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OPTIMAL CONTROL OF LIPSCHITZIAN AND DISCONTINUOUS DIFFERENTIAL INCLUSIONS WITH VARIOUS APPLICATIONS

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Abstract. This paper is devoted to optimal control of dynamical systems governed by differential inclusions in both frameworks of Lipschitz continuous and discontinuous velocity mappings. The latter framework mostly concerns a new class of optimal control problems described by various versions of the so-called sweeping/Moreau processes that are very challenging mathematically and highly important in applications to mechanics, engineering, economics, robotics, etc. Our approach is based on developing the method of discrete approximations for optimal control problems of such differential inclusions that addresses both numerical and qualitative aspects of optimal control. In this way we derive necessary optimality conditions for optimal solutions to differential inclusions and discuss their various applications. Deriving necessary optimality conditions strongly involves advanced tools of first-order and second-order variational analysis and generalized differentiation.

1. Introduction and Overview

This paper is devoted to the study of differential inclusions given by

$$\dot{x}(t) \in F\big(t, x(t)\big) \quad \text{a.e.} \quad t \in [a, b], \tag{1.1}$$

where $F : [a, b] \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued mapping/multifunction acting in finite-dimensional spaces, and where $\dot{x}(t)$ denotes the standard derivative in time for absolutely continuous vector functions $x(t)$. Such dynamical systems arise as extensions of the *controlled differential equations* described by

$$\dot{x}(t) = f\big(t, x(t), u(t)\big), \quad u(t) \in U \quad \text{a.e.} \quad t \in [a, b] \tag{1.2}$$

with $F(t, x) := f(t, x, U) = \{v \in \mathbb{R}^n |\ v = f(t, x, u) \ \text{ for some } \ u \in U\}$ in (1.1). We refer the reader to the classical monograph by Pontryagin et al. [44] on optimal control theory for systems (1.2) governed by velocity mappings f continuously differentiable in state variables x and to the author's book [38] and the bibliographies where differential inclusions of type (1.1) with Lipschitzian set-valued mappings $x \mapsto F(t, x)$ are investigated from the viewpoint of modern

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variational analysis and generalized differentiation. Note that the differential inclusion formalism (1.1) covers not only the standard program control setting (1.1) with constant control sets U, or the setting when $U = U(t)$, but also much more involved case where the moving control set depends on state variables $U = U(t, x)$. The latter case corresponds to $F(t, x) = f(t, x, U(t, x))$ in (1.1) and reflects a *feedback control* effect that is crucial for engineering design and other applications. Furthermore, differential inclusions of type (1.1) naturally arise not only in the parameterized control framework as in (1.2) with U depending or not depending on (t, x), but also without any control parameterizations, which is often the case of dynamical systems appearing in applied models from mechanics, economics, and behavioral sciences; see, e.g., the sweeping processes considered below.

Our approach to study differential inclusions and optimization problems for them is based on the *method of discrete approximations*. The idea is simple and actually goes back to Euler [20]: replace the derivative $\dot{x}$ by some finite difference, then consider optimization of the obtained family of discrete-time problems with a fixed step of discretization, and finally investigate what happens when the discretization step is diminishing. Euler himself dealt with minimizing a specific integral functional dependent on $\dot{x}$ and observed in this way a certain necessary condition for optimality, which was a prototype of what is now called "Euler equation" or "Euler-Lagrange equation" in the calculus of variations.

The implementation of this idea in the presence of dynamic constraints of type (1.1), or even of type (1.2) with smooth mappings f, is dramatically more involved. We refer the reader to the author's books [33, 38] with the extended bibliographies therein for realizations of this approach in dynamical systems of various kinds governed by ordinary differential equations and inclusions, delay-differential and neutral-type inclusions, partial differential equations and inclusions of the parabolic type, etc. Using the method of discrete approximations married to appropriate robust tools of variational analysis and generalized differentiation allowed us to establish the required well-posedness and convergence of discrete approximations, to derive necessary optimality conditions for discrete-time systems, and then to justify limiting procedures of obtaining necessary optimality conditions of the *Euler-Lagrange* and *Hamiltonian* types (including counterparts of the *Pontryagin maximum principle*) for the corresponding continuous-time problems of dynamic optimization with finite-dimensional and infinite-dimensional state spaces.

In the framework of optimization problems for differential inclusions (1.1), the most essential assumption imposed in the aforementioned publication was the *Lipschitz continuity* of the velocity mapping F with respect to the state variable x. This assumption was crucial for both convergence analysis of discrete approximations and realization of the limiting procedure to derive necessary optimality conditions in continuous-time systems by employing machinery of variational analysis; see Section 3.

Although the aforementioned Lipschitzian assumption seems to be rather general, it heavily fails for recently discovered classes of optimal control problems governed by differential inclusions, which are associated with the so-called *sweeping processes*. The basic sweeping process ("processus de rafle") was introduced by Moreau in the 1970s to describe some mechanical problems mainly related to elastoplasticity; see [43] and the book [30] for more details and references. In a parallel way, similar processes were considered in the Soviet literature by Krasnosel'skii and Pokrovskii in connection with systems of hysteresis; see, e.g., their book [27]. Besides the original motivations, models of this type have found numerous applications to electric circuits [1], traffic equilibria [29, 49], and various other areas of applied sciences. For its own sake, sweeping process theory has become an important area of nonlinear analysis with impressive mathematical achievements; see, e.g., [16] with a large list of references.

The basic *sweeping process* of Moreau is described by the differential inclusion

$$\dot{x}(t) \in -N(x(t); C(t)) \ \text{ a.e. } \ t \in [0,T], \quad x(0) := x_0 \in C(0), \tag{1.3}$$

where $N(x;\Omega) = N_\Omega(x)$ stands for the *normal cone* to a *convex* set $\Omega \subset \mathbb{R}^n$ at x defined by

$$N(x;\Omega) := \begin{cases} \{v \in \mathbb{R}^n |\ \langle v, u - x\rangle \le 0 \ \text{ for all } u \in \Omega\} & \text{if } \ x \in \Omega, \\ \emptyset & \text{otherwise,} \end{cases} \tag{1.4}$$

and where the moving convex set $C(t)$ nicely (in a certain continuous way) evolves in time on the fixed interval $[0,T]$. We see that the velocity set $F(x) := -N(x; C(t))$ on the right-hand side of (1.3) has *unbounded* values and is *never Lipschitz continuous*; in fact, it is even discontinuous. Thus the aforementioned theory for (1.1) cannot be applied to (1.3). On the other hand, for each $t \in [0,T]$ the mapping $x \mapsto N(x; C(t))$ is *maximal monotone* due to a fundamental result of convex analysis [46], and so the differential inclusion in (1.3) is *dissipative.* It stays behind a central result of sweeping process theory saying that the Cauchy problem in (1.3) admits a *unique* solution; see [43] and also [16] for further developments. This clearly excludes the possibility to consider any optimization problem for the sweeping differential inclusion (1.3) in contrast to model (1.1) with a Lipschitzian velocity mapping F.

In [11] we initiated a new approach to sweeping process theory that made it possible to control and optimize the sweeping dynamics. The idea in [11] was to enter control actions into the moving set by

$$C(t) := C\big(u(t)\big) \ \text{ for all } \ t \in [0,T] \tag{1.5}$$

and to regulate in this way the sweeping dynamics by the choice of appropriate control functions $u(t)$ under certain constraints while changing via (1.5) the shape of the moving sets $C(t)$ and hence the corresponding trajectories $x(t)$ of (1.3) in order to minimize a given cost functional. By the second line in the normal cone definition (1.4), the control parametrization (1.5) of the moving set in (1.3) always contains implicitly the pointwise *state-control constraints*

$$x(t) \in C\big(u(t)\big) \ \text{ for a.e. } \ t \in [0,T], \tag{1.6}$$

which are among the most difficult in control theory even for standard ODE control systems in form (1.2) with smooth velocity mappings f. Remembering the discontinuity of the set-valued mapping $x \mapsto -N(x; C(t))$ on the right-hand side of (1.3), we arrive therefore at a class of highly challenging and previously unsolved problems in systems control that are strongly motivated by applications.

