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DIAGONAL LIFTS OF METRICS TO COFRAME BUNDLE

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Abstract. In this paper the diagonal lift ${}^{D}g$ of a Riemannian metric g of a manifold M_n to the coframe bundle $F^*(M_n)$ is defined, Levi-Civita connection, Killing vector fields with respect to the metric ${}^{D}g$ and also an almost paracomplex structures in the coframe bundle are studied.

1. Introduction

The Riemannian metrics in the tangent bundle firstly has been investigated by the Sasaki [14]. Tondeur [16] and Sato [15] have constructed Riemannian metrics on the cotangent bundle, the construction being the analogue of the metric Sasaki for the tangent bundle. Mok [7] has defined so-called the diagonal lift of metric to the linear frame bundle, which is a Riemannian metric resembles the Sasaki metric of tangent bundle. Some properties and applications for the Riemannian metrics of the tangent, cotangent, linear frame and tensor bundles are given in [1-4,7-9,12,13]. This paper is devoted to the investigation of Riemannian metrics in the coframe bundle. In 2 we briefly describe the definitions and results that are needed later, after which the diagonal lift ${}^{D}g$ of a Riemannian metric g is constructed in 3. The Levi-Civita connection of the metric ${}^{D}g$ is determined in In 4. In 5 we consider Killing vector fields in coframe bundle with respect to Riemannian metric ${}^{D}g$. An almost paracomplex structures in the coframe bundle equipped with metric ${}^{D}g$ are studied in 6.

2. Preliminaries

We shall summarize briefly the basic definitions and results which be used later. Let M_n be an $n-$dimensional differentiable manifold of class C^∞ and $F^*(M_n)$ its coframe bundle (see, [10, 11]). The coframe bundle $F^*(M_n)$ over M_n consists of all pairs (x, u^*), where x is a point of M_n and u^* is a basis (coframe) for the cotangent space $T^*_x M$. We denote by π the natural projection of $F^*(M_n)$ to M_n defined by $\pi(x, u^*) = x$. If $(U; x^1, x^2, ..., x^n)$ is a system of local coordinates in M_n, then a coframe $u^* = (X^\alpha) = (X^1, X^2, ..., X^n)$ for $T^*_x M_n$ can be expressed uniquely in the form $X^\alpha = X^\alpha_i (dx^i)_x$ and hence

$$\left(\pi^{-1}(U); x^1, x^2, ..., x^n, X^1_1, X^1_2, ..., X^n_n\right)$$

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is a system of local coordinates in $F^*(M_n)$ (see, [10]). Indices $i, j, k, ..., \alpha, \beta, \gamma, ...$ have range in $\{1, 2, ..., n\}$, while indices $A, B, C, ...$ have range in

$$\left\{1, ..., n, n+1, ..., n+n^2\right\}.$$

We put $h_\alpha = \alpha \cdot n + h$. Summation over repeated indices is always implied.

We denote by $\Im^r_s(M_n)$ the set of all differentiable tensor fields of type (r, s) on M_n. Let $V = V^i\partial_i$ and $\omega = \omega_i dx^i$ be the local expressions in $U \subset M_n$ of a vector and a covector (1-form) fields $V \in \Im^1_0(M_n)$ and $\omega \in \Im^0_1(M_n)$, respectively. Then the complete and horizontal lifts ${}^CV, {}^HV \in \Im^1_0(F^*M_n)$ of V and the $\beta-$th vertical lifts ${}^{V_\beta}\omega \in \Im^1_0(F^*M_n)$ $(\beta = 1, 2, ..., n)$ of ω have respectively,

$${}^CV = \begin{pmatrix} V^i \\ -X^\alpha_j\partial_i V^j \end{pmatrix}, \qquad {}^HV = \begin{pmatrix} V^i \\ X^\alpha_j\Gamma^j_{ik}V^k \end{pmatrix}, \qquad {}^{V_\beta}\omega = \begin{pmatrix} 0 \\ \delta^\alpha_\beta\omega_i \end{pmatrix} \quad (2.1)$$

with respect to the natural frame $\{\partial_i, \partial_{i_\alpha}\} = \left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial X^\alpha_i}\right\}$, (see [10] for more details).

The vertical lift of a smooth function f on M_n is a function ${}^Vf$ on $F^*(M_n)$ defined by ${}^Vf = f \circ \pi$.

Let (U, x^i) be a coordinate system in M_n. In $U \in M_n$, we put

$$X_{(i)} = \frac{\partial}{\partial x^i}, \quad \theta^{(i)} = dx^i, i = 1, 2, ..., n.$$

Taking account of (2.1), we easily see that the components of ${}^HX_{(i)}$ and ${}^{V_\alpha}\theta^{(i)}$ are respectively, given by

$$D_i = {}^HX_{(i)} = \left(A^H_i\right) = \begin{pmatrix} \delta^h_i \\ X^\alpha_j\Gamma^j_{ih} \end{pmatrix}, \quad (2.2)$$

$$D_{i_\alpha} = {}^{V_\alpha}\theta^{(i)} = \left(A^H_{i_\alpha}\right) = \begin{pmatrix} 0 \\ \delta^\alpha_\beta\delta^i_h \end{pmatrix} \quad (2.3)$$

with respect to the natural frame $\{\partial_i, \partial_{i_\alpha}\}$. We call the set $\left\{{}^HX_{(i)}, {}^{V_\alpha}\theta^{(i)}\right\}$ the frame adapted to the Levi-Civita connection ∇_g. On putting

$$D_i = {}^HX_{(i)}, \quad D_{i_\alpha} = {}^{V_\alpha}\theta^{(i)},$$

we write the adapted frame as $\{D_I\} = \{D_i, D_{i_\alpha}\}$. From equations (2.2), (2.3) and (2.1) we see that ${}^HV$ and ${}^{V_\alpha}\omega$ have respectively, components

$${}^HV = V^iD_i, \ {}^HV = \left({}^HV^I\right) = \begin{pmatrix} V^i \\ 0 \end{pmatrix}, \quad (2.4)$$

$${}^{V_\alpha}\omega = \sum_i \omega_i\delta^\beta_\alpha D_{i_\alpha}, \ {}^{V_\alpha}\omega = \left({}^{V_\alpha}\omega^I\right) = \begin{pmatrix} 0 \\ \delta^\beta_\alpha\omega_i \end{pmatrix} \quad (2.5)$$

with respect to the adapted frame $\{D_I\}$, where V^i and ω_i being local components of $V \in \Im^1_0(M_n)$ and $\omega \in \Im^0_1(M_n)$, respectively.

