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AN APPROACH FOR SOLVING NONLINEARLY LOADED PROBLEMS FOR LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. In the paper we study linear systems of ordinary differential equations with nonlinear point loadings and multipoint conditions. The conditions of existence and uniqueness of the solution of the problems under consideration are given. An approach for solving the problem is suggested. Application of the method is shown in illustrated problems.

1. Introduction

In the paper we consider loaded ordinary differential equations. The main feature of the problems under consideration is the linearity of the main part of the equations and nonlinear participation the values of the sought function at loading points. Another feature of the studied problems are non-separated (nonlocal) conditions. They include not only linear values of the sought functions both at arbitrary points and at loading points, but also the same as in differential equations, the nonlinear values of the sought function at loading points.

The case when the values of the sought function at loading points linearly enter the equation and boundary conditions was studied in a number of works [1]-[3] [9, 16, 17, 19]. The conditions of existence and uniqueness of their solutions were obtained for them and the methods for their numerical solution were suggested. The optimal control and parametric identification problems were studied for these problems [4, 11]. It should be noted that the important source of the problems considered in this paper are loaded partial differential equations [4, 12, 14, 15, 18, 22]. Using the method of lines, i.e. difference approximation of partial derivatives in all variables except one, the loaded boundary value problem with respect to partial differential equations is reduced to the loaded problem for the system of ordinary differential equations [6].

The works devoted to problems with nonlinear loadings; especially the methods for solving them are few. Such problems may arise when using nonlinear loading functions, for example, in earlier studied linear cases to increase adequacy of mathematical models of the considered processes. Such problems arise in optimal feedback control problems for objects with distributed parameters in the case

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when a nonlinear state function is used for feedback and the method of lines is applied for solving them. In this case, the process with distributed parameters is reduced to the problem described by the system of nonlinearly loaded ordinary differential equations [5, 8, 10]. Note that in [1] the linearization method was used for solving the system of nonlinear differential equations with linear loadings.

The important result of the paper is the suggested constructive approach to the solution of the problems under consideration. It is based on the solution of auxiliary systems of linear differential equations with multipoint non-separated conditions without loading. The finite-dimensional system of nonlinear equations with respect to the values of the sought vector-function at the loading points is formed from the obtained solutions. Application of the suggested approach is shown by two illustrative problems.

2. Problem statement

We consider the following nonlinearly loaded problem with respect to the linear system of ordinary differential equations of n-th order:

$$\dot{x}(t) = A(t)\, x(t) + \sum_{i=1}^{l} B^i(t)\, g_i\left(x(\bar{t}_i)\right) + f(t), \quad t \in [t_0, t_f], \tag{2.1}$$

$$\sum_{i=1}^{l} \left[\alpha_i^1 x\left(\bar{t}_i\right) + \alpha_i^2 g_i(x\left(\bar{t}_i\right)\right] = \alpha^0. \tag{2.2}$$

Here: T is a transposition $sign$; $x(t)$ is n-dimensional sought function; given are: n-dimensional continuous square matrix function $A(t)$; $(n \times m_i)$-dimensional continuous matrix function $B^i(t)$; n-dimensional continuous vector-function $f(\cdot)$; in the general case, nonlinear continuously differentiable m_i-dimensional vector-functions $g_i(x)$, $0 \le m_i \le n$, $i = 1, \ldots, l$; the points $\bar{t}_i \in (t_0, t_f)$ and their number l; constant n -dimensional quadratic matrices α_i^1, $i = 1, \ldots, l$, $(n \times m_i)$-dimensional matrices α_i^2, $i = 1, \ldots, l$; n-dimensional vector α^0.

Some of the matrix functions $B^i(t)$, $i = 1, \ldots l$, can be identically equal to zero and therefore the points $\bar{t}_i$ will not be the loading points in the equations, but only will participate in non-separated conditions (2.2). This case is equivalent to the fact that the corresponding $m_i = 0$.

In the general case the system of differential equations are assumed to be non-autonomous, i.e. $A(t) \ne const$ and rank $A(t) = n$ for $t \in [t_0, t_f]$. Note that for autonomous systems the application of the approach suggested below for studying and solving them becomes more easier.