Starting in [11] with the case where $C(u)$ is given by a half-space in $\mathbb{R}^n$, we obtained in [13] a set of necessary optimality conditions to solve optimal control problems of the generalized Bolza type under the general *polyhedral* description of the moving sets in (1.3) presented in the form

$$C(t) := \big\{x \in \mathbb{R}^n \big|\ \langle u_i(t), x\rangle \le w_i(t),\ \ i = 1, \ldots, m\big\} \tag{1.7}$$

$$\text{with}\ \ \|u_i(t)\| = 1\ \ \text{for all}\ \ t \in [0, T],\ \ i = 1, \ldots, m, \tag{1.8}$$

where the control functions $u(\cdot) = (u_1(\cdot), \ldots, u_m(\cdot))$ and $w(\cdot) = (w_1(\cdot), \ldots, w_m(\cdot))$ are *absolutely continuous* on $[0, T]$. Taking into account the structure of $C(t)$ in (1.7) and the absolute continuity assumptions on the feasible controls $(u(t), w(t))$, which are needed for the existence of the corresponding trajectories in (1.3), the state-control constraints in (1.6) are written now as

$$x(t) \in C\big(u(t), w(t)\big)\ \ \text{for all}\ \ t \in [0, T] \tag{1.9}$$

with $u(t)$ satisfying the equality constraints (1.8) required by applications. The approach used in [11, 13] is based on constructing well-posed and largely modified in comparison with [36, 38] *discrete approximations* as in [12], establishing their strong convergence in Sobolev space $W^{1,2}([0, T]; \mathbb{R}^n)$, and then deriving necessary optimality conditions in the extended *Euler-Lagrange form* expressed entirely via the given data of the problem and the local optimal solution under consideration. In Section 4 we present more general necessary optimality conditions obtained not only in the Euler-Lagrange form but also in the new *Hamiltonian* form with a novel version of the *maximum principle* for a more general class of controlled sweeping processes. The derivation of the obtained results is based on the appropriate development of the method of discrete approximations as given in the recent paper [26].

Another class of controlled sweeping processes was introduced and studied in [5] by using the method of discrete approximations. The difference from [11, 13, 26] is that—along with controls $u(\cdot) \in W^{1,2}([0, T]; \mathbb{R}^n)$ entering the moving set $C(t)$ as

$$C(t) := C + u(t)\ \ \text{where}\ \ C := \big\{x \in \mathbb{R}^n \big|\ \langle x_i^*, x\rangle \le 0\ \ \text{for all}\ \ i = 1, \ldots, m\big\} \tag{1.10}$$

with the fixed generating vectors x_i^* of the convex polyhedron C in (1.10)-we also employ the other type of controls $v(\cdot) \in W^{1,2}([0, T]; \mathbb{R}^d)$ in the additive *perturbation* of (1.3) given by

$$-\dot{x}(t) \in N\big(x(t); C(t)\big) + g\big(x(t), v(t)\big)\ \ \text{for a.e.}\ \ t \in [0, T], x(0) := x_0 \in C(0) \tag{1.11}$$

within a controlled external force $g\colon \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^d$. The necessary optimality conditions from [5] were then applied in [6] to solving some optimal control problems for the *corridor version* of the *crowd motion model* of traffic equilibria. The dynamics of the aforementioned model was described as a sweeping process in [29, 49], while our optimal control approach and the results obtained open the gate to find the best strategies for regulating the pedestrian traffic and related dynamical processes.

However, the polyhedral description of the moving set in (1.10) does not allow us to cover a more realistic *planar crowd motion model* the dynamic of which was described in [29, 49]. This was the main motivation for us to consider in [7] nonconvex (and hence nonpolyhedral) sweeping control problems and derive necessary optimality conditions for them by employing the method of discrete approximations. Some applications of the obtained results to the planar crowd motion model were given in [8]. We discuss these and related topics in more detail in Sections 5 and 6.

Finally, in Section 7 we present very recent results on controlled sweeping process with control actions entering the external force as in (1.11), which may be *discontinuous* while satisfying certain *pointwise/hard constraints*. The results obtained therein by using a new version of the discrete approximation method from [14] contain necessary optimality conditions including the maximization of the corresponding Hamiltonian function (i.e., the maximum principle) were then applied in [15] to some practical models arising in robotics and traffic flow models different from those for crowd motions mentioned above.

The structure of the rest of the paper is as follows. Section 2 contains basic preliminaries from variational analysis and generalized differentiation that are widely used in the text. Section 3 is devoted to optimization problems for *Lipschitzian* differential inclusions. Each of the subsequent sections presents the most advanced results on various types of controlled sweeping processes and also discusses some *open questions* for further research and possible interest of the readers.

Section 4 is mainly based on the recent paper [26], where we consider optimization problems for a general class of sweeping processes with $W^{1,2}$-*smooth* controls entering *moving sets*. It contains discrete approximations and necessary optimality conditions in both extended Euler-Lagrange and Hamiltonian forms with a new version of the maximum principle that is different from the expected form, which actually fails. Applications to hysteresis systems and elastoplasticity problems are also discussed there.

In Section 5 is devoted to optimization of sweeping processes with $W^{1,2}$-smooth controls in both *moving sets* and *external perturbations*. We consider control problems for polyhedral and nonpolyhedral (prox-regular) moving sets. The obtained necessary optimality conditions are applied to optimal control of the microscopical crowd motion model in the corridor and planar versions. The planar version of this model is presented in Section 6.

The last Section 7 concerns a class of optimal control problems for a perturbed sweeping process with *constrained discontinuous* controls located only in the perturbation term. By using an extended version of the discrete approximation approach, we establish a certain strong convergence of discrete optimal solution to a given local minimizer and derive a new set of necessary optimality conditions including the maximum principle. Then obtained results are applied to determine optimal strategies in some constrained models of robotics and traffic equilibria.

Throughout the paper we use the standard notation of variational analysis and control theory; see, e.g., [39, 47, 50]. Recall that $\mathbb{N} := \{1, 2, \ldots\}$, that A^* stands for the transposed/adjoint matrix to A, and that $\mathbb{B}$ denotes the closed unit ball of the space in question. The symbol $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ indicates that F may be a set-valued mapping/multifunction (i.e., takes values in the collections of all subsets of $\mathbb{R}^m$), in contrast to the usual notation $F\colon \mathbb{R}^n \to \mathbb{R}^m$ for single-valued mappings.

2. Tools of Variational Analysis and Generalized Differentiation

Modern machinery of variational analysis and optimal control, especially for differential inclusions, are largely based on generalized differentiation; see, e.g., the books [37, 38, 39, 50] and the references therein. Nonsmoothness comes naturally and frequently even for problems with smooth initial data due to the very essence of constraints of inequality and inclusion types and via using variational principles and techniques. Thus appropriate tools of generalized differentiation are required in the study of control problems and particularly in the derivation (and often in formulation) of necessary optimality conditions for them.

In this section we briefly overview some basic notions of generalized differentiation for nonsmooth functions, sets, and set-valued mappings needed for our purposes while referring to the books [33, 37, 39, 47] and the bibliographies therein for more details. Our approach to generalized differentiation is *geometric* as developed in [33, 37, 39] and numerous papers. This means that we start with generalized normals to sets, then proceed with generalized derivatives (coderivatives) of set-valued or single-valued mappings, and finally end up with first-order and second-order subdifferentials of extended-real-valued (in particular, of usual—nonsmooth at the first or second order—real-valued) functions.

Given a set $\Omega \subset \mathbb{R}^n$, which is locally closed around a point $\overline{x} \in \Omega$, the *normal cone* to Ω is defined by

$$N(\overline{x};\Omega) = N_\Omega(\overline{x}) := \begin{cases} \{v \in R^n \mid \exists\ x_k \to \overline{x},\ \alpha_k \geq 0,\ w_k \in \Pi(x_k;\Omega), \\ \qquad \alpha_k(x_k - w_k) \to v \ \text{ if } \ \overline{x} \in \Omega, \\ \emptyset \qquad\qquad\qquad\qquad\qquad \textit{otherwise}, \end{cases} \tag{2.1}$$

where $\Pi(x;\Omega)$ stands for the Euclidean projector of x onto Ω. This cone has been widely spread in variational analysis and its numerous applications as the basic or limiting normal cone by Mordukhovich; it was introduced by the author in [31]. If Ω is convex, the normal cone (2.1) reduces to the classical normal cone of convex analysis (1.4). If Ω is not convex, then the normal cone (2.1) is often nonconvex as well, even for very simple sets as, e.g., the graph of the function $\varphi(x) := |x|$ or epigraph of the function $-|x|$ at $(0,0) \in \mathbb{R}^2$. This tells us that the collections of normals in (2.1) *cannot* be generated by *any* tangential approximation of Ω via the polarity/duality relation, since polarity always yields convexity. In spite of (actually due to) it, the normal cone (2.1) and the associated constructions of subdifferentials for lower semicontinuous (l.s.c.) functions and coderivatives of set-valued mappings enjoy *full calculus* based on *variational/extremal principles* of variational analysis; see [37, 39, 47] for more details.