Let us consider local 1−forms $\tilde{\eta}^I$ in $\pi^{-1}(U)$ defined by

$$\tilde{\eta}^I = \bar{A}^I{}_J dx^J,$$

where

$$A^{-1} = (\bar{A}^I{}_J) = \begin{pmatrix} \bar{A}^i{}_j & \bar{A}^i{}_{j_\beta} \\ \bar{A}^{i_\alpha}{}_j & \bar{A}^{i_\alpha}{}_{j_\beta} \end{pmatrix} = \begin{pmatrix} \delta^i_j & 0 \\ -X^\alpha_m\Gamma^m_{ij} & \delta^\alpha_\beta\delta^j_i \end{pmatrix}. \quad (2.6)$$

The matrix (2.6) is the inverse of the matrix

$$A = (A_K{}^J) = \begin{pmatrix} A_k{}^j & A_{k_\gamma}{}^j \\ A_k{}^{j_\beta} & A_{k_\gamma}{}^{j_\beta} \end{pmatrix} = \begin{pmatrix} \delta_k^j & 0 \\ X_m^\beta \Gamma_{jk}^m & \delta_\gamma^\beta \delta_j^k \end{pmatrix} \tag{2.7}$$

of the transformation $D_K = A_K{}^J \partial_J$ (see (2.2) and (2.3)). It is easy to establish that the set $\{\tilde{\eta}^I\}$ is the coframe dual to the adapted frame $\{D_K\}$, i.e.

$$\tilde{\eta}^I(D_K) = \bar{A}^I{}_J A_K{}^J = \delta_K^I.$$

3. Diagonal lift ${}^D g$ of a Riemannian metric g to the coframe bundle

On putting locally

$${}^D g = g_{ij}\tilde{\eta}^i \otimes \tilde{\eta}^j + \delta_{\alpha\beta} \sum_{i,j}^{n} g^{ij} \tilde{\eta}^{i_\alpha} \otimes \tilde{\eta}^{j_\beta} \tag{3.1}$$

in the coframe bundle $F^*(M_n)$, we see that ${}^D g$ defines a tensor field of type $(0,2)$ in $F^*(M_n)$ which called the diagonal lift of the tensor field g to $F^*(M_n)$ with respect to Γ_{ij}^k. From (2.6), (2.7) and (3.1) we prove that ${}^D g$ has components of the form

$${}^D g = \begin{pmatrix} g_{ij} & 0 \\ 0 & \delta_{\alpha\beta} g^{ij} \end{pmatrix} \tag{3.2}$$

with respect to the adapted frame $\{D_I\}$ in $F^*(M_n)$ and components

$${}^D g = \begin{pmatrix} g_{ij} + \sum_{\alpha=1}^n g^{ks} X_m^\alpha X_l^\alpha \Gamma_{ik}^m \Gamma_{js}^l & -g^{js} X_l^\beta \Gamma_{is}^l \\ -g^{is} X_l^\alpha \Gamma_{js}^l & \delta_{\alpha\beta} g^{ij} \end{pmatrix} \tag{3.3}$$

with respect to the natural frame $\{\partial_i, \partial_{i_\alpha}\}$, where g^{ij} denote contravariant components of g.

From (3.2) it easily follows that if g is a Riemannian metric in M_n, then ${}^D g$ is a Riemannian metric in $F^*(M_n)$. The metric ${}^D g$ is similar to that of the Riemannian metric studied by Sasaki in tangent bundle $T(M_n)$[14] (for the cotangent bundle $T^*(M_n)$and linear frame bundle $F(M_n)$ see [15],[7], respectively).

From (2.1) and (3.3) we have

$${}^D g({}^H X, {}^H Y) = {}^V(g(X,Y)) = g(X,Y) \circ \pi. \tag{3.4}$$

Therefore we have as follows.

Theorem 3.1. *Let $X, Y \in \Im_0^1(M_n)$. Then the inner product of the horizontal lifts ${}^H X$and ${}^H Y$ to $F^*(M_n)$with the metric ${}^D g$ is equal to the vertical lift of the inner product of $X$and Yin M_n.*

From (2.1) and (3.3) we have also

$${}^D g({}^{V_\alpha}\omega, {}^{V_\beta}\theta) = \delta^{\alpha\beta V}(g^{-1}(\omega,\theta)) = \delta^{\alpha\beta}(g^{-1}(\omega,\theta) \circ \pi), \tag{3.5}$$

$${}^D g({}^H X, {}^{V_\beta}\theta) = 0 \tag{3.6}$$

for all $X \in \Im_0^1(M_n)$ and $\omega, \theta \in \Im_1^0(M_n)$. We recall that any element $t \in \Im_2^0(F^*(M_n))$ is completely determined by its action on vector fields of type

${}^{H}X$ and ${}^{V_\alpha}\omega$. From this it follows that ${}^{D}g$ is completely determined by its eqs (3.4), (3.5) and (3.6).