We will use the following notations:0_n for n-dimensional vector and $0_{n\times m}$ for $n \times m$-dimensional matrix whose elements are equal to zero. We will assume that for the extended matrix $\mathcal{A} = \left[\alpha_1^1, \ldots, \alpha_l^1\right]$ the following holds:

$$rank \mathcal{A} = n, \tag{2.3}$$

i.e. non-separated values of phase variables $x(t)$ necessarily participate in (2.2) in some given moments of time $\bar{t}_i$. Participation in (1.1), (2.2) nonlinear terms is not necessary, i.e. it is possible that some or all matrices $B^i(t)$, α_i^2, $i = 1, \ldots, l$, contain null elements or are generally null.

Condition (2.3) is not principle, as we can increase the Jacobian rank of the linear part of constraints (2.2) in many ways due to their nonlinear part. Let the rank of the matrix $\mathcal{A}$ be equal to n_1 and $n_1 < n$, and let, not loosing generality, the rank of the linear part of the first n_1 of constraints be equal to n_1. Consequently, the last $(n - n_1)$ rows of the extended matrix $\mathcal{A}$ are linearly dependent on its first n_1 rows. Let us introduce for example the functions

$$g^1_{n_1+i}\left(x\left(\bar{t}_1\right)\right) = \gamma_i x_i\left(\bar{t}_1\right), \quad i = 1, \dots, n - n_1. \tag{2.4}$$

Here the choice of the values of the coefficients γ_i, $i = 1, \dots, n - n_1$, is arbitrary. To the $(n_1 + i)$-th constraints of (2.2) we add summand equal to zero:

$$\gamma_i x_i\left(\bar{t}_1\right) - g^1_{n_1+i}\left(x\left(\bar{t}_1\right)\right) = 0, \quad i = 1, \dots, n - n_1, \tag{2.5}$$

which will lead to the change in matrices α^1_1 and α^2_1. As a result the extended matrix $\mathcal{A}$ will also change and taking into account arbitrariness in the choice of the values of the coefficients γ_i, it is highly likely that its rank will also change. As the end of the paper we give an illustrative example where we use this method for increasing the rank of the linear part of the problem constraints .

3. Problem solution

Let us consider the following essentially used further two auxiliary problems that are the special cases of problem (2.1),(2.2):

$$\dot{x}(t) = A(t)x(t) + f(t), \quad t \in [t_0, t_f], \tag{3.1}$$

$$\sum_{i=1}^{l} \alpha^1_i x\left(\bar{t}_i\right) = \alpha^0, \tag{3.2}$$

$$\dot{x}(t) = A(t)x(t) + \sum_{i=1}^{l} B^i(t)\lambda_i + f(t), \tag{3.3}$$

$$\sum_{i=1}^{l} \left[\alpha^1_i x\left(\bar{t}_i\right) + \alpha^2_i \lambda_i\right] = \alpha^0. \tag{3.4}$$

All the matrices, vectors and functions participating in (3.1)-(3.4) have the same meaning as in problem (2.1),(2.2). The vectors λ_i of dimension m_i, $i = 1, \dots, l$, are arbitrary, real and are free (independent) parameters of problem (3.3),(3.4).

Denote by $\Phi(t,\tau)$ $n \times n$ dimensional matrix being a fundamental matrix of the solutions of the system of equations (3.1), i.e.

$$\dot{\Phi}_t(t,\tau) = \mathrm{A}(t)\Phi(t,\tau), \qquad \Phi(\tau,\tau) = I_n, \quad t,\tau \in (t_0\,, t_f],$$

where I_n is n-dimensional identity matrix.

We have the following theorem.

Theorem 3.1. *Let all the conditions imposed above on the functions and matrices participating in problem (3.1), (3.2) are satisfied and the condition holds true:*

$$rank\left(\sum_{i=1}^{l} \alpha^1_i \Phi\left(\bar{t}_i, t_0\right)\right) = n. \tag{3.5}$$

Then the solution of problem (3.1), (3.2) exists and is unique.

Proof. By the Cauchy formula we have:

$$x\left(\bar{t}_i\right) = \Phi\left(\bar{t}_i, t_0\right) x\left(t_0\right) + \int_{t_0}^{\bar{t}_i} \Phi\left(\bar{t}_i, \tau\text{up}\right) f\left(\tau\right) d\tau, \quad i = 1, \ldots, l. \tag{3.6}$$

Substitute (3.6) in (3.2):

$$\left(\sum_{i=1}^{l} \alpha_i^1 \Phi\left(\bar{t}_i, t_0\right)\right) x\left(t_0\right) + \sum_{i=1}^{l} \alpha_i \int_{t_0}^{\bar{t}_i} \Phi\left(\bar{t}_i, \tau\text{up}\right) f\left(\tau\right) d\tau = \alpha^0.$$