Given next a set-valued (in particular, single-valued) mapping $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and a point $(\bar{x},\bar{y}) \in \operatorname{gph} F$ from the (locally closed) graph of F which is defined by

$$\operatorname{gph} F := \big\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \big|\ y \in F(x))\big\},$$

the *coderivative* [32] of F at $(\bar{x},\bar{y})$ is the set-valued mapping $D^*F(\bar{x},\bar{y})\colon \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ with the values

$$D^*F(\bar{x},\bar{y})(u) := \big\{v \in \mathbb{R}^n \big|\ (v,-u) \in N\big((\bar{x},\bar{y});\operatorname{gph} F\big)\big\} \ \text{ for all } \ u \in \mathbb{R}^m, \tag{2.2}$$

while we drop $\overline{y}$ in (2.2) when F is single-valued. If in this case F is C^1-smooth around $\overline{x}$, then

$$D^*F(\bar{x})(u) = \big\{\nabla F(\bar{x})^*u\big\} \text{ for all } u \in \mathbb{R}^m$$

expressed via the transpose Jacobian matrix. In the nonsmooth and set-valued cases of F, the coderivative (2.2) is a positively homogeneous and always set-valued mapping enjoying comprehensive calculus rules and providing complete characterizations (called the "Mordukhovich criteria" in [47]) of the major *well-posedness* properties in nonlinear analysis related to Lipschitzian stability, metric regularity, and linear openness/covering of multifunctions; see [35] and then [37, 39, 47] for different proofs and applications.

Consider finally an extended-real-valued function $\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}} := (\infty, \infty]$, which may take the (plus) infinity value along with real numbers. This framework has been recognized, starting from convex analysis [46], as a convenient language in, e.g., constrained optimization. Define the *domain* and *epigraph* of φ by

$$\operatorname{dom}\varphi := \big\{x \in \mathbb{R}^n \big|\ \varphi(x) < \infty\big\} \text{ and } \operatorname{epi}\varphi := \big\{(x,\alpha) \in \mathbb{R}^{n+1} \big|\ \alpha \geq \varphi(x)\big\}.$$

Picking $\bar{x} \in \operatorname{dom}\varphi$, the (first-order) *subdifferential* of φ at $\bar{x}$ is introduced geometrically as in [31] by

$$\partial\varphi(\bar{x}) := \big\{v \in \mathbb{R}^n \big|\ (v,-1) \in N\big((\bar{x},\varphi(\bar{x}));\operatorname{epi}\varphi\big)\big\} \tag{2.3}$$

via the normal cone (2.1) to the epigraph of φ at $(\bar{x},\varphi(\bar{x}))$. In the case where $\varphi(x) := \delta_\Omega(x)$ is the indicator function of a set Ω that equals 0 for $x \in \Omega$ and ∞ otherwise, we have $\partial\varphi(\bar{x}) = N(\bar{x};\Omega)$.

To proceed further with second-order constructions, we employ the (dual) generalized "derivative-of-derivative" approach, which is implemented as follows [34]. Picking $\bar{v} \in \partial\varphi(\bar{x})$, the *second-order subdifferential* (or *generalized Hessian*) $\partial^2\varphi(\bar{x},\bar{v})\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ of φ at $\bar{x}$ relative to $\bar{v}$ is defined as the coderivative of the first-order subgradient mapping by

$$\partial^2\varphi(\bar{x},\bar{v})(u) := (D^*\partial\varphi)(\bar{x},\bar{v})(u), \quad u \in \mathbb{R}^n, \tag{2.4}$$

where we drop $\bar{v} = \nabla\varphi(\bar{x})$ when φ is differentiable at $\bar{x}$. If φ is C^2-smooth around $\bar{x}$, then (2.4) reduces to the classical (symmetric) Hessian matrix of φ at $\bar{x}$:

$$\partial^2\varphi(\bar{x})(u) = \big\{\nabla^2\varphi(\bar{x})u\big\} \text{ for all } u \in \mathbb{R}^n.$$

Dealing with the sweeping processes defined via the normal cone mappings as in (1.3) and (1.7) and deriving necessary optimality conditions for them which involve appropriate *adjoint systems*, we naturally arrive at considering coderivatives of normal cone mappings, i.e., second-order subdifferentials. To be more precise, define the set-valued mapping $N\colon \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ by

$$N(x,w) := N\big(x;S(w)\big) \text{ for } x \in S(w) := \big\{x \in \mathbb{R}^n \big|\ \theta(x,w) \in \Theta\big\}, \quad w \in \mathbb{R}^m, \tag{2.5}$$

where $S(w)$ is a paremeter/control dependent solution map to the constraint system $\theta(x,w) \in \Theta \subset \mathbb{R}^d$, which covers many important models that frequently appear in applications; see below. Considering further an arbitrary function $\varphi\colon \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ of two variables, define the *partial second-order subdifferential* of φ with respect to x at $(\bar{x},\bar{w})$ relative to $\bar{v}$ by

$$\partial^2_x\varphi(\bar{x},\bar{w},\bar{v})(u) := \big(D^*\partial_x\varphi\big)(\bar{x},\bar{w},\bar{v})(u) \text{ for all } u \in \mathbb{R}^n \tag{2.6}$$

via the coderivative (2.2) of the corresponding first-order partial subdifferential given by

$$\partial_x\varphi(x,w):=\partial\varphi_w(x)\ \text{ with }\ \varphi_w(x):=\varphi(x,w). \tag{2.7}$$

In the case where φ is C^2-smooth around $(\bar{x},\bar{w})$, the partial second-order subdifferential (2.6) reduces to

$$\partial^2\varphi(\bar{x},\bar{w})(u)=\big\{\big(\nabla^2_{xx}\varphi(\bar{x},\bar{w})^*u,\nabla^2_{xw}\varphi(\bar{x},\bar{w})^*u\big)\big\}\ \text{ for all }\ u\in\mathbb{R}^n.$$

However, smoothness is absolutely not the case to be suitable for applications to sweeping processes, which are described by highly discontinuous extended-real-valued functions. Indeed, it is easy to observe that the normal cone mapping (2.5) admits the representation $N(x,w)=\partial_x\varphi(x,w)$ via the composite function $\varphi(x,w):=(\delta_\Theta\circ\theta)(x,w)$ with θ and Θ taken from (2.5). This gives us the following expression of the coderivative of (2.5) as the partial second-order subdifferential (2.6) of the above composite function:

$$D^*N(\bar{x},\bar{w},\bar{v})(u)=\partial^2_x\varphi(\bar{x},\bar{w},\bar{v})(u)\ \text{ whenever }\ \bar{v}\in N(\bar{x},\bar{w})\ \text{ and }\ u\in\mathbb{R}^n. \tag{2.8}$$

Due to the composite structure of φ in (2.8) we need a *second-order chain rule* to calculate the coderivative of the normal cone mapping in (2.8). The most appropriate result for our applications is the one obtained by the author and Rockafellar in [42, Theorem 3.1]:

Generalized Second-Order Chain Rule. *Assume that $\theta\colon\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^d$ be a vector function which is C^2-smooth around $(\bar{x},\bar{w})$ with the surjective partial Jacobian operator $\nabla_x\theta(\bar{x},\bar{w})$. Then for each $\bar{v}\in N(\bar{x},\bar{w})$ there exists a unique vector $\bar{q}\in N_\Theta(\theta(\bar{x},\bar{w}))$ satisfying the equation $\nabla_x\theta(\bar{x},\bar{w})^*\bar{q}=\bar{v}$ and such that the coderivative* (2.2) *of the normal cone mapping* (2.5) *is computed for all $u\in\mathbb{R}^n$ by*

$$D^*N(\bar{x},\bar{w},\bar{v})(u)=\begin{bmatrix}\nabla^2_{xx}\langle\bar{q},\theta\rangle(\bar{x},\bar{w})\\ \nabla^2_{xw}\langle\bar{q},\theta\rangle(\bar{x},\bar{w})\end{bmatrix}u+\nabla\theta(\bar{x},\bar{w})^*D^*N_\Theta\big(\theta(\bar{x},\bar{w}),\bar{q}\big)\big(\nabla_x\theta(\bar{x},\bar{w})u\big). \tag{2.9}$$

The obtained second-order chain rule allows us to reduce the calculation of the coderivative of the normal cone mapping associated with the solution map S in (2.5) to, besides the classical Jacobian and Hessian terms, the coderivative of the normal cone mapping associated with the given set Θ. Constructive computation of D^*N_Θ for various classes of sets Θ, which are overwhelmingly encountered in optimization and control theories and their applications, can be found in [23, 39, 41, 42] and the references therein.