4. Levi-Civita connection of ${}^{D}g$

In section 2 we see that the components of the adapted frame $\{D_I\}$ are given by (2.2) and (2.3). On the other hand the components of the dual coframe $\left\{\tilde{\eta}^I\right\}$ are given by matrix (2.6).

Since the adapted frame is non-holonomic, we put

$$[D_I, D_J] = \Omega_{IJ}{}^{K} D_K$$

from which we have

$$\Omega_{IJ}{}^{K} = \left(D_I A_J^L - D_J A_I^L\right) \bar{A}_L^K.$$

According to (2.2), (2.3), (2.6) and (2.7), the components of non-holonomic object $\Omega_{IJ}{}^{K}$ are given by

$$\begin{cases} \Omega_{ij_\beta}{}^{k_\gamma} = -\Omega_{j_\beta i}{}^{k_\gamma} = -\delta_\beta^\gamma \Gamma_{ik}^j, \\ \Omega_{ij}{}^{k_\gamma} = X_m^\gamma R_{ijk}^m \end{cases} \tag{4.1}$$

all the others being zero, where R_{ijk}^m local components of the curvature tensor field R of ∇_g.

Let ${}^{D}\nabla$ be the Levi-Civita connection determined by the metric ${}^{D}g$ on the coframe bundle $F^*(M_n)$. We put

$${}^{D}\nabla_{D_I} D_J = {}^{D}\Gamma_{IJ}^K D_K.$$

From the equation

$${}^{D}\nabla_X Y - {}^{D}\nabla_Y X = [X, Y], \forall X, Y \in \Im_0^1(F^*(M_n))$$

we have

$${}^{D}\Gamma_{IJ}^K - {}^{D}\Gamma_{JI}^K = \Omega_{IJ}{}^{K}. \tag{4.2}$$

The equation

$$\left({}^{D}\nabla_X {}^{D}g\right)(Y, Z) = 0$$

has form

$$D_L {}^{D}g_{IJ} - {}^{D}\Gamma_{LI}^K {}^{D}g_{KJ} - {}^{D}\Gamma_{LJ}^K {}^{D}g_{IK} = 0 \tag{4.3}$$

with respect to the adapted frame $\{D_K\}$. By using (4.1) and (4.2) we obtain:

$$\begin{array}{l} {}^{D}\Gamma_{IJ}^K = \frac{1}{2} {}^{D}g^{KL}\left(D_I {}^{D}g_{LJ} + D_J {}^{D}g_{IL} - D_L {}^{D}g_{IJ}\right) \\ +\frac{1}{2}\left(\Omega_{IJ}{}^{K} + \Omega_{\cdot\ \ IJ}^{K} + \Omega_{\cdot\ \ JI}^{K}\right), \end{array} \tag{4.4}$$

where $\Omega_{\cdot\ \ IJ}^{K} = {}^{D}g^{KL\,D}g_{PJ}\Omega_{LI}{}^{P}$ and

$$\left({}^{D}g\right)^{-1} = \left({}^{D}g^{KJ}\right) = \begin{pmatrix} g^{kj} & 0 \\ 0 & \delta_{\gamma\beta} g_{kj} \end{pmatrix}. \tag{4.5}$$

Taking account (2.2), (2.3), (3.2), (4.1), (4.3) and (4.5), we obtain from (4.4)

$$\begin{cases} {}^{D}\Gamma_{ij}^k = \Gamma_{ij}^k, & {}^{D}\Gamma_{i_\alpha j_\beta}^k = {}^{D}\Gamma_{i_\alpha j}^{k_\gamma} = {}^{D}\Gamma_{i_\alpha j_\beta}^{k_\gamma} = 0, \\ {}^{D}\Gamma_{ij_\beta}^k = \frac{1}{2} X_m^\beta R_{\cdot\, i\, \cdot}^{k\, jm}, & {}^{D}\Gamma_{i_\alpha j}^k = \frac{1}{2} X_m^\alpha R_{\cdot\, j\, \cdot}^{k\ \ im}, \\ {}^{D}\Gamma_{ij}^{k_\gamma} = \frac{1}{2} X_m^\gamma R_{ijk}^m, & {}^{D}\Gamma_{ij_\beta}^{k_\gamma} = -\delta_\gamma^\beta \Gamma_{ik}^j. \end{cases} \tag{4.6}$$

Let $\tilde{X}, \tilde{Y} \in \Im_0^1(F^*(M_n))$ and $\tilde{X} = \tilde{X}^I D_I, \tilde{Y} = \tilde{Y}^J D_J$. Then the covariant derivative ${}^D\nabla_{\tilde{Y}}\tilde{X}$ along $\tilde{Y}$ has components in the form

$$ {}^D\nabla_{\tilde{Y}}\tilde{X}^I = \tilde{Y}^J D_J \tilde{X}^I + {}^D\Gamma_{JK}^I \tilde{X}^K \tilde{Y}^J \tag{4.7}$$

with respect to the adapted frame $\{D_I\}$.
Using (2.2), (2.3), (2.4), (2.5), (4.6) and (4.7) we have