Then by (3.5) the system

$$Fx\left(t_0\right) = E,$$

where

$$F = \sum_{i=1}^{l} \alpha_i^1 \Phi\left(\bar{t}_i, t_0\right), \quad E = \alpha^0 - \sum_{i=1}^{l} \alpha_i^1 \int_{t_0}^{\bar{t}_i} \Phi\left(\bar{t}_i, \tau\text{up}\right) f\left(\tau\right) d\tau,$$

for arbitrary given vector α^0 and continuous vector-function $f\left(t\right)$, $t \in [t_0, t_f]$, has the unique solution $x(t_0)$. After determining $x\left(t_0\right)$ the Cauchy problem for differential equation (3.1) will also have a unique solution.

For problem (3.3),(3.4) we prove the following theorem. □

Theorem 3.2. *Let all the conditions imposed above on the functions and matrices participating in problem (3.3),(3.4) be fulfilled. Then for the arbitrarily given parameters* $\lambda_j \in R^{m_j}$, $j = 1, \ldots, l$, *the solution of problem (3.3), (3.4) is representable in the form*

$$x\left(t\right) = x^0\left(t\right) + \sum_{j=1}^{l} x^j\left(t\right) \lambda_j, \tag{3.7}$$

where $x^0\left(t\right)$ *is* n*-dimensional vector-function,* $x^j\left(t\right)$, $j = 1, \ldots, l$,*are* $n \times m_i$ *- dimensional matrix functions are the solutions of the following problems:*

$$\dot{x}^0\left(t\right) = A\left(t\right) x^0\left(t\right) + f\left(t\right), \quad t \in \left[t_0, t_f\right], \tag{3.8}$$

$$\sum_{i=1}^{l} \alpha_i^1 x^0\left(\bar{t}_i\right) = \alpha^0, \tag{3.9}$$

$$\dot{x}^j\left(t\right) = A\left(t\right) x^j\left(t\right) + B^j\left(t\right), \ \ j = 1, \ldots, l, \tag{3.10}$$

$$\sum_{i=1}^{l} \alpha_i^1 x^j\left(\bar{t}_i\right) = -\alpha_j^2, \quad j = 1, \ldots, l. \tag{3.11}$$

Proof. We differentiate the both hand sides of (3.7) with respect to t:

$$\dot{x}\left(t\right) = \dot{x}^0\left(t\right) + \sum_{j=1}^{l} \dot{x}^j\left(t\right) \lambda_j,$$

and substitute the result in differential equation (3.6). After simple transformations and grouping we get:

$$\left[\dot{x}\left(t\right) - A\left(t\right) x^0\left(t\right) - f(t)\right] + \sum_{i=1}^{l} \left[\dot{x}^j\left(t\right) - A\left(t\right) x^j\left(t\right) - B^j\left(t\right)\right] \lambda_j = 0_n.$$

By arbitrariness of the vectors λ_j, $j = 1,\ldots,l$, we require equality to zero of the expressions in square brackets. Hence it follows that the functions $x^i(t)$, $i = 1,\ldots,l$, should satisfy differential equations (3.8), (3.10).

Now substitute representation (3.7) in condition (3.4). After simple transformations we will have

$$\left[\sum_{i=1}^{l}\alpha_i^1 x^0\left(\bar{t}_i\right) - \alpha^0\right] + \sum_{i=1}^{l}\alpha_i^1\sum_{j=1}^{l}x^j\left(\bar{t}_i\right)\lambda_j + \sum_{i=1}^{l}\alpha_i^2\lambda_j = 0_n.$$

Changing the summation order in the second summand and replacing index i by j, in the third one we get:

$$\left[\sum_{i=1}^{l}\alpha_i^1 x^0\left(\bar{t}_i\right) - \alpha^0\right] + \sum_{j=1}^{l}\left[\sum_{i=1}^{l}\alpha_i^1 x^j\left(\bar{t}_i\right) + \alpha_j^2\right]\lambda_j = 0_n.$$

Because of the arbitrariness of the vectors λ_j, $j = 1,\ldots,l$, we require equality to zero of the expressions in square brackets, from which the required conditions (3.9), (3.11) follow. □

Theorem 3.3. *The vector-function $x^0(t)$ and matrix functions $x^i(t)$, $i = 0,\ldots,l$, participating in representation (3.7) of the solution of problem (3.3), (3.4), are unique.*