3. Lipschitzian Differential Inclusions

In this section we consider optimal control problems for general differential inclusions of type (1.1) defined by a Lipschitzian set-valued mapping F with respect to state variables. For simplicity and having in mind the subsequent material on optimal control of sweeping processes, we concentrate here only on autonomous problems in finite-dimensional spaces under convexity assumptions. In [38, Chapter 6] and further publications the reader can find results for nonautonomous and nonconvex problems where state spaces are Banach. The problem under consideration in this section is as follows:

$$\text{minimize} \;\; J[x] := \varphi\big(x(0), x(T)\big) + \int_0^T \ell\big(x(t), \dot{x}(t)\big)dt \tag{3.1}$$

over absolutely continuous trajectories $x\colon [0,T] \to \mathbb{R}^n$ of the autonomous differential inclusion

$$\dot{x}(t) \in F\big(x(t)\big) \;\; \text{a.e.} \;\; t \in [0,T] \tag{3.2}$$

subject to the geometric endpoint constraints

$$\big(x(0), x(T)\big) \in \Omega. \tag{3.3}$$

We assume in this section that the mapping $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is locally Lipschitz continuous, convex-valued, and bounded; that the terminal cost function $\varphi\colon \mathbb{R}^{2n} \to \mathbb{R}$ is locally Lipschitzian; that the integrand $\ell(x,v)$ is locally Lipschitzian in (x,v) and convex with respect to the velocity variable $v = \dot{x}$; and that the set Ω is closed. As mentioned above, problem (3.1)–(3.3) can be treated under much more general assumptions, while the (local) Lipschitz continuity of F in x is essential for the obtained results.

Our approach to study problems of type (3.1)–(3.3) is based on the *method of discrete approximations*, which consists of the following major steps:

Step A: Construct a *well-posed family* of discrete approximations involving a finite-difference replacement of the derivative $\dot{x}$ in (3.2) and a matched perturbation of the endpoint constraints in (3.3). The key issue of this step is to verify the possibility to approximate any *feasible* trajectory of (3.2) by feasible trajectories of discrete systems in a topology yielding (along a subsequence if needed) the a.e. convergence of the extended discrete derivatives. We address here not only *qualitative* aspects of well-posedness but also *quantitative* ones with estimating error bounds, convergence rates, etc. Achieving this (in [38] it was done in the norm topology of $W^{1,2}$) leads us then to the strong $W^{1,2}$-norm approximation of a given *local minimizer* for the continuous-time problem (3.1)–(3.3) by a sequence of optimal solutions to the discrete-time problems that are piecewise linearly extended to the whole interval $[0,T]$. In [38] it was done for a class of the so-called "intermediate local minimizers" introduced in [36]. This class includes strong local minimizers while occupying an intermediate position between the latter and weak local minimizers in dynamic optimization; see Definition 3.1 and the discussion after it.

Step B: For each fixed step of discretization, the approximating discrete-time problems can be reduced to nondynamic problems of *mathematical programming* with increasingly many *geometric constraints.* The latter problems are *finite-dimensional* (of increasing dimensions) provided that the dimension of the state space in the original infinite-dimensional optimization problem for (3.2) is finite. Powerful tools of generalized differentiation in variational analysis can be employed for deriving *discrete-time necessary optimality conditions* in the approximation problems in full generality, without any Lipschitzian and convexity assumptions, by using appropriate *calculus rules.* However, dealing with the *graphical structure* of the geometric constraints in the approximation problems requires robust generalized differential constructions, which—besides enjoying comprehensive calculus rules—should be subtle and small enough to handle graphical sets. In particular, applying the *convexified* normal cone and related constructions by Clarke [10] to graphs of mappings often gives us the whole space or its subspace of maximal dimension; see [37, 39, 47] for more details. On the other hand, the nonconvex generalized differential constructions presented in Section 2 satisfy all the required properties and can be successfully used for our purposes.

Step C: The final step in deriving necessary optimality conditions for optimal solutions of (3.1)–(3.3) is the passage to the limit from those for discrete approximation problems obtained in Step B with taking into account the strong convergence of discrete optimal solutions established in Step A. This step is highly involved, since it requires justifying an appropriate convergence of *dual* arcs as trajectories of *adjoint differential inclusions*. For the case of Lipschitzian differential inclusions considered in this section while following [38], it is done by using the Mordukhovich coderivative criteria for the Lipschitz continuity and metric regularity of set-valued mappings mentioned in Section 2. In this way we arrive at the *continuous-time necessary optimality conditions* presented below in Theorem 3.1.

Recall first the appropriate notion of local minimizers for problem (3.1)–(3.3) taken from [36].

Definition 3.1. (intermediate local minimizers for differential inclusions). Let $\bar{x}(\cdot)$ be an absolutely continuous vector function satisfying the differential inclusion (3.2) and the endpoint constraint (3.3). We say that $\bar{x}(\cdot)$ is an INTERMEDIATE LOCAL MINIMIZER of rank $p \in [1, \infty)$ for the dynamic optimization problem (3.1)–(3.3) if there exist real numbers $\varepsilon > 0$ and $\alpha \geq 0$ such that $J[\bar{x}] \leq J[x]$ for any feasible solution to this problem localized by the conditions

$$\|x(t) - \bar{x}(t)\| < \varepsilon \text{ for all } t \in [0, T] \text{ and } \alpha \int_0^T \|\dot{x}(t) - \dot{\bar{x}}(t)\|^p dt < \varepsilon. \tag{3.4}$$

The conditions in (3.4) tell us that a neighborhood of $\bar{x}(\cdot)$ in the space $W^{1,p}\big([0, T]; \mathbb{R}^n\big)$ is actually considered. If $\alpha = 0$ in (3.4), we get the classical *strong* local minimum corresponding to a neighborhood of $\bar{x}$ in the norm topology of $C([0, T]; \mathbb{R}^n)$. If we replace (3.4) by the more restrictive requirement

$$\|\dot{x}(t) - \dot{\bar{x}}(t)\| < \varepsilon \text{ a.e. } t \in [0, T],$$

then it gives us the classical *weak* local minimum in the framework of Definition 3.1. This corresponds to considering a neighborhood of $\bar{x}(\cdot)$ in the norm topology of $W^{1,\infty}([0, T]; \mathbb{R}^n)$. We refer the reader to [36, 50] and particularly to [38, Subsection 6.1.2] for various examples showing that the intermediate notion of Definition 3.1 is properly different from both strong and weak local minimizers of (3.1)–(3.3) for the autonomous differential inclusion under the convexity assumptions imposed above.

Now we are ready to present the main necessary optimality conditions for problem (3.1)–(3.3) obtained via the method of discrete approximations by implementing the procedure outlined in Steps A–C. Under the assumptions imposed in this section, it is sufficient to consider the case where $p = 2$ in Definition 3.1 when we refer to $\bar{x}(\cdot)$ as simply to an intermediate local minimizer.

Theorem 3.1. (extended Euler-Lagrange and maximum conditions for intermediate local minimizers). *Let $\bar{x}(\cdot)$ be an intermediate local minimizer for problem (3.1)–(3.3) under the assumptions made. Then there exist a number $\lambda \geq 0$ and an absolutely continuous vector function $p\colon [0,T] \to \mathbb{R}^n$, which are not equal to zero simultaneously, satisfying the extended Euler-Lagrange inclusion*

$$\dot{p}(t) \in \operatorname{co}\Big\{u \in \mathbb{R}^n \Big|\ \big(u, p(t)\big) \in \lambda\partial\ell\big(\bar{x}(t), \dot{\bar{x}}(t)\big) + N\big((\bar{x}(t), \dot{\bar{x}}(t)); \operatorname{gph} F\big)\Big\} \quad \text{a.e. } t \in [0,T], \tag{3.5}$$

the Weierstrass-Pontryagin maximum condition

$$\big\langle p(t), \dot{\bar{x}}(t)\big\rangle - \lambda\vartheta\big(\bar{x}(t), \dot{\bar{x}}(t)\big) = \max_{v \in F(\bar{x}(t))}\Big\{\big\langle p(t), v\big\rangle - \lambda\ell\big(\bar{x}(t), v\big)\Big\} \quad \text{a.e. } t \in [0,T], \tag{3.6}$$

and the transversality inclusion at both endpoints

$$\big(p(0), -p(T)\big) \in \lambda\partial\varphi\big(\bar{x}(0), \bar{x}(T)\big) + N\big((\bar{x}(0), \bar{x}(T)); \Omega\big). \tag{3.7}$$

When the convexity assumptions do not hold (but the Lipschitzian one still do), we can proceed with *relaxed* intermediate local minimizers as in [38]. In many settings concerning mostly strong minimizers, we automatically have the *relaxation stability* telling us that (local) optimal values in the original and *convexified* problems agree, and thus the given local minimizer for the original problems provides a local minimum to the convex problem as well. This is due to the *continuity/nonatomicity* of the Lebesgue measure on $[0,T]$; see [18, 19, 38, 39, 48, 50] for more details and references.

