ \ \theta_{k,\,\tau,\,\varepsilon}(x)=\theta(w_{k,\,\tau,\,\varepsilon},x),$$

and satisfy boundary conditions

$$\theta_k(0)=\theta_{k,\,\tau,\,\varepsilon}(0)=-\,\alpha,\ \ \theta_k(\pi)=\varrho(\lambda_k)+k\pi,\ \ \theta_{k,\,\tau,\,\varepsilon}(\pi)=\varrho(\lambda_{k,\,\tau,\,\varepsilon})+k\pi.\tag{3.8}$$

Integrating both sides of (3.6) and (3.7) in the range from 0 to π and using (3.8), we obtain

$$\rho(\lambda_k)+k\pi+\alpha=\lambda_k\pi+\frac{1}{2}\int_0^\pi\{p(x)+r(x)\}\,dx+\frac{1}{2}\int_0^\pi\{p(x)-r(x)\}\cos 2\,\theta_k(x)dx,$$

$$\begin{gathered}
\rho(\lambda_{k,\,\tau,\,\varepsilon})+k\pi+\alpha=\lambda_{k,\,\tau,\,\varepsilon}\pi+\frac{1}{2}\int_0^\pi\{p(x)+r(x)+\varphi_{\tau,\,\varepsilon}(x)+\omega_{\tau,\,\varepsilon}(x)\}\,dx+\\
\frac{1}{2}\int_0^\pi\{p(x)+\varphi_{\tau,\,\varepsilon}(x)-r(x)-\omega_{\tau,\,\varepsilon}(x)\}\cos 2\,\theta_{k,\,\tau,\,\varepsilon}(x)dx+\\
\frac{1}{2}\int_0^\pi\{\psi_{\tau,\,\varepsilon}(x)+\phi_{\tau,\,\varepsilon}(x)\}\sin 2\,\theta_{k,\,\tau,\,\varepsilon}(x)dx,
\end{gathered}$$

respectively. Subtracting the first equality from the second equality we obtain

$$\begin{gathered}\rho(\lambda_{k,\tau,\varepsilon})-\rho(\lambda_k)=(\lambda_{k,\tau,\varepsilon}-\lambda_k)\,\pi+\tfrac{1}{2}\int_0^\pi\{\varphi_{\tau,\varepsilon}(x)+\omega_{\tau,\varepsilon}(x)\}\,dx+\\ \tfrac{1}{2}\int_0^\pi\{p(x)+\varphi_{\tau,\varepsilon}(x)-r(x)-\omega_{\tau,\varepsilon}(x)\}\cos 2\,\theta_{k,\tau,\varepsilon}(x)dx+\\ \tfrac{1}{2}\int_0^\pi\{\psi_{k,\tau,\varepsilon}(x)+\phi_{\tau,\varepsilon}(x)\}\sin 2\,\theta_{k,\tau,\varepsilon}(x)dx-\\ \tfrac{1}{2}\int_0^\pi\{p(x)-r(x)\}\cos 2\,\theta_k(x)dx.\end{gathered} \tag{3.9}$$

By virtue of [13, Lemma 4.3] for sufficiently large $|k|$ we have the following relations:

$$\int_0^\pi\{p(x)+\varphi_{\tau,\varepsilon}(x)-r(x)-\omega_{\tau,\varepsilon}(x)\}\cos 2\,\theta_{k,\tau,\varepsilon}(x)dx=O\left(\frac{1}{k}\right), \tag{3.10}$$

$$\int_0^\pi\{\psi_{\tau,\varepsilon}(x)+\phi_{\tau,\varepsilon}(x)\}\sin 2\,\theta_{k,\tau,\varepsilon}dx=O\left(\frac{1}{k}\right). \tag{3.11}$$

$$\int_0^\pi\{p(x)-r(x)\}\cos 2\,\theta_k(x)dx=O\left(\frac{1}{k}\right) \tag{3.12}$$

By (1.4) the function $\rho(\lambda)$ is strictly decreasing on $\mathbb{R}$. Then it follows from (2.8) that

$$|\rho(\lambda_{k,\tau,\varepsilon})-\rho(\lambda_k)|<\pi. \tag{3.13}$$

Using (3.5) and (3.10)-(3.13) from (3.9) we obtain

$$|\lambda_{k,\tau,\varepsilon}-\lambda_k|\le\frac{1}{2}\,(M+K)+1+c_k \tag{3.14}$$

where $c_k=O\left(\frac{1}{k}\right)$.