Proof. Let along with the functions $x^i(t)$, $i = 0,\ldots,l$, $y^0(t)$ there exist such a vector-function and matrix functions $y^i(t)$, $i = 1,\ldots,l$, that the vector-function,

$$x(t) = y^0(t) + \sum_{j=1}^{l} y^i(t)\lambda_j$$

is the solution of problem (3.3), (3.4). Then, repeating the calculations of Theorem 3.3, we get that the functions $y^i(t)$, $i = 0,\ldots,l$, should be the solutions of the following problems:

$$\dot{y}^0(t) = A(t)\,y^0(t) + f(t), \quad t \in [t_0, t_f], \tag{3.12}$$

$$\sum_{i=1}^{l}\alpha_i^1 y^0\left(\bar{t}_i\right) = \alpha^0, \tag{3.13}$$

$$\dot{y}^j(t) = A(t)\,y^j(t) + B^j(t), \quad j = 1,\ldots,l, \quad t \in [t_0, t_f], \tag{3.14}$$

$$\sum_{i=1}^{l}\alpha_i^1 y^j\left(\bar{t}_i\right) = -\alpha_j^2, \quad j = 1,\ldots,l. \tag{3.15}$$

It is clear that statements of problems (3.12),(3.13) and (3.14),(3.15) completely coincide with the statements of problem (3.8),(3.9) and (3.10),(3.11) respectively. The required statement follows from the uniqueness of the solution of problem (3.12),(3.13), coinciding by the statement with problem (3.12),(3.13) and (3.14),(3.15), proved in theorem 3.1. □

We now formulate the basic theorem for the solution of the considered problem (2.1),(2.2).

Theorem 3.4. *Under the above requirements imposed on the function and matrix participating in the problem statement, problem (2.1),(2.2) has a unique solution in the case when the system of nonlinear ln equations with ln unknowns*

$$z^j - \sum_{i=1}^{l} x^i\left(\bar{t}_j\right) g_i(z^i) = x^0\left(\bar{t}_j\right), j = 1, \ldots, l, \tag{3.16}$$

has a unique solution, with respect to $z^j \in R^n$, $j = 1, \ldots, l$, where $x^j(t)$, $j = 0, \ldots, l$ are the solutions of auxiliary problems (3.8)-(3.11).

Proof. Accepting the notations

$$z^j = x\left(\bar{t}_j\right), \lambda_j = g_j\left(x\left(\bar{t}_j\right)\right) = g_j\left(z^j\right), j = 1, \ldots, l, \tag{3.17}$$

using the representation (3.7) for the points $t = \bar{t}_i$, $i = 1, \ldots, l$, we get the nonlinear system:

$$z^j = x^0\left(\bar{t}_j\right) + \sum_{i=1}^{l} x^i\left(\bar{t}_j\right) g_i\left(x\left(\bar{t}_i\right)\right), \quad j = 1, \ldots, l, \tag{3.18}$$

where $x^i\left(\bar{t}_j\right)$, $i = 0, \ldots, l$, $j = 1, \ldots, l$, are the solutions of problems (3.8)-(3.11).

The proof of the theorem directly follows from Theorems 3.1-3.3. Using representation (3.7) and notation (3.17), we get a system of (ln) nonlinear equations with respect to (ln) unknowns $z^j = x\left(\bar{t}_j\right)$, $j = 1, \ldots, l$.

The existence and uniqueness of the solution of the original problem (2.1),(2.2) directly depends on the existence and uniqueness of the solution of the nonlinear system of equations (3.18). It is clear that if the vectors $z^j = x\left(\bar{t}_j\right)$ are uniquely determined from system (3.18), then $g_i\left(x\left(\bar{t}_i\right)\right)$, and by theorem 3.1, the solution of the original problem (2.1),(2.2) is determined uniquely. □

Using the fundamental matrix of the solutions $\Phi(t, \tau)$ we can write nonlinear system (3.16) in the following form:

$$z^j - \sum_{i=1}^{l} G_j^i g_i\left(z^i\right) = Q_j, \ j = 1, \ldots, l, \tag{3.19}$$

where

$$G_j^i = \int_{t_0}^{\bar{t}_j} \Phi\left(\bar{t}_j, \tau \mathrm{up}\right) B^i(\tau)\, d\tau + \Phi\left(\bar{t}_j, t_0\right) x^i(t_0), \ \ i = 1, \ldots, l,$$

$$Q_j = \Phi\left(\bar{t}_j, t_0\right) x^0(t_0) + \int_{t_0}^{\bar{t}_j} \Phi\left(\bar{t}_j, \tau \mathrm{up}\right) f(\tau)\, d\tau, j = 1, \ldots, l.$$

Note that for finite-dimensional systems of nonlinear equations there are no global constructive conditions for the existence and uniqueness of the solution, and the existing conditions are only local. In practice it is necessary to carry on these studies directly for each problem using computer experiments applying different numerical methods for solving the systems.