Theorem 4.1. *Let M_n be a Riemannian manifold with metric g and ${}^D\nabla$ be the Levi-Civita connection of the coframe bundle $F^*(M_n)$ equipped with metric ${}^Dg$. Then ${}^D\nabla$ satisfies*

$$ i) {}^D\nabla_{V_\alpha \omega}{}^{V_\beta}\theta = 0, $$
$$ ii) {}^D\nabla_{V_\alpha \omega}{}^H Y = \tfrac{1}{2}{}^H\left(R(\tilde{X}^\alpha, \tilde{\omega})Y\right), $$
$$ iii) {}^D\nabla_{{}^H X}{}^{V_\beta}\theta = {}^{V_\beta}\left(\nabla_X \theta\right) + \tfrac{1}{2}{}^H\left(R(\tilde{X}^\beta, \tilde{\theta})X\right), $$
$$ iv) {}^D\nabla_{{}^H X}{}^H Y = {}^H\left(\nabla_X Y\right) + \tfrac{1}{2}\gamma(R(X,Y)) $$

for all $X, Y \in \Im_0^1(M_n)$ and $\omega, \theta \in \Im_1^0(M_n)$, where $\tilde{\omega} = g^{-1} \circ \omega \in \Im_0^1(M_n)$, $\tilde{\theta} = g^{-1} \circ \theta \in \Im_0^1(M_n)$, $\tilde{X}^\alpha = g^{-1} \circ X^\alpha \in \Im_0^1(M_n)$ and $\gamma(R(X,Y))$ the vertical vector field having components

$$ \gamma(R(X,Y)) = \begin{pmatrix} 0 \\ X_m^\alpha R_{jki}^m X^j Y^k \end{pmatrix} $$

with respect to the adapted frame $\{D_I\}$.

We note that the analogue of Theorem 4.1 in the case of cotangent bundle is proved in [9].

5. Killing vector fields

A vector field $X \in \Im_0^1(M_n)$is said to be an infinitesimal isometry or a Killing vector field of a Riemannian manifold M_n with metric g, if $L_X g = 0$[17, p.78]. In terms of components g_{ij} of g, X is a Killing vector field if and only if

$$ L_X g_{ij} = X^m \nabla_m g_{ij} + g_{mj}\nabla_i X^m + g_{im}\nabla_j X^m = \nabla_i X_j + \nabla_j X_i = 0, $$

where X^m and X_i being the contravariant and covariant components of X, respectively and ∇ is the Riemannian connection of the metric g.

Let $\tilde{X}$ be a covector field in $F^*(M_n)$ and $(\tilde{X}_I) = (\tilde{X}_i, \tilde{X}_{i_\alpha})$ its components with respects to the adapted frame $\{D_I\}$. Then the covariant derivative ${}^D\nabla\tilde{X}$ has components

$$ {}^D\nabla_I \tilde{X}_J = D_I \tilde{X}_J + {}^D\Gamma_{IJ}^K \tilde{X}_K, \tag{5.1}$$

${}^D\Gamma_{IJ}^K$ being given by (4.6), with respect to the adapted frame $\{D_I\}$.

The associated covector fields of the complete, horizontal and vertical lifts to the coframe bundle $F^*(M_n)$ with the metric ${}^Dg$ are given respectively by

$$ \begin{aligned} \left({}^C V_J\right) &= \left({}^D g_{IJ}{}^C V^I\right) = \left(V_j, -g^{ij} X_m^\alpha V^m\right), \\ \left({}^H V_J\right) &= \left({}^D g_{IJ}{}^H V^I\right) = \left(V_j, 0\right), \\ \left({}^{V_\gamma}\omega_J\right) &= \left({}^D g_{IJ}{}^{V_\gamma}\omega^I\right) = \left(0, \delta_\gamma^\beta \omega^j\right) \end{aligned} \tag{5.2}$$

with respect to the adapted frame $\{D_I\}$, where $V_j = g_{ji}V^i$ and $\omega^j = g^{ji}\omega_i$.

We now compute Lie derivatives of the metric ${}^{D}g$ with respect to the vector fields ${}^{C}V, {}^{H}V$ and ${}^{V_\alpha}\omega$, by means (5.1) and (5.2). The Lie derivatives of ${}^{D}g$ with respect to ${}^{C}V, {}^{H}V$ and ${}^{V_\alpha}\omega$ have respectively components

$$\left(L_{{}^{C}V}{}^{D}g_{IJ}\right) = \left({}^{D}\nabla_I{}^{C}V_J + {}^{D}\nabla_J{}^{C}V_I\right) =$$

$$= \begin{pmatrix} \nabla_i V_j + \nabla_j V_i & -X_m^\beta g^{jl}g^{mt}\left(\nabla_i\nabla_l V_t - R_{silt}V^s\right) \\ -X_m^\alpha g^{il}g^{mt}\left(\nabla_j\nabla_l V_t - R_{sjlt}V^s\right) & -\left(g^{mj}\delta_{\beta\alpha}\nabla_m V^i + g^{mi}\delta_{\alpha\beta}\nabla_m V^j\right) \end{pmatrix},$$

$$\left(L_{{}^{H}V}{}^{D}g_{IJ}\right) = \left({}^{D}\nabla_I{}^{H}V_J + {}^{D}\nabla_J{}^{H}V_I\right) = \begin{pmatrix} \nabla_i V_j + \nabla_j V_i & X_m^\beta g^{js}R_{lis}{}^{m}V^l \\ X_m^\alpha g^{is}R_{ljs}{}^{m}V^l & 0 \end{pmatrix}, \quad (5.3)$$

$$\left(L_{{}^{V_\gamma}\omega}{}^{D}g_{IJ}\right) = \left({}^{D}\nabla_I{}^{V_\gamma}\omega_J + {}^{D}\nabla_J{}^{V_\gamma}\omega_I\right) = \begin{pmatrix} 0 & \delta_\gamma^\beta g^{js}\nabla_i\omega_s \\ \delta_\gamma^\alpha g^{is}\nabla_j\omega_s & 0 \end{pmatrix}$$

with respect to the adapted frame in $F^*(M_n)$.