4. Sweeping Processes with Controlled Moving Sets

In this section we start the study of optimal control problems for sweeping differential inclusions where the Lipschitz continuity of the velocity mapping F in (3.2) dramatically fails. The main attention here is paid to problems of the following type:

$$\text{minimize } J[x,u] := \varphi\big(x(T)\big) + \int_0^T \ell\big(x(t), u(t), \dot{x}(t), \dot{u}(t)\big)dt \tag{4.1}$$

over absolutely continuous control actions $u\colon [0,T] \to \mathbb{R}^m$ and the corresponding absolutely continuous trajectories $x\colon [0,T] \to \mathbb{R}^n$ of the sweeping differential inclusion

$$\dot{x}(t) \in g\big(x(t)\big) - N\big(h(x(t)); C(u(t))\big) \quad \text{a.e. } t \in [0,T], \quad x(0) = x_0 \in C\big(u(0)\big) \tag{4.2}$$

with the controlled moving set defined by

$$C(u) := \big\{x \in \mathbb{R}^n \big|\ \theta(x,u) \in \Theta\big\}, \quad u \in \mathbb{R}^m, \tag{4.3}$$

where $g\colon \mathbb{R}^n \to \mathbb{R}^n$ and $h\colon \mathbb{R}^n \to \mathbb{R}^n$ are C^1-smooth mapping, while $\theta\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^d$ is C^2-smooth mapping around the references points and its partial Jacobian matrix $\nabla_x\theta$ is of full rank therein. We keep the same assumptions on the cost functions φ and ℓ from (4.1) as in Section 3 and also suppose that the set Θ in (4.3) is locally closed and may not be convex, and thus the set $C(u)$ therein is generally nonconvex as well. The normal cone in (4.2) is understood in the sense of (2.1). Among other areas, our investigation of such control problems is motivated by applications to *rate-independent* systems arising in *hysteresis* and related areas. We refer to [1, 2, 4, 25, 27] for sweeping process descriptions of the dynamics in systems of this type. Note also that controlled moving sets (4.3) surely cover the polyhedral ones defined in (1.7), which were considered in [13].

It follows from (4.2) due to (2.1) that problem (4.1)–(4.3) automatically contains the *irregular* pointwise state-control constraints given by the functional inclusion

$$\theta\big(g(x(t)), u(t)\big) \in \Theta \ \text{ for all } \ t \in [0,T],$$

which are the most challenging even in standard optimal control theory for ODE systems with smooth dynamics and are being largely underinvestigated.

Besides the assumptions on the initial data of (4.1)–(4.3) formulated above, we suppose that **(H)**: there are a number $r > 0$ and a mapping $\vartheta\colon \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ locally Lipschitz continuous and uniformly bounded on bounded sets such that for all $\bar{v} \in N(\theta_{\bar{u}}(\bar{x}); \Theta)$ and $x \in \theta_u^{-1}(\Theta)$ with $u := \bar{u} + \vartheta(x - \bar{x}, x, \bar{x}, \bar{u})$ there exists a vector $v \in N(\theta_u(x); \Theta)$ satisfying $\|v - \bar{v}\| \le r\|x - \bar{x}\|$. This assumption is rather technical, while it automatically holds in the polyhedral setting of [13] as well as in simple nonconvex settings; see [26] for more details, discussions, and examples.

We now present by following [26] necessary optimality conditions for local minimizers of problem (4.1)–(4.3) in both *extended Euler-Lagrange* and *Hamiltonian forms* with significantly new elements. Our approach is based again on the method of discrete approximations, which allows us to investigate two new types of local minimizers for controlled sweeping processes that surely cover conventional ones.

Definition 4.1. (local minimizers for controlled sweeping processes). Under the assumptions imposed above in this section, fix a feasible pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ to problem (4.1)–(4.3).

(i) We say that $(\bar{x}(\cdot), \bar{u}(\cdot))$ be a LOCAL $W^{1,2} \times W^{1,2}$-MINIMIZER for this problem if $\bar{x}(\cdot) \in W^{1,2}([0,T]; \mathbb{R}^n)$, $\bar{u}(\cdot) \in W^{1,2}([0,T]; \mathbb{R}^m)$, and it holds

$$\begin{aligned} &J[\bar{x}, \bar{u}] \le J[x,u] \ \text{ whenever } \ x(\cdot) \in W^{1,2}([0,T]; \mathbb{R}^n) \\ &\text{and } \ u(\cdot) \in W^{1,2}([0,T]; \mathbb{R}^m) \end{aligned} \tag{4.4}$$

are sufficiently close to $(\bar{x}(\cdot), \bar{u}(\cdot))$ in the norm topology of the $W^{1,2}$ spaces in (4.4).

(ii) In the case where the integrand $\ell(\cdot)$ in (4.1) does not depend on $\dot{u}$, we say that $(\bar{x}(\cdot), \bar{u}(\cdot))$ be a LOCAL $W^{1,2} \times C$-MINIMIZER for the optimal control problem under consideration if $\bar{x}(\cdot) \in W^{1,2}([0,T]; \mathbb{R}^n)$, $\bar{u}(\cdot) \in C([0,T]; \mathbb{R}^m)$, and it holds

$$J[\bar{x}, \bar{u}] \le J[x,u] \ \text{ whenever } \ x(\cdot) \in W^{1,2}([0,T]; \mathbb{R}^n) \ \text{ and } \ u(\cdot) \in C([0,T]; \mathbb{R}^m) \tag{4.5}$$

are sufficiently close to $(\bar{x}(\cdot), \bar{u}(\cdot))$ in the norm topology of the corresponding spaces in (4.5).

Note that the notion of local minimizers from Definition 4.1(i) can be viewed as a counterpart for the sweeping control problem (4.1)–(4.3) of the intermediate local minimizer notion for general differential inclusions taken from Definition 3.1 with respect to both (x, u) variables. The notion in Definition 4.1(ii) is specific for controlled sweeping processes and first appeared in [26].

The following major result can be treated as a far-going extension of the Euler-Lagrange conditions in the form specific for the controlled sweeping process under consideration with new phenomena established in this framework. Its proof is given in [26, Theorem 4.3] by using the method of discrete approximations and employing the generalized second-order calculus rule given in (2.9). For simplicity we present the theorem in the case where $g(x) := 0$ and $h(x) := x$ for all x.

Theorem 4.1. (extended Euler-Lagrange formalism for sweeping processes with controlled moving sets). *Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be a local minimizer for problem (4.1)–(4.3). Then we have:*

(i) *If $(\bar{x}(\cdot), \bar{u}(\cdot))$ is a local $W^{1,2} \times W^{1,2}$-minimizer, then there exist a multiplier $\lambda \ge 0$, an adjoint arc $p(\cdot) = (p^x, p^u) \in W^{1,2}([0,T]; \mathbb{R}^n \times \mathbb{R}^m)$, a vector measure $\gamma \in C^*([0,T]; \mathbb{R}^d)$, as well as pairs $(w^x(\cdot), w^u(\cdot)) \in L^2([0,T]; \mathbb{R}^n \times \mathbb{R}^m)$ and $(v^x(\cdot), v^u(\cdot)) \in L^\infty([0,T]; \mathbb{R}^n \times \mathbb{R}^m)$ with*

$$\big(w^x(t), w^u(t), v^x(t), v^u(t)\big) \in \operatorname{co} \partial \ell\big(\bar{x}(t), \bar{u}(t), \dot{\bar{x}}(t), \dot{\bar{u}}(t)\big) \quad a.e. \quad t \in [0,T] \tag{4.6}$$

satisfying the following collection of necessary optimality conditions:

• Primal-dual dynamic relationships*:*

$$\dot{p}(t) = \lambda w(t) + \begin{bmatrix} \nabla^2_{xx}\langle \eta(t), \theta \rangle \big(\bar{x}(t), \bar{u}(t)\big) \\ \nabla^2_{xw}\langle \eta(t), \theta \rangle \big(\bar{x}(t), \bar{u}(t)\big) \end{bmatrix} \big(-\lambda v^x(t) + q^x(t)\big) \quad a.e. \quad t \in [0,T], \tag{4.7}$$

$$q^u(t) = \lambda v^u(t) \quad a.e. \quad t \in [0,T], \tag{4.8}$$

where $\eta(\cdot) \in L^2([0,T]; \mathbb{R}^s)$ is a uniquely defined vector function determined by the representation

$$\dot{\bar{x}}(t) = -\nabla_x \theta\big(\bar{x}(t), \bar{u}(t)\big)^* \eta(t) \quad a.e. \quad t \in [0,T] \tag{4.9}$$

with $\eta(t) \in N(\theta(\bar{x}(t), \bar{u}(t)); \Theta)$, and where $q \colon [0,T] \to \mathbb{R}^n \times \mathbb{R}^m$ is a function of bounded variation on $[0,T]$ with its left-continuous representative given, for all $t \in [0,T]$ except at most a countable subset, by

$$q(t) = p(t) - \int_{[t,T]} \nabla \theta\big(\bar{x}(\tau), \bar{u}(\tau)\big)^* d\gamma(\tau). \tag{4.10}$$