Clearly, the case of non-uniqueness of the solution of the system of nonlinear equations (3.16) or (3.19) corresponds to the possibility of existence of several solutions of the input nonlinearly loaded problem (2.1), (2.2). In this case two problems arise: finding all the solutions of the system and determination of the most acceptable solution. The first problem is practically not solvable in real problems at least due to the fact that even the number of solutions is unknown for nonlinear systems. The solution of the second problem is possible if there is expert information on real values of the state of the process at loading points (i.e. $x\left(\bar{t}_i\right)$) or on the values of the loading (i.e. $g_i\left(x\left(\bar{t}_i\right)\right)$). This information may be used in iterative methods for numerical solution of the systems of nonlinear equations (3.16), (3.19) to form an initial approximation.

Let us consider a very important and frequently encountered special case of the problem (2.1), (2.2), when we have the Cauchy initial conditions hold instead of conditions (2.2):

$$x\left(t_0\right) = \alpha^0, \qquad \alpha^0 \in R^n. \tag{3.20}$$

Then, using the scheme of the proof of theorem 3.2, instead of the conditions (3.9), (3.11) it is easy to get the following initial conditions:

$$x^0\left(t_0\right) = \alpha^0, \tag{3.21}$$

$$x^j\left(t_0\right) = 0_{n\times m_j}, \qquad j = 1,\ldots,l. \tag{3.22}$$

Consequently, the auxiliary problems for the initial nonlinear loaded Cauchy problem (2.1),(3.20) are the linear Cauchy problems (3.8),(3.21) and (3.10),(3.22). Concerning theorem 3.4, it remains valid.

Another special case of problem (2.1),(2.2) is the specification of conditions (2.2) in the form:

$$\sum_{i=1}^{l} \alpha_i^2 g_i\left(x\left(\bar{t}_i\right)\right) = 0_n, \tag{3.23}$$

i.e. all $\alpha_i^1 = 0_{n\times n}$, $\quad i = 1,\ldots,l$, and $\alpha_0 = 0_n$. Then in theorem 2 (how difficult to get them) the conditions (3.9),(3.11) will be homogeneous, i.e.

$$x^0\left(t_0\right) = 0_n, \tag{3.24}$$

$$x^j\left(t_0\right) = 0_{n\times m_j}, \quad j = 1,\ldots,l. \tag{3.25}$$

Consequently, auxiliary problems (3.8),(3.24) and (3.10),(3.25) are the Cauchy problems with respect to nonlinear differential equations. Theorem 3.4 remain in this case unchanged.

We give general description of the application of the suggested approach (algorithm) to the numerical solution of problem (2.1),(2.2).

At first auxiliary liner vector (3.8),(3.9) and matrix problems (3.10),(3.11) with non-separated multipoint conditions are solved. Here the methods for example in [1, 3, 5, 6, 8, 12] can be used. It is possible to reduce the problems with multipoint conditions to two point ones [20] by increasing the order of the system of differential equations with further application of the sweep method possible [7, 21].

The values of the obtained solutions of the auxiliary problems at the loading points $\bar{t}_i$, $j = 1,\ldots,l$, are used to form nonlinear finite-dimensional system (3.16) or what is the same (3.19) with ln unknowns $\quad z^j = x\left(\bar{t}_j\right)$, $\quad j = 1,\ldots,l$. Taking into account that as a rule the functions $g_j(.)$, $\quad j = 1,\ldots,l$, have sufficiently simple form, are easily differentiable, then for the system it is not difficult to construct the Jacobian matrix in the analytical (finite) form. Consequently, for the numerical solution of the system (3.16) it is possible to use effective methods of second order convergence (e.g. Newtonian-type methods).

Having defined the values of the loading $x\left(\bar{t}_j\right)$ from the system of nonlinear equations if the solutions of the auxiliary problems $x^i\left(t\right), \quad i=0,\ldots,l, \quad t\in\left[t_0,t_f\right]$, are have been saved in computer memory one can obtain the solution of the original problem from representation (3.7), taking into account notation (3.17), but if it is impossible to store the trajectories $x^i\left(t\right), \quad t\in[t_0,t_f], i=0,\ldots,l$, in computer memory we do as follows: we substitute the obtained values of the loading $x\left(\bar{t}_j\right)$ in differential equations and boundary conditions, we solve ordinary (without loading) system of linear differential equations with ordinary (without loading) multipoint boundary conditions.