Since we have

$$\nabla_i\nabla_l V_t + R_{silt}V^s = 0, \quad L_V g^{ij} = -g^{mj}\nabla_m V^i - g^{im}\nabla_m V^j = 0$$

as a consequence of $L_V g_{ij} = \nabla_i V_j + \nabla_j V_i = 0$ (see [6, p.17]), we conclude by means of (5.3) that the complete lift ${}^{C}V$ is a Killing vector field in $F^*(M_n)$ if and only if V is a Killing vector field in M_n.

We next have $R_{lik}{}^{m}V^l = 0$ as a consequence of the vanishing of the second covariant derivation of V. Conversely, the condition $L_V g_{ij} = \nabla_i V_j + \nabla_j V_i = 0$ and $R_{lik}{}^{m}V^l = 0$ imply that the second covariant derivation of V vanishes. Summing up these results we have

Theorem 5.1. *Necessary and sufficient conditions in order that the*

(1) *complete* ${}^{C}V \in \Im_0^1(F^*(M_n))$,
(2) *horizontal* ${}^{H}V \in \Im_0^1(F^*(M_n))$ *and*
(3) *vertical* ${}^{V_\alpha}\omega \in \Im_0^1(F^*(M_n))$

lifts in $F^*(M_n)$ with the metric ${}^{D}g$, of a vector field $V \in \Im_0^1(M_n)$ and 1-form $\omega \in \Im_1^0(M_n)$ be a Killing vector field in $F^*(M_n)$ are that

a) V is a Killing vector field in M_n,
b) V is a Killing vector field with vanishing second covariant derivative in M_n and
c) ω is parallel in M_n.

A vector field $V \in \Im_0^1(M_n)$ is called the infinitesimal affine transformation if $L_V\nabla_g = 0$ (see [17, p.67]). A Killing vector field is necessarily an infinitesimal affine transformation, i.e. we have $L_V\nabla_g = 0$ as a consequence of $L_V g = 0$ (see [13, p.79]). Let V and W be vector fields in M_n. If V and W are Killing vector fields in M_n, from the definition of the Killing vector field, we have

$$L_{[V,W]}g = L_V(L_W g) - L_W(L_V g) = 0,$$

i.e. $[V, W]$ is an infinitesimal isometry in M_n.

It is well known that [10]:

$$\left[{}^{C}V, {}^{C}W\right] = {}^{C}[V, W] \quad (5.4)$$

for all $V, W \in \Im_0^1(M_n)$

The following theorem is hold.

Theorem 5.2. *Let M_n be a differentiable manifold and $F^*(M_n)$ be its coframe bundle. Then*

$$i)\ \left[{}^{C}V, {}^{H}W\right] = {}^{H}[V, W] + \gamma\left(L_V\nabla\right)W \quad (5.5)$$

$$ii)\ \left[{}^{C}V, {}^{V_\gamma}\omega\right] = {}^{V_\gamma}(L_V\omega). \tag{5.6}$$

Proof. i) In the case when $I = i$, by using (2.1) we see that the left hand side of (5.5) reduces to

$$\left[{}^{C}V, {}^{H}W\right]^i = {}^{C}V^K \partial_K {}^{H}W^i - {}^{H}W^K \partial_K {}^{C}V^i = {}^{C}V^k \partial_k {}^{H}W^i - {}^{H}W^k \partial_k {}^{C}V^i$$
$$+{}^{C}V^{k_\gamma} \partial_{k_\gamma} {}^{H}W^i - {}^{H}W^{k_\gamma} \partial_{k_\gamma} {}^{C}V^i = V^k \partial_k W^i - W^k \partial_k V^i = [V, W]^i .$$

In the case $I = i_\alpha$ we have

$$\left[{}^{C}V, {}^{H}W\right]^{i_\alpha} = {}^{C}V^K \partial_K {}^{H}W^{i_\alpha} - {}^{H}W^K \partial_K {}^{C}V^{i_\alpha} = {}^{C}V^k \partial_k {}^{H}W^{i_\alpha} - {}^{H}W^k \partial_k {}^{C}V^{i_\alpha}$$
$$+{}^{C}V^{k_\gamma} \partial_{k_\gamma} {}^{H}W^{i_\alpha} - {}^{H}W^{k_\gamma} \partial_{k_\gamma} {}^{C}V^{i_\alpha} = V^k \partial_k (X^\alpha_m \Gamma^m_{li} W^l)$$
$$+(-X^\gamma_s \partial_k V^s) \partial_{k_\gamma} (X^\alpha_m \Gamma^m_{li} W^l) - W^k \partial_k (-X^\alpha_m \partial_i V^m)$$
$$-(X^\gamma_s \Gamma^s_{lk} W^l) \partial_{k_\gamma} (-X^\alpha_m \partial_i V^m) = \Gamma^m_{li} V^k X^\alpha_m \partial_k W^l + V^k W^l X^\alpha_m \partial_k \Gamma^m_{li}$$
$$-X^\gamma_s \partial_k V^s \delta^\alpha_\gamma \delta^k_m \Gamma^m_{li} W^l + X^\alpha_m W^k \partial_k \partial_i V^m + X^\gamma_s \Gamma^s_{lk} W^l \delta^\alpha_\gamma \delta^k_m \partial_i V^m$$
$$= \Gamma^m_{li} X^k_m (V^k \partial_k W^l - W^k \partial_k V^l) + \Gamma^m_{li} X^\alpha_m W^k \partial_k V^l + V^k W^l X^\alpha_m (\partial_k \Gamma^m_{li} - \partial_l \Gamma^m_{ki}$$
$$+\Gamma^m_{ks} \Gamma^s_{li} - \Gamma^m_{ls} \Gamma^s_{ki}) + V^k W^l X^\alpha_m \partial_l \Gamma^m_{ki} - V^k W^l X^\alpha_m \Gamma^m_{ks} \Gamma^s_{li}$$
$$+V^k W^l X^\alpha_m \Gamma^m_{ls} \Gamma^s_{ki} - X^\alpha_s W^l \Gamma^m_{li} \partial_m V^s + X^\alpha_m W^k \partial_k \partial_i V^m + X^\alpha_s W^l \Gamma^s_{lm} \partial_i V^m$$
$$= {}^{H}[V, W]^{i_\alpha} + X^\alpha_m V^k W^l R_{kli}{}^m + X^\alpha_m W^l \nabla_l \nabla_i V^m.$$