Since $||w_{\tau,k,\varepsilon}||=\tau$ for $0<\varepsilon\le 1$, $f\in C([0,\pi]\times\mathbb{R}^2\times\mathbb{R};\mathbb{R}^2)$ and $\lambda_{\tau,k,\varepsilon}\in[\lambda_k-\tilde{c}_k,\lambda_k+\tilde{c}_k]$ for $0<\varepsilon\le 1$ (by (3.15)), it follows from (3.1) that the set $\{w_{\tau,\varepsilon}\in E\ :\ 0<\varepsilon\le 1\}$ is bounded in $C^1([0,\pi];\mathbb{R}^2)$. Then it is precompact in E by the Arzelà-Ascoli theorem.

Let $\{\varepsilon_n\}_{n=1}^\infty\subset(0,1)$ be a sequence such that $\varepsilon_n\to 0$ and $(\lambda_{\tau,k,\varepsilon_n},w_{\tau,k,\varepsilon_n})\to(\lambda_{\tau,k},w_{\tau,k})$ in $\mathbb{R}\times E$. Taking the limit in (3.11) we see that $(\lambda_{\tau,k},w_{\tau,k})$ is a solution of the nonlinear problem (1.1)-(1.3). Note that $\lambda_{\tau,k}\in I_k$, $w_{\tau,k}\in\overline{S_k^\nu}=S_k^\nu\cup\partial S_k^\nu$. Since $||w_{\tau,k}||=\tau$ it follows from Lemma 2.1 that $w_{\tau,k}\in S_k^\nu$. The proof of this lemma is complete.

Corollary 3.1 *The set of bifurcation points of problem* (1.1)-(1.3) *with respect to the set* $\mathbb{R}\times S_k^\nu$ *is nonempty, and if* $(\lambda,0)$ *is a bifurcation point of problem* (1.1)-(1.3) *with respect to the set* S_k^ν*, then* $\lambda\in I_k$.

For each $k\in\mathbb{Z}$ and each ν, let $\tilde{\mathcal{D}}_k^\nu\subset\mathcal{C}$ denote the union of all connected components $\mathcal{D}_{k,\lambda}^\nu$ of $\mathcal{C}$ emanating from bifurcation points $(\lambda,0)\in I_k\times\{0\}$ with respect to the set $\mathbb{R}\times S_k^\nu$. It follows from Corollary 3.1 that $\tilde{\mathcal{D}}_k^\nu\ne\emptyset$. Note that $\mathcal{D}_k^\nu=\tilde{\mathcal{D}}_k^\nu\cup(I_k\times\{0\})$ is a connected subset of $\mathbb{R}\times E$, but $\tilde{\mathcal{D}}_k^\nu$ is not necessarily connected in $\mathbb{R}\times E$.

Theorem 3.1 *For each $k \in \mathbb{Z}$ and each ν, the connected component $\mathcal{D}_k^\nu$ of $\mathcal{C}$ lies in $(\mathbb{R} \times S_k^\nu) \cup (I_k \times \{0\})$ and is unbounded in $\mathbb{R} \times E$.*

The proof of this theorem is similar to that of [2, Theorem 1.3] using Lemma 3.1 and Corollary 3.1.

Since between solutions of problem (1.1)-(1.3) and (2.12) there exists an isomorphism $(\lambda, w) \leftrightarrow (\lambda, \hat{w})$ Theorem 3.1 yields the following result.