4. ILLUSTRATIVE EXAMPLES

Example 1. Let us consider a special case of problem (2.1),(2.2)in which:

$$n=1, \quad l=2, \quad A\left(t\right)=8t, \quad B^n\left(t\right)=0, \quad B^2\left(t\right)=t, \quad g\left(x\right)=x^2+x,$$

$$f\left(t\right)=-\left(e^2+3e+10\right), \quad t_0=\ t_1=0, \ \ \bar{t}_2=0,5, \ \ t_f=1,$$

$$\alpha^0=6+e^2+3e, \quad \alpha_1^1=2, \quad \alpha_1^2=1.$$

The following problem corresponds to these parameters:

$$\dot{x}\left(t\right)=8tx\left(t\right)+t\left(x^2\left(0.5\right)+x\left(0.5\right)\right)-\left(e^2+3e+10\right)t, \quad t\in\left[0;1\right],$$

$$2x\left(0\right)+g\left(x\left(0,5\right)\right)=6+e^2+3e.$$

By theorem 3.2 and above scheme for using the suggested approach, at first we solve auxiliary problems (3.8),(3.11) that for the problem under consideration have the following form:

$$\dot{x}^0\left(t\right)=8tx^0\left(t\right)-\left(e^2+3e+10\right)t, \quad t\in\left[0;1\right],$$

$$2x^0\left(0\right)=6+e^2+3e,$$

$$\dot{x}^1\left(t\right)=8tx^1\left(t\right)+t, \quad t\in\left[0;1\right],$$

$$2x^1\left(0\right)=-1.$$

Their solutions are respectively the functions

$$x^0\left(t\right)=\ \left(0,375e^2+1,25e+1,875\right)e^{4t^2}+0,125\left(e^2+3e+10\right),$$

$$x^1\left(t\right)=-0,375e^{4t^2}-0,125, t\in\left[0;1\right].$$

Hence it follows that the values of these functions at the loading point $t=0,5$ are equal to

$$x^0\left(0,5\right)=0,375e^3+1,25\ e^2+2,125e+1,25,$$

$$x^1\left(0,5\right)=-0,375e-0,125.$$

Denote the value of the solution at the loading point by $z=x(0,5)$. Write out the nonlinear (in the given case quadratic) equation (3.16) subject to calculated values of the solutions of auxiliary problems:

$$z+0,125\left(3e+1\right)\left(z^2+z\right)=0,375e^3+1,25e^2+2,125e+1,25.$$

For this equation it is easy to determine the following two solutions

$$z_1 = e + 1, \quad z_2 = -e - 2 - \frac{8}{3e+1} \quad .$$

These solutions correspond to the values of the loading

$$g\left(z_1\right) = z_1^2 + z_1 = e^2 + 3e + 2.$$

$$g\left(z_2\right) = z_2^2 + z_2 = e^2 + 3e + 2 + \frac{16\mathrm{e}}{3e+1} + \frac{24}{3e+1} + \frac{64}{\left(3e+1\right)^2} \quad .$$

Having substituted these values in the representation of the original problem:

$$x\left(t\right) = x^0\left(t\right) + x^1\left(t\right) g\left(x\left(0,5\right)\right),$$

we get two possible solutions of the original problem

$$x\left(t\right) = \left(1 - \frac{6e}{3e+1} - \frac{9}{3e+1} - \frac{24}{\left(3e+1\right)^2}\right) e^{4t^2} +$$

$$+1 - \frac{2e}{3e+1} - \frac{3}{3e+1} - \frac{8}{\left(3e+1\right)^2}.$$

$$x\left(t\right) = e^{4t^2} + 1, \quad t \in \left[0;1\right].$$

By substituting these functions in the problem under consideration we can see that both of the functions satisfy both the loaded differential equation and non-separated boundary condition.