• Measured coderivative condition*: Considering the t-dependent outer limit*

$$\begin{aligned} \operatorname*{Lim\,sup}_{|B| \to 0} \frac{\gamma(B)}{|B|}(t) := \Big\{ y \in \mathbb{R}^s \Big| \; \exists \; sequence \;\; B_k \subset [0,1] \\ with \;\; t \in I\!B_k, \; |B_k| \to 0, \; \frac{\gamma(B_k)}{|B_k|} \to y \Big\} \end{aligned} \tag{4.11}$$

over Borel subsets $B \subset [0,1]$ with the Lebesgue measure $|B|$, for a.e. $t \in [0,T]$ we have

$$\begin{aligned} D^* N_\Theta\big(\theta(\bar{x}(t), \bar{u}(t)), \eta(t)\big)\big(\nabla_x \theta(\bar{x}(t), \bar{u}(t))(q^x(t) - \lambda v^x(t))\big) \cap \\ \cap \operatorname*{Lim\,sup}_{|B| \to 0} \frac{\gamma(B)}{|B|}(t) \neq \emptyset. \end{aligned} \tag{4.12}$$

• Transversality condition *at the right endpoint:*

$$-\big(p^x(T), p^u(T)\big) \in \lambda\big(\partial \varphi(\bar{x}(T)), 0\big) + \nabla \theta\big(\bar{x}(T), \bar{u}(T)\big) N_\Theta\big((\bar{x}(T), \bar{u}(T)\big). \tag{4.13}$$

• MEASURE NONATOMICITY CONDITION: *Whenever* $t \in [0,T)$ *with* $\theta(\bar{x}(t),\bar{u}(t)) \in \operatorname{int}\Theta$ *there is a neighborhood* V_t *of* t *in* $[0,T]$ *such that* $\gamma(V) = 0$ *for any Borel subset* V *of* V_t.

• NONTRIVIALITY CONDITION:

$$\lambda + \sup_{t\in[0,T]} \|p(t)\| + \|\gamma\| \neq 0 \quad \text{with} \quad \|\gamma\| := \sup_{\|x\|_{C([0,T])}=1} \int_{[0,T]} x(s)d\gamma. \tag{4.14}$$

(ii) *If* $(\bar{x}(\cdot),\bar{u}(\cdot))$ *is a local* $W^{1,2}\times C$*-minimizer, then all the conditions* (4.7)–(4.14) *in* (i) *hold with the replacement of the quadruple* $(w^x(\cdot),w^u(\cdot),v^x(\cdot),v^u(\cdot))$ *in* (4.6) *by the triple* $(w^x(\cdot),w^u(\cdot),v^x(\cdot)) \in L^2([0,T];\mathbb{R}^n)\times L^2([0,T];\mathbb{R}^m)\times$

$L^\infty([0,T];\mathbb{R}^n)$ *satisfying the inclusion*

$$\big(w^x(t),w^u(t),v^x(t)\big) \in \operatorname{co}\partial\ell\big(\bar{x}(t),\bar{u}(t),\dot{\bar{x}}(t)\big) \quad \text{a.e.} \quad t\in[0,T].$$

Observe that the extended Euler-Lagrange optimality conditions of Theorem 4.1, as well as the previous results from [11] and [13] obtained for polyhedral sweeping control problems with movings sets defined in (1.7) and (1.8), do *not* contain a *maximization condition* like in the Pontryagin maximum principle for standard control problems governed by the ODE systems (1.2) with smooth dynamics and also like for Lipschitzian differential inclusions as given in (3.6) of Theorem 3.1. Necessary optimality conditions in sweeping control theory containing the maximization of the corresponding Hamiltonian were first obtained in [4] for (global) optimal solutions to a sweeping process of another type with an *uncontrolled* strictly smooth and convex set $C(t) \equiv C$ having nonempty interior and control functions that linearly enter an adjacent ordinary differential equation. Quite recent results in this direction were derived in the case of the sweeping control system given by

$$\dot{x}(t) \in g\big(x(t),u(t)\big) - N\big(x(t);C(t)\big) \quad \text{a.e.} \quad t\in[0,T], \tag{4.15}$$

where measurable controls $u(t)$ enter the additive smooth term g while the uncontrolled moving set $C(t)$ is compact, convex or mildly nonconvex, and possesses a C^3-smooth boundary for each $t\in[0,T]$ along with some additional assumptions. The very recent paper [17] also deals with a sweeping control system of type (4.15) and establishes necessary optimality conditions for global minimizers involving the maximization of the usual Hamiltonian function, provided that the convex and compact set $C(t)\equiv C$ of nonempty interior given by $C := \{x\in\mathbb{R}^n \mid \psi(x)\le 0\}$ via a C^2-smooth function ψ under other assumptions, which are partly differ from [3]. Certain penalty-type approximation methods developed in [3], [4], and [17] are different from each other, significantly based on the *smoothness* of *uncontrolled moving sets* while being sharply disparate from the method of discrete approximations discussed above.

Let us now present necessary optimality conditions in the modified *Hamiltonian form*, which is complemented to Theorem 4.1 and contains a maximization condition of the new type appeared in our papers [26, 40] as the first version of the maximum principle for sweeping process with controlled moving sets. To proceed, consider problem (4.1)–(4.3) with $\Theta = \mathbb{R}^d_-$ being a nonpositive orthant in $\mathbb{R}^d$. This result is based on the generalized second-order chain rule given in

Section 2 and the precise calculation of the second-order construction $D^*N_{\mathbb{R}^d_-}$ taken from [41]. When $\Theta = \mathbb{R}^d_-$ in (4.3), define the active index set

$$I(x,u) := \big\{ i \in \{1,\ldots,d\} \big|\ \theta_i(x,u) = 0 \big\}$$

and observe that under the standing surjectivity assumption on the partial Jacobian operator $\nabla_x\theta$ for each $v \in -N(x;C(u))$ there is a unique collection $\{\alpha_i\}_{i\in I(x,u)}$ with $\alpha_i \le 0$ and $v = \sum_{i\in I(x,u)} \alpha_i [\nabla_x\theta(x,u)]_i$. Given $\nu \in \mathbb{R}^d$, define further the vector $[\nu,v] \in \mathbb{R}^n$ by

$$[\nu,v] := \sum_{i\in I(x,u)} \nu_i\alpha_i \big[\nabla_x\theta(x,u)\big]_i$$

and introduce the *new Hamiltonian function* by

$$H_\nu(x,u,p) := \sup\big\{ \big\langle [\nu,v],p \big\rangle \big|\ v \in -N\big(x;C(u)\big)\big\}, \qquad (x,u,p) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n. \tag{4.16}$$

Now we are ready to present a new form of the maximum principle, along with the extended Euler-Lagrange conditions, in sweeping optimal control with controlled moving sets.

Theorem 4.2. (new maximum principle in sweeping optimal control). *Consider the optimal control problem* (4.1)–(4.3) *in the frameworks of Theorem* 4.1 *with* $\Theta = \mathbb{R}^d_-$. *Then, in addition to all the conditions in assertions* (i) *and* (ii) *of Theorem* 4.1*, we have the maximization condition*

$$\big\langle \big[\nu(t),\dot{\bar{x}}(t)\big], q^x(t) - \lambda v^x(t)\big\rangle = H_{\nu(t)}\big(\bar{x}(t),\bar{u}(t),q^x(t)-\lambda v^x(t)\big) = 0 \quad a.e.\ \ t \in [0,T].$$

holds with a measurable vector function $\nu\colon [0,T] \to \mathbb{R}^d$ *satisfying the inclusion*

$$\nu(t) \in D^*N_{\mathbb{R}^d_-}\big(\theta(\bar{x}(t),\bar{u}(t)),\mu(t)\big)\times$$

$$\times\big(\nabla_x\theta(\bar{x}(t),\bar{u}(t))(q^x(t)-\lambda v^x(t))\big) \cap \operatorname*{Lim\,sup}_{|B|\to 0} \frac{\gamma(B)}{|B|}(t)$$

for a.e. $t \in [0,T]$*, where* Lim sup *is defined in* (4.11).