Thus,

$$\left[{}^{C}V, {}^{H}W\right] = {}^{H}[V, W] + \gamma\,(L_V \nabla)\, W.$$

ii) In the case $I = i$ from (2.1) we have

$$\left[{}^{C}V, {}^{V_\gamma}\omega\right]^i = {}^{C}V^K \partial_K {}^{V_\gamma}\omega^i - {}^{V_\gamma}\omega^K \partial_K {}^{C}V^i = -{}^{V_\gamma}\omega^k \partial_k {}^{C}V^i - {}^{V_\gamma}\omega^{k_\sigma} \partial_{k_\sigma} {}^{C}V^i$$
$$= 0.$$

In the case $I = i_\alpha$ we obtain

$$\left[{}^{C}V, {}^{V_\gamma}\omega\right]^{i_\alpha} = {}^{C}V^K \partial_K {}^{V_\gamma}\omega^{i_\alpha} - {}^{V_\gamma}\omega^K \partial_K {}^{C}V^{i_\alpha} = {}^{C}V^k \partial_k {}^{V_\gamma}\omega^{i_\alpha} + {}^{C}V^{k_\sigma} \partial_{k_\sigma} {}^{V_\gamma}\omega^{i_\alpha}$$
$$-{}^{V_\gamma}\omega^k \partial_k {}^{C}V^{i_\alpha} - {}^{V_\gamma}\omega^{k_\sigma} \partial_{k_\sigma} {}^{C}V^{i_\alpha} = \delta^\alpha_\gamma (V^k \partial_k \omega_i - \omega_k \partial_i V^m) = \delta^\alpha_\gamma\, (L_V \omega)_i\,,$$

from which follows that

$$\left[{}^{C}V, {}^{V_\gamma}\omega\right] = {}^{V_\gamma}(L_V\omega).$$

Thus Theorem 5.2 is proved. □

Using (5.4) and (5.6), we compute the Lie derivatives of the metric ${}^{D}g$ with respect to ${}^{C}[V, W]$ and ${}^{V_\gamma}(L_V\omega)$:

$$\begin{aligned} L_{{}^{C}[V,W]}{}^{D}g &= L_{[{}^{C}V, {}^{C}W]}{}^{D}g = L_{{}^{C}V}(L_{{}^{C}W}{}^{D}g) - L_{{}^{C}W}(L_{{}^{C}V}{}^{D}g), \\ L_{{}^{V_\gamma}(L_V\omega)}{}^{D}g &= L_{\left[{}^{C}V, {}^{V_\gamma}\omega\right]}{}^{D}g = L_{{}^{C}V}(L_{{}^{V_\gamma}\omega}{}^{D}g) - L_{{}^{V_\gamma}\omega}(L_{{}^{C}V}{}^{D}g). \end{aligned} \tag{5.7}$$

Using Theorem 5.1 and (5.7), we get as follows.

Theorem 5.3. *Sufficient conditions in order that the complete lift of a vector field $[V, W]$ and the vertical lift of a covector field $L_V\omega$ in M_n to the coframe bundle $F^*(M_n)$ be a Killing vector fields with metric ${}^{D}g$ are that V and $W$are a Killing vector fields with vanishing second covariant derivative and ω is parallel in M_n.*

Let V be infinitesimal affine transformation in M_n ($L_V\nabla = 0$). Then ${}^{C}\left[V, {}^{H}W\right] = {}^{H}[V, W]$ and

$$L_{{}^{H}[V,W]}{}^{D}g = L_{[{}^{C}V, {}^{H}W]}{}^{D}g = L_{{}^{C}V}(L_{{}^{H}W}{}^{D}g) - L_{{}^{H}W}(L_{{}^{C}V}{}^{D}g). \tag{5.8}$$

Using Theorem 5.1 and equation (5.8), we have

Theorem 5.4. *Sufficient conditions in order that the horizontal lift of a vector field* $[V, W]$ *in* M_n *to the coframe bundle* $F^*(M_n)$*be a Killing vector field with metric* ${}^Dg$ *are that* V *and* W*are a Killing vector fields with vanishing second covariant derivative in* M_n.

6. Almost paracomplex structures in the coframe bundle

An almost paracomplex manifold is an almost product manifold $(M_n, \varphi), \varphi^2 = I$, such that the two eigenbundles $T^+(M_n)$ and $T^-(M_n)$ associated to the two eigenvalues $+1$ and -1 of φ, respectively, have the same rank. The dimension of an almost paracomplex manifold is necessarily even.

A tensor field $t \in \Im_q^0(M_{2n})$ is said to be a pure with respect to the para-complex structure φ, if

$$t(\varphi X_1, X_2, ..., X_q) = t(X_1, \varphi X_2, ..., X_q) = t(X_1, X_2, ..., \varphi X_q)$$

for any $X_1, X_2, ..., X_q \in \Im_0^1(M_{2n})$.

We define the following operator ϕ_φ associated with φ and apply to the pure tensor field t :

$$\begin{aligned}(\phi_\varphi t)(Y, X_1, X_2, ..., X_q) &= (\varphi Y)(t(X_1, X_2, ..., X_q)) \\ &-Y(t(\varphi X_1, X_2, ..., X_q)) + t((L_{X_1}\varphi)Y, X_2, ..., X_q)) \\ &+ \cdots + t(X_1, X_2, ..., (L_{X_q}\varphi)Y).\end{aligned} \tag{6.1}$$

We note that $\phi_\varphi t \in \Im_q^0(M_{2n})$. If $\phi_\varphi t = 0$ then t is said to be an almost paraholomorphic with respect to the paracomplex algebra $R(j), j^2 = 1$ (see [5, 9,12]).