Example 2. We give numerical solution of the following loaded Cauchy problem with respect to second order differential equation:

$$u^{''}\left(t\right) + u\left(t\right) + \sum_{i=1}^{l} b_i\left(t\right) g_i\left(u\left(\bar{t}_i\right)\right) + f\left(t\right) = 0, \quad t \in \left[0, T\right],$$

$$u\left(0\right) = \alpha^{0_1}, \qquad u_t^{'}\left(0\right) = \alpha^{0_2},$$

under the following values of data and functions participating in the problem: $l = 3$, $T = \pi$, $\bar{t}_i = i\frac{\pi}{4}$, $i = 1,2,3$, $b_1\left(t\right) = t^2$, $b_2\left(t\right) = -t$, $b_3\left(t\right) = t^3$, $f\left(t\right) = -\frac{t^2+t^3}{2\sqrt{2}}$, $g_1\left(u\left(\cdot\right)\right) = u^3$, $g_2(u(\cdot)) = \left(u^{'}\left(\cdot\right)\right)^2$, $g_3\left(u\left(\cdot\right)\right) = u^3$, $\alpha_1^0 = 0$, $\alpha_2^0 = 1$.

Introducing the variables: $x_1\left(t\right) = u(t)$, $x_2\left(t\right) = u^{'}(t)$, we get a Cauchy problem of the form (2.1),(3.20), wherein

$$A\left(t\right) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B_1\left(t\right) = \begin{pmatrix} 0 \\ -t^2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ t \end{pmatrix},$$

$$B_3\left(t\right) = \begin{pmatrix} 0 \\ -t^3 \end{pmatrix}, \ f\left(t\right) = \begin{pmatrix} 0 \\ -\frac{t^2+t^3}{2\sqrt{2}} \end{pmatrix}, \quad \alpha^0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$g_1\left(x\left(\bar{t}_1\right)\right) = x_1^3\left(\bar{t}_1\right), \qquad g_2\left(x\left(\bar{t}_2\right)\right) = x_2^2\left(\bar{t}_2\right), \qquad g_3\left(x\left(\bar{t}_3\right)\right) = x_1^3\left(\bar{t}_3\right).$$

For numerical solution of auxiliary Cauchy problems (3.8),(3.9) and (3.10),(3.25) the Runge-Kutta method of fourth order with the step $h = \pi/40$ was used. The results of solutions of these systems at the loading points (all results below were rounded to the third digit after the decimal point) are given in table 1.

These results were used to form the following nonlinear system corresponding to (3.16), with respect to six unknowns: $z = \left(z^1, z^2, z^3\right)$,

$$z^i = \left(x_1\left(\bar{t}_i\right),\ x_2(\bar{t}_i)\right), i = 1, \ldots, 3:$$

$$z_1^1 = 0.723 - 0.031\left(z_1^1\right)^3 + 0.078\left(z_2^2\right)^2 - 0.015\left(z_1^3\right)^3,$$
$$z_2^1 = 0.795 - 0.157\left(z_1^1\right)^3 + 0.293\left(z_2^2\right)^2 - 0.093\left(z_1^3\right)^3,$$
$$z_1^2 = 1.325 - 0.467\left(z_1^1\right)^3 + 0.571\left(z_2^2\right)^2 - 0.451\left(z_1^3\right)^3,$$
$$z_2^2 = 0.899 - 1.142\left(z_1^1\right)^3 + \left(z_2^2\right)^2 - 1.402\left(z_1^3\right)^2,$$
$$z_1^3 = 2.589 - 2.137\left(z_1^1\right)^3 + 1.649\left(z_2^2\right)^2 - 3.196\left(z_1^3\right)^3,$$
$$z_2^3 = 2.726 - 3.298\left(z_1^1\right)^3 + 1.707\left(z_2^3\right)^2 - 6.411\left(z_1^3\right)^3.$$

For numerical solution of this nonlinear system of equations the Newton iterative method from the Matlab package was used. As initial approximations for the unknowns z many different vectors were chosen, and six of them are given below
1) (1;1;1;1;1;1); 2) (0;2;0;4;5;0); 3) (-5;4;-2;0.5;-3;12);
4) (-1;0;2;3;1;4); 5) (3;2;1;4;5;6); 6) (14;21;16;11;7;-4).

In all experiments the nonlinear system was solved with accuracy about $10^{-10} \div 10^{-8}$ with respect to the magnitude of equations residual. The following values of the unknown function at the loading points were obtained by the Newton method starting from the first three initial points:

$$u\left(\bar{t}_1\right) = 0.707, \quad u'\left(\bar{t}_2\right) = 0, \quad u\left(\bar{t}_3\right) = 0.707.$$

They were substituted in the original equation and the Cauchy problem was solved. The obtained result was given in Fig 1 (the first one).