Furthermore, it is shown in [26] that a conventional form of the maximum principle with replacing the new Hamiltonian function (4.16) by the standard one

$$H(x,u,p) := \sup\big\{\langle p,v\rangle \big|\ v \in -N\big(x;C(u)\big)\big\}, \quad (x,u,p) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n.$$

fails as a necessary optimality condition even for global minimizers of (4.1)–(4.3).

The obtained necessary optimality conditions for the sweeping control problem (4.1)–(4.3) and its specifications admit various *applications* to practical models. We refer the reader to [13] for applications of the results established for controlled polyhedral moving sets of type (1.7), (1.8) to *quasistatic elastoplasticity models with hardening* the sweeping dynamics of which is described in [21]. Other applications to nonpolyhedral models of *elastoplasticity* and *hysteresis* can be found in [26], where the necessary optimality conditions from Theorems 4.1 and 4.2 are used for complete calculations of optimal solutions in the controlled hysteresis model which dynamics is described in the sweeping form (4.2) in [2]. Further applications in this direction, including hysteresis models that arise in problems of contact and nonsmooth mechanics [1, 45], require elaborations of the results presented above; in particular, a serious relaxation of assumption **(H)** imposed at the beginning of this section.

5. Sweeping Processes with Smooth Controlled Perturbations

As mentioned in Section 1, another interesting class of controlled sweeping processes involves control actions not only in the moving set, but also in additive perturbations. Such problems were first formulated and studied in [5, 6], where the controlled sweeping dynamics was described by polyhedral systems of type (1.11), (1.10). The discrete approximation result and necessary optimality conditions obtained in [5, 6] were then applied in [6] to optimal control of the *crowd motion model in a corridor* the dynamics of which was described in [29, 49] as a sweeping process with a polyhedral moving set. Since the polyhedral description does not cover a more realistic *planar crowd motion model* from [29, 49], in the recent paper [7] we considered a more advanced and challenging optimal control problem for the sweeping process (1.10) with the *nonconvex* (and hence nonpolyhedral) moving set

$$\begin{cases} C(t) := C + u(t) = \bigcap_{i=1}^m C_i + u(t) \ \text{ with} \\ C_i := \big\{x \in \mathbb{R}^n \big|\ \vartheta_i(x) \geq 0\big\} \ \text{ for all } \ i = 1, \ldots, m, \end{cases} \tag{5.1}$$

defined via some convex and C^2-smooth functions $\vartheta_i \colon \mathbb{R}^n \to \mathbb{R}$. In this case we replace the convex normal cone (1.4) by the nonconvex one from (2.1). Furthermore, the assumptions imposed in [49] to describe the dynamics of the planar crowd motion model ensure that the set $C(t)$ in (5.1) is *uniformly prox-regular* (as in [7, Definition 2.1]), which is the notion that has been well understood in variational analysis and geometric measure theory; see [16] for more details and references.

The following optimal control problem is investigated in our newly published paper [7], where the obtained results cover the previous ones from [5, 6] for polyhedral moving sets:

$$\text{minimize} \ \ J[x,u,v] := \varphi\big(x(T)\big) + \int_0^T \ell\big(x(t), u(t), v(t), \dot{x}(t), \dot{u}(t), \dot{v}(t)\big)\, dt \tag{5.2}$$

over *smooth* control pairs $u(\cdot) \in W^{1,2}([0,T];\mathbb{R}^n)$, $v(\cdot) \in W^{1,2}([0,T];\mathbb{R}^d)$ and the corresponding trajectories $x(\cdot) \in W^{1,2}([0,T];\mathbb{R}^n)$ of the perturbed sweeping differential inclusion (1.11) with the controlled moving set (5.1). Suppose that the time final time $T > 0$ and the initial vector $x_0 \in \mathbb{R}^n$ are fixed. Besides the dynamic constraints (1.11), we imposed the pointwise/hard constraints on u-controls defined by

$$r_1 \leq \|u(t)\| \leq r_2 \ \text{ for all } \ t \in [0,T] \tag{5.3}$$

with the given constraint bounds $0 < r_1 \leq r_2$ separated from zero; this is largely due the physical sense of the introduced model. It follows (1.11) and the second line in the normal cone definition (2.1) that the formulated problem automatically contains the *pointwise mixed state-control constraints*

$$\vartheta_i\big(x(t) - u(t)\big) \geq 0 \ \text{ for all } \ t \in [0,T] \ \text{ and } \ i = 1, \ldots, m, \tag{5.4}$$

which are *irregular* while being the most difficult in optimal control theory.

We are able to deal with this problem by using the method of *discrete approximations*. The aforementioned paper [7] contains, under some technical assumptions in addition to the main ones presented above, *existence theorems* and verifiable *necessary optimality conditions* for intermediate local minimizers of the sweeping optimal control problem formulated in (1.11), (5.2)–(5.4). Besides the discrete approximation machinery, our developments in [7] are strongly based on the constructions and calculus rules of *second-order* variational analysis and generalized differentiation that are briefly discussed in Section 2.

The major application (and motivation) of the necessary optimality conditions obtained in [7] are given to the planar crowd motion model in the fresh paper [8]. Let us discuss this model next.

6. Controlled Crowd Motion Model

The original developments on the crowd motion model concerns local interactions between participants in order to describe the dynamics of pedestrian traffic. Nowadays this model is successfully used to study more general classes of problems in socioeconomics, operations research, etc.

The so-called *microscopic form* of the crowd motion model is based on the following *two postulates*. Firstly, each individual has a *spontaneous* velocity that he/she intends to implement in the absence of other participants. However, in reality the *actual* velocity must be taken into account. The latter one is incorporated via a projection of the spontaneous velocity into the set of admissible/feasible velocities, i.e., those which do not violate certain *nonoverlapping constraints*. A mathematical description of the *uncontrolled* microscopic crowd motion model was given in [29, 49] as a *sweeping process*, and then it was used these and other papers for numerical simulations and various applications.

In [6] we formulated an *optimal control version* of the crowed motion model in a *corridor* and developed efficient procedure to solve it on the basic of necessary optimality conditions obtained in [5, 6] for a sweeping process with polyhedral moving set. In contrast to the corridor model, the major overlapping condition in the more realistic *planar* crowd motion model is not polyhedral anymore while being represented in the following form of (5.1):

$$\big\{x \in \mathbb{R}^{2n}\big|\ D_{ij}(x) \ge 0 \ \text{ for all } \ i \ne j\big\},$$

where $D_{ij}(x) := \|x_i - x_j\| - 2R$ is the signed distance between the disks i and j of the same radius R identified with $n \ge 2$ participants on the plane. The corresponding optimal control problem formulated and investigated in [8] is described via the sweeping dynamics as follows: minimize the cost functional of type (5.2) over the constrained controlled sweeping process

$$\begin{cases} -\dot{x}(t) \in N\big(x(t); C(t)\big) + g\big(x(t), v(t)\big) & \text{for a.e. } \ t \in [0,T], \\ C(t) := C + \bar{u}(t), \ \|\bar{u}(t)\| = r \in [r_1, r_2] \ \text{ on } \ [0,T], \ x(0) = x_0 \in C(0), \end{cases}$$

where the initial data and constraints are given by

$$g\big(x(t), v(t)\big) := \big(s_1 v_1(t)\cos\theta_1(t), s_1 v_1(t)\sin\theta_1(t), \ldots,$$

$$\ldots, s_n v_n(t)\cos\theta_n(t), s_n v_n(t)\sin\theta_n(t)\big),$$

$$\bar{u}_{i+1}(t) = \bar{u}_i(t) := \left(\frac{r}{\sqrt{2n}}, \frac{r}{\sqrt{2n}}\right), \quad i = 1, \ldots, n-1,$$

$$C := \left\{x = (x_1, \ldots, x_n) \in \mathbb{R}^{2n} \middle|\ \vartheta_{ij}(x) \geq 0 \text{ for all } i \neq j \text{ as } i, j = 1, \ldots, n\right\}$$

with the functions $\vartheta_{ij}(x) := D_{ij}(x) = \|x_i - x_j\| - 2R$, and with

$$x(t) - \bar{u}(t) \in C \text{ for all } t \in [0, T].$$

This model belongs to optimal control theory for sweeping processes governed by *prox-regular moving sets* which was discussed in Section 5. Applying the necessary optimality conditions for such problems developed in [7] allowed us to obtain in [8] a complete solution to this model in the case of lower numbers of participants and also to establish efficient relationships to determine optimal parameters in the general crowd model setting with finitely many participants. On the other hand, further algorithmic developments are needed in the case of many participants in crowd motion modeling.