Definition 6.1. In a manifold with almost paracomplex structure φ, a vector field $X$is called an almost paraholomorphic vector field if $L_X\varphi = 0$.

Let $F \in \Im_1^1(M_n)$. We define a tensor field ${}^DF$ of type $(1,1)$ in $F^*(M_n)$by

$${}^DF^HX = {}^H(FX), {}^DF^{V_\beta}\omega = -{}^{V_\beta}(\omega \circ F) = -{}^{V_\beta}(\omega F)$$

for any $X \in \Im_0^1(M_n)$ and $\omega \in \Im_1^0(M_n)$. We call ${}^DF$ the diagonal lift of the tensor field F. ${}^DF$ has components

$${}^DF = \begin{pmatrix} F_j^i & 0 \\ 0 & -\delta_\beta^\alpha F_i^j \end{pmatrix}$$

with respect to the adapted frame $\{D_I\}$. The diagonal lift ${}^DI$ of identity tensor field $I$of type $(1,1)$ has components

$${}^DI = \begin{pmatrix} I_j^i & 0 \\ X_m^\alpha \Gamma_{ij}^m & -\delta_\beta^\alpha \delta_i^j \end{pmatrix}$$

with respect to the induced coordinates $\{\partial_i, \partial_{i_\alpha}\}$ and satisfies ${}^DI^2 = I$. Thus ${}^DI$ is an almost product structure in the coframe bundle $F^*(M_n)$.

We put

$$S(\tilde{X}, \tilde{Y}) = {}^Dg({}^DI\tilde{X}, \tilde{Y}) - {}^Dg(\tilde{X}, {}^DI\tilde{Y}).$$

If $S(\tilde{X}, \tilde{Y}) = 0$ for all vector fields $\tilde{X}$ and $\tilde{Y}$ which are of the form ${}^{V_\alpha}\omega, {}^{V_\beta}\theta$ or ${}^HX, {}^HY$, then $S = 0$.By virtue of ${}^DI^{V_\alpha}\omega = -{}^{V_\alpha}\omega$, ${}^DI^HX = {}^HX$, (3.4),(3.5) and (3.6) we have

$$\begin{aligned} S({}^{V_\alpha}\omega, {}^{V_\beta}\theta) &= {}^Dg(-{}^{V_\alpha}\omega, {}^{V_\beta}\theta) - {}^Dg({}^{V_\alpha}\omega, -{}^{V_\beta}\theta) = 0, \\ S({}^{V_\alpha}\omega, {}^HX) &= {}^Dg(-{}^{V_\alpha}\omega, {}^HX) - {}^Dg({}^{V_\alpha}\omega, {}^HX) = 0, \\ S({}^HX, {}^{V_\beta}\theta) &= {}^Dg({}^HX, {}^{V_\beta}\theta) - {}^Dg({}^HX, -{}^{V_\beta}\theta) = 0, \\ S({}^HX, {}^HY) &= {}^Dg({}^HX, {}^HY) - {}^Dg({}^HX, {}^HY) = 0 \end{aligned}$$

i.e. ${}^Dg$ is pure metric with respect to ${}^DI$. Hence, the following theorem holds.

Theorem 6.1. *$(F^*(M_n), {}^DI, {}^Dg)$ is an almost paracomplex Riemannian manifold.*

If $X, Y, Z \in \Im_0^1(M_n)$, $\omega, \theta \in \Im_1^0(M_n)$ and f a smooth function on the M_n. Then the following equations are satisfies:

$$\begin{gathered} {}^{V_\alpha}\omega^V f = 0, {}^HX^V f = {}^V(Xf), \\ \left[{}^HX, {}^{V_\alpha}\omega\right] = {}^{V_\alpha}(\nabla_X\omega), \left[{}^{V_\alpha}\omega, {}^{V_\beta}\theta\right] = 0, \\ \left[{}^HX, {}^HY\right] = {}^H[X, Y] + \gamma(R(X, Y)), \\ {}^DI(\gamma(R(X, Y))) = -\gamma(R(X, Y)), \\ {}^Dg(\gamma(R(X, Y), {}^HZ)) = 0, \end{gathered} \tag{6.2}$$

where $\gamma(R(X, Y)) = X_h^\alpha R_{kli}^h X^k Y^l D_{i_\alpha}$.