Theorem 3.2 *For each $k \in \mathbb{Z}$ and each ν, the connected component $\hat{\mathcal{D}}_k^\nu = \{\hat{w} \in \hat{E} : w \in \mathcal{D}_k^\nu\}$ of $\hat{\mathcal{C}}$ lies in $(\mathbb{R} \times \hat{S}_k^\nu) \cup (I_k \times \{0\})$ and is unbounded in $\mathbb{R} \times \hat{E}$.*

New assume that the vector-function $f(x, w, \lambda)$ satisfies (1.3) for all $x \in [0, \pi]$ and $(w, \lambda) \in \mathbb{R}^2 \times \mathbb{R}$. Then we have the following statement.

Lemma 3.2 *If $(\lambda, w) \in \mathbb{R} \times E$ is a solution of problem (1.1)-(1.3), then $w \in \bigcup\limits_{n=-\infty}^{+\infty} S_n$. Moreover, if $w \in S_k$, then $\lambda \in I_k$.*

Proof. Let $(\lambda, w) \in \mathbb{R} \times E$ be a solution of problem (1.1)-(1.3). Then (λ, w) is a solution of the linear problem

$$\begin{array}{l} \ell(w)(x) + H(x) = \lambda w(x),\ 0 < x < \pi, \\ U(\lambda, w) = 0, \end{array} \tag{3.15}$$

where $H(x) = \begin{pmatrix} \varphi(x) & \psi(x) \\ -\phi(x) & -\,\omega(x) \end{pmatrix}$ and the functions $\varphi(x)$, $\psi(x)$, $\phi(x)$ and $\tau(x)$ are determined as follows:

$$\begin{array}{l} \varphi(x) = \frac{f_1(x, u(x), v(x), \lambda)\, u(x)}{u^2(x)+v^2(x)},\ \ \psi(x) = \frac{f_1(x, u(x), v(x), \lambda)\, v(x)}{u^2(x)+v^2(x)}, \\ \\ \phi(x) = \frac{f_2(x, u(x), v(x), \lambda)\, u(x)}{u^2(x)+v^2(x)},\ \ \omega(x) = \frac{f_2(x, u(x), v(x), \lambda)\, v(x)}{u^2(x)+v^2(x)}. \end{array} \tag{3.16}$$

It follows from (3.16) that $\varphi(x)$, $\psi(x)$, $\phi(x)$, $\omega(x) \in C([0, \pi] : \mathbb{R}^2)$. Then by Lemma 3 from [1] and Theorem 2.1 we have $w \in \bigcup\limits_{n=-\infty}^{+\infty} S_n$.

In view (1.5) from (3.16) we obtain

$$|\varphi(x)|,\ |\psi(x)| \le M,\ |\phi(x)|,\ |\omega(x)| \le K,\ \ x \in [0, \pi]. \tag{3.17}$$

Let $w \in S_k$ for some $k \in \mathbb{Z}$. Then by Theorem 2.1 λ is the kth eigenvalue of problem (3.15). By virtue of (3.17) it follows from proof of Lemma 3.1 that $\lambda \in I_k$ and the proof is complete.

In view of Lemma 5.2 from Theorem 3.1 and Theorem 3.2 we obtain the next results.

Theorem 3.3 *Let the condition (1.5) satisfies for all $(x, w, \lambda) \in [0, \pi] \times \mathbb{R}^2 \times \mathbb{R}$. Then for each $k \in \mathbb{Z}$ and each ν, the connected component $\mathcal{D}_k^\nu$ of $\mathcal{C}$ lies in strip $(I_k \times S_k^\nu) \cup (I_k \times \{0\})$ and is unbounded in $\mathbb{R} \times E$.*

Theorem 3.4 *Let the condition (1.5) satisfies for all $(x, w, \lambda) \in [0, \pi] \times \mathbb{R}^2 \times \mathbb{R}$. Then for each $k \in \mathbb{Z}$ and each ν, the connected component $\hat{\mathcal{D}}_k^\nu$ of $\hat{\mathcal{C}}$ lies in strip $(I_k \times \hat{S}_k^\nu) \cup (I_k \times \{\hat{0}\})$ and is unbounded in $\mathbb{R} \times \hat{E}$.*

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