When using the fourth and fifth points as initial approximations, for the Newton method, the following solution was obtained:

$$u\left(\bar{t}_1\right) = 1.952, \quad u'\left(\bar{t}_2\right) = 4.435, \quad u\left(\bar{t}_3\right) = 1.760.$$

After substituting there values in the equation, the obtained solution of the Cauchy problem was given in Fig 1 (the second one).

When using the sixth initial point, the solution of the system of equations at which

$$u\left(\bar{t}_1\right) = 2.729, \quad u'\left(\bar{t}_2\right) = 5.878, \quad u\left(\bar{t}_3\right) = 1.655,$$

was obtained, and the graph of the function $u(x)$, obtained from the solution of the Cauchy problem was given in Fig.1 (the third one).

j	***0***	***1***	***2***	***3***
$x^j(\bar{t}_1)$	0.72329; 0.79542	-0.03107; -0.15658	0.07829; 0.29289	-0.01472; -0.09319
$x^j(\bar{t}_2)$	1.32471; 0.89937	-0.46741; -1.14158	0.57080; 0.99999	-0.45101; -1.40219
$x^j(\bar{t}_3)$	2.58932; 2.72605	-2.13744; -3.29815	1.64908; 1.70710	-3.18626; -6.41226

Table 1. The values of auxiliary problems at loading points.

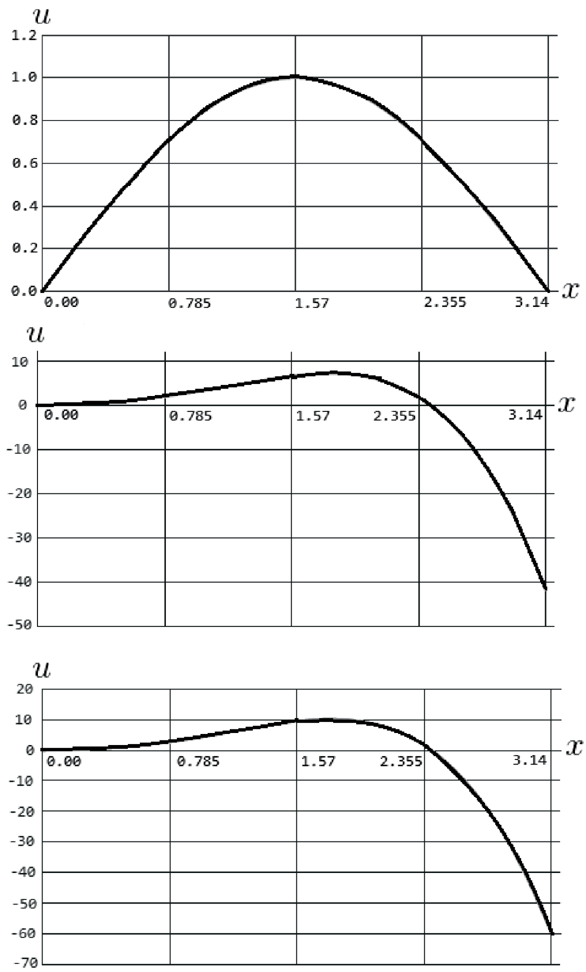

Fig.1. Graphs of the functions $u(x)$ corresponding to different solutions.

In all these experiments, the goal was not to find all the solutions of the system of nonlinear equations. It was interesting to clarify "region of attraction" of the solutions of the system, i.e. stability of the solution in relation to the choice of initial approximation. As can be seen from the cited results, this stability holds for this problem.

We should note the most important thing for this problem: the parameters and functions participating in it were chosen so that the function $u(x) = \sin(x)$ would be its solution. In experiments carried out with the first three initial approximations which were quite far from each other, this function was determined with high accuracy. Clearly, in real problems the final choice of the solution depends on the sense of the problem and existing conditions and constrains imposed on the sought function related to it.

5. CONCLUSION

An approach to solving ordinary differential equations with a linear principal part and with terms involving nonlinear functions of the values of the sought functions at the loading points, was suggested. Boundary conditions were determined by linear combination of the values of the sought function at intermediate points and terms with the same nonlinear functions of the values of the unknown function at the loading points that participate in equations.

At first the solution of the input problem is reduced to the solution of auxiliary linear boundary value problems without loading, then a finite-dimensional nonlinear system of equations is solved and the values of the sought function at loading points are determined.

Some special cases of defining boundary conditions in the problem under consideration were considered.

The approach suggested in the paper for solving nonlinearly loaded linear ordinary differential equations may be easily extended to some classes of nonlinearly loaded boundary value problems with respect to nonlinearly loaded linear partial differential equations, in particular, of elliptic type.

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