From (6.1), (6.2) we have

$$\begin{gathered} (\phi_{{}^DI}{}^Dg)({}^{V_\alpha}\omega, {}^{V_\beta}\theta, {}^{V_\gamma}\xi) = ({}^DI^{V_\alpha}\omega)({}^Dg({}^{V_\beta}\theta, {}^{V_\gamma}\xi)) - {}^{V_\alpha}\omega({}^Dg({}^DI^{V_\beta}\theta, {}^{V_\gamma}\xi)) \\ + {}^Dg((L_{{}^{V_\beta}\theta}{}^DI)^{V_\alpha}\omega, {}^{V_\gamma}\xi) + {}^Dg({}^{V_\beta}\theta, ((L_{{}^{V_\gamma}\xi}{}^DI)^{V_\alpha}\omega) = 0, \\ (\phi_{{}^DI}{}^Dg)({}^{V_\alpha}\omega, {}^{V_\beta}\theta, {}^HX) = ({}^DI^{V_\alpha}\omega)({}^Dg({}^{V_\beta}\theta, {}^HX)) - {}^{V_\alpha}\omega({}^Dg({}^DI^{V_\beta}\theta, {}^HX)) \\ + {}^Dg((L_{{}^{V_\beta}\theta}{}^DI)^{V_\alpha}\omega, {}^HX) + {}^Dg({}^{V_\beta}\theta, ((L_{{}^HX}{}^DI)^{V_\alpha}\omega) = 0, \\ (\phi_{{}^DI}{}^Dg)({}^{V_\alpha}\omega, {}^HX, {}^{V_\gamma}\xi) = ({}^DI^{V_\alpha}\omega)({}^Dg({}^HX, {}^{V_\gamma}\xi)) - {}^{V_\alpha}\omega({}^Dg({}^DI^HX, {}^{V_\gamma}\xi)) \\ + {}^Dg((L_{{}^HX}{}^DI)^{V_\alpha}\omega, {}^{V_\gamma}\xi) + {}^Dg({}^HX, ((L_{{}^{V_\gamma}\xi}{}^DI)^{V_\alpha}\omega) = 0, \\ (\phi_{{}^DI}{}^Dg)({}^HX, {}^{V_\beta}\theta, {}^{V_\gamma}\xi) = ({}^DI^HX)({}^Dg({}^{V_\beta}\theta, {}^{V_\gamma}\xi)) - {}^HX({}^Dg({}^DI^{V_\beta}\theta, {}^{V_\gamma}\xi)) \\ + {}^Dg((L_{{}^{V_\beta}\theta}{}^DI)^HX, {}^{V_\gamma}\xi) + {}^Dg({}^{V_\beta}\theta, ((L_{{}^{V_\gamma}\xi}{}^DI)^HX) = 0, \\ (\phi_{{}^DI}{}^Dg)({}^HX, {}^HY, {}^HZ) = ({}^DI^HX)({}^Dg({}^HY, {}^HZ)) - {}^HX({}^Dg({}^DI^HY, {}^HZ)) \\ + {}^Dg((L_{{}^HY}{}^DI)^HX, {}^HZ) + {}^Dg({}^HY, ((L_{{}^HZ}{}^DI)^HX) = 0, \\ (\phi_{{}^DI}{}^Dg)({}^{V_\alpha}\omega, {}^HY, {}^HZ) = ({}^DI^{V_\alpha}\omega)({}^Dg({}^HY, {}^HZ)) - {}^{V_\alpha}\omega({}^Dg({}^DI^HY, {}^HZ)) \\ + {}^Dg((L_{{}^HY}{}^DI)^{V_\alpha}\omega, {}^HZ) + {}^Dg({}^HY, ((L_{{}^HZ}{}^DI)^{V_\alpha}\omega) = 0 \\ (\phi_{{}^DI}{}^Dg)({}^HX, {}^HY, {}^{V_\alpha}\omega) = ({}^DI^HX)({}^Dg({}^HY, {}^{V_\alpha}\omega)) - {}^HX({}^Dg({}^DI^HY, {}^{V_\alpha}\omega)) \\ + {}^Dg((L_{{}^HY}{}^DI)^HX, {}^{V_\alpha}\omega) + {}^Dg({}^HY, ((L_{{}^{V_\alpha}\omega}{}^DI)^HX) \\ = 2{}^Dg({}^{V_\alpha}\omega, \gamma(R(X, Y))), \\ (\phi_{{}^DI}{}^Dg)({}^HX, {}^{V_\beta}\theta, {}^HZ) = ({}^DI^HX)({}^Dg({}^{V_\beta}\theta, {}^HZ)) - {}^HX({}^Dg({}^DI^{V_\beta}\theta, {}^HZ)) \\ + {}^Dg((L_{{}^{V_\beta}\theta}{}^DI)^HX, {}^HZ) + {}^Dg({}^{V_\beta}\theta, ((L_{{}^HZ}{}^DI)^HX) \\ = 2{}^Dg({}^{V_\beta}\theta, \gamma(R(X, Z))), \end{gathered}$$

for all $X, Y, Z \in \Im_0^1(M_n)$ and $\omega, \theta, \xi \in \Im_1^0(M_n)$.

Therefore we have as follows.

Theorem 6.2. *The almost paracomplex Riemannian manifold $(F^*(M_n), {}^DI, {}^Dg)$ is paraholomorphic if and only if M_n is flat.*

Let us consider the Lie derivative of ${}^DI$ with respect to the complete lift ${}^CX$. Then we obtain:

$$\begin{gathered} (L_{{}^CX}{}^DI)^{V_\beta}\theta = L_{{}^CX}({}^DI^{V_\beta}\theta) - {}^DI(L_{{}^CX}{}^{V_\beta}\theta) = -L_{{}^CX}{}^{V_\beta}\theta - {}^DI(L_{{}^CX}{}^{V_\beta}\theta) \\ = -{}^{V_\beta}(L_X\theta) - {}^DI^{V_\beta}(L_X\theta) = 0, \\ (L_{{}^CX}{}^DI)^HY = L_{{}^CX}({}^DI^HY) - {}^DI(L_{{}^CX}{}^HY) = L_{{}^CX}{}^HY - {}^DI(L_{{}^CX}{}^HY) \\ = {}^H[X, Y] + \gamma(L_X\nabla)Y - {}^H[X, Y] + \gamma(L_X\nabla)Y = 2\gamma(L_X\nabla)Y. \end{gathered}$$

By using Definition 6.1, we have

Theorem 6.3. *If X is an infinitesimal affine transformation of a Riemannian manifold M_n, i.e., if $L_X\nabla = 0$, then its complete lift ${}^CX$ to the coframe bundle $F^*(M_n)$ is an almost paraholomorphicvector field with respect to the almost paracomplex structure $({}^DI, {}^Dg)$.*

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