7. Sweeping Processes with Constrained Discontinuous Controls

Yet another class of optimal control problems for controlled sweeping processes have been studied in our new papers [14, 15] with applications to some practical models in robotics and traffic flow dynamics. The description of this class of problems is as follows:

$$\text{minimize} \quad J[x, u] := \varphi\big(x(T)\big) \tag{7.1}$$

over pairs $(x(\cdot), u(\cdot))$ of *measurable* controls $u(t)$ and absolutely continuous trajectories $x(t)$ on the fixed time interval $[0, T]$ satisfying the controlled sweeping differential inclusion

$$\dot{x}(t) \in -N\big(x(t); C\big) + g\big(x(t), u(t)\big) \text{ a.e. } t \in [0, T],\ x(0) := x_0 \in C \subset \mathbb{R}^n, \tag{7.2}$$

subject to the pointwise *control constraints* given by

$$u(t) \in U \subset \mathbb{R}^d \text{ a.e. } t \in [0, T]. \tag{7.3}$$

The set C in (7.2) is a convex polyhedron given by

$$C := \bigcap_{i=1}^{m} C_i \text{ with } C_i := \left\{x \in \mathbb{R}^n \middle|\ \langle x_i^*, x \rangle \leq c_i\right\}. \tag{7.4}$$

Note that the main difference between problem (7.1)–(7.4) and that considered in [5, 6] (see Section 1) is that controls are not $W^{1,2}$-smooth while being *discontinuous* and *constrained* by (7.3). On the hand, the set C in (7.2) given by (7.4) does not depend on control and time variables as in (1.10).

Problems of type (7.1)–(7.4) have been already considered in the literature, but only from viewpoints of the existence of solutions and relaxation stability; see [9, 19, 48]. Deriving necessary optimality conditions is a highly challenging issue with even more increasing complications in comparison with those mentioned above for other controlled sweeping models. Our approach is based again on the method of discrete approximations, which requires a serious modification in this case.

We say that a feasible pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ for (7.1)–(7.4) is a $W^{1,2} \times L^2$-*local minimizer* for this problem if there is $\varepsilon > 0$ such that $J[\bar{x}, \bar{u}] \le J[x, u]$ for all the feasible pairs $(x(\cdot), u(\cdot))$ satisfying

$$\int_0^T \big(\| \cdot x(t) - \dot{\bar{x}}(t)\|^2 + \|u(t) - \bar{u}(t)\|^2\big)\, dt < \varepsilon.$$

For simplicity we assume the following while referring the reader to [14] for more general settings:

(H1) The control set U is compact and convex in $\mathbb{R}^d$, and the image set $g(x, U)$ is convex in $\mathbb{R}^n$.
(H2) The cost function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ in (5.2) is C^1-smooth around $\bar{x}(T)$.
(H3) The perturbation mapping $g\colon \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$ in (7.2) is C^1-smooth around $(\bar{x}(\cdot), \bar{u}(\cdot))$ and satisfies the sublinear growth condition

$$\|g(x,u)\| \le \beta\big(1 + \|x\|\big) \ \text{ for all } \ u \in U \ \text{ with some } \ \beta > 0.$$

(H4) The vertices x_i^* of (7.4) satisfy the linear independence constraint qualification

$$\Big[\sum_{i \in I(\bar{x})} \alpha_i x_i^* = 0,\ \alpha_i \in \mathbb{R}\Big] \Longrightarrow \big[\alpha_i = 0 \ \text{ for all } \ i \in I(\bar{x})\big\}$$

along the trajectory $\bar{x} = \bar{x}(t)$ as $t \in [0, T]$, where $I(\bar{x}) := \{i \in \{1, \ldots, m\} \mid \langle x_i^*, \bar{x}\rangle = c_i\}$.

Among other results of [14], we present the necessary optimality conditions obtained by using discrete approximations together with advanced tools of variational analysis and generalized differentiation.

Theorem 7.1. (necessary optimality conditions for constrained sweeping processes). *Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be a $W^{1,2} \times L^2$-local minimizer for problem* (7.1)–(7.4) *under the assumptions in* (H1)–(H4)*, where $\bar{u}(\cdot)$ is of bounded variation (BV) with a right continuous representative on $[0, T]$. Then there exist a multiplier $\lambda \ge 0$, a measure $\gamma = (\gamma_1, \ldots, \gamma_n) \in C^*([0, T]; \mathbb{R}^n)$ as well as adjoint arcs $p(\cdot) \in W^{1,2}([0, T]; \mathbb{R}^n)$ and $q(\cdot) \in BV([0, T]; \mathbb{R}^n)$ such that $\lambda + \|q(t)\|_{L^\infty} + \|p(T)\| > 0$ and the following conditions are satisfied:*

• Primal velocity representation:

$$-\dot{\bar{x}}(t) = \sum_{i=1}^m \eta_i(t) x_i^* - g\big(\bar{x}(t), \bar{u}(t)\big) \ \textit{ for a.e. } \ t \in [0, T], \tag{7.5}$$

where $\eta^i(\cdot) \in L^2([0, T]; \mathbb{R}_+)$ being uniquely determined by (7.5) *and well defined at $t = T$.*

• Adjoint system:

$$\dot{p}(t) = -\nabla_x g\big(\bar{x}(t), \bar{u}(t)\big)^* q(t) \ \textit{ for a.e. } \ t \in [0, T],$$

where the dual arcs $q(\cdot)$ and $p(\cdot)$ are precisely connected by the equation

$$q(t) = p(t) - \int_{(t,T]} d\gamma(\tau)$$

that holds for all $t \in [0,T]$ except at most a countable subset.

• Maximization condition*:*

$$\big\langle \psi(t), \bar{u}(t) \big\rangle = \max\big\{ \big\langle \psi(t), u \big\rangle \big|\ u \in U \big\} \quad \textit{with}$$

$$\psi(t) := \nabla_u g\big(\bar{x}(t), \bar{u}(t)\big)^* q(t) \quad \textit{for a.e.} \quad t \in [0,T].$$

• Complementarity conditions:

$$\big\langle x_i^*, \bar{x}(t) \big\rangle < c_i \Longrightarrow \eta_i(t) = 0 \quad \textit{and} \quad \eta_i(t) > 0 \Longrightarrow \big\langle x_i^*, q(t) \big\rangle = c_i$$

for a.e. $t \in [0,T]$ including $t = T$ and for all $i = 1, \ldots, m$.

• Right endpoint transversality conditions*:*

$$-p(T) = \lambda \nabla \varphi\big(\bar{x}(T)\big) + \sum_{i \in I(\bar{x}(T))} \eta_i(T) x_i^* \quad \textit{with} \quad \sum_{i \in I(\bar{x}(T))} \eta_i(T) x_i^* \in N\big(\bar{x}(T); C\big).$$

• Measure nonatomicity condition: *If $t \in [0,T)$ and $\langle x_i^*, \bar{x}(t) \rangle < c_i$ for all $i = 1, \ldots, m$, then there is a neighborhood V_t of t in $[0,T]$ such that $\gamma(V) = 0$ for all the Borel subsets V of V_t.*

The subsequent paper [15] contains applications of the results from Theorem 7.1 to two practical models, which can be written in the form of the constrained controlled sweeping process (7.1)–(7.4). The first model is an optimal control version of the *mobile robot model with obstacles* the dynamics of which was described as a sweeping process in [22]. The second one is a continuous-time, deterministic, and optimal control version of the *pedestrian traffic flow model through a doorway* for which a stochastic, discrete-time, and simulation (uncontrolled) counterpart was originated in [28].

The application of Theorem 7.1 leads us in [15] to complete calculations of optimal solutions for both models in several important settings, but many unsolved issues remain in numerical implementations.

Major *research goals* concerning these models include developing efficient *numerical algorithms* to solve optimal control problems for them with *large numbers of participants*. It could be done, in particular, by using an appropriate discretization and employing numerical algorithms of finite-dimensional optimization to the discrete-time problems obtained in this way. We also believe that the developed necessary optimality conditions for the perturbed sweeping processes would be useful to investigate other practical model with a sweeping process dynamics that frequently appear in various branches of mechanics, engineering, economics, behavioral sciences, etc.

Regarding theoretical developments on constrained controlled sweeping processes of type (7.1)–(7.4), the main *open questions* include the following issues:

• Derive necessary optimality conditions for *nonpolyhedral* descriptions of the sweeping set C. Prox-regularity considered in Section 5 is an appropriate substitution of polyhedrality to begin with.

• Investigate sweeping control models with *two groups of controls* similarly to those considered above, but now with *discontinuous* controls in perturbations and subject to control *constraints*. Involving *controlled moving sets* in the constrained models is an important factor for further applications.

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