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GLOBAL BIFURCATION FROM INFINITY IN NONLINEAR ONE DIMENSIONAL DIRAC PROBLEMS

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Abstract. In this paper, we study the asymptotic bifurcation in nonlinear eigenvalue problems for the one-dimensional Dirac equation. For an asymptotically linear nonlinearity, we shows the existence of two families of unbounded continua of solutions emanating from asymptotically bifurcation points and having usual nodal properties near these points.

1. Introduction

We consider the nonlinear Dirac problem

$$\ell w(x) \equiv B w'(x) - P(x) w(x) = \lambda w(x) + g(x, w(x), \lambda), \;\; 0 < x < \pi, \tag{1.1}$$

$$U_1(w) := (\sin\alpha,\, \cos\alpha)\, w(0) = v(0)\cos\alpha + u(0)\sin\alpha = 0, \tag{1.2}$$

$$U_2(w) := (\sin\beta,\, \cos\beta)\, w(\pi) = v(\pi)\cos\beta + u(\pi)\sin\beta = 0, \tag{1.3}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\; P(x) = \begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix},\; w(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix},$$

where $\lambda \in \mathbb{R}$ is a spectral parameter, $p(x), r(x) \in C([0,\pi];\mathbb{R})$, α and β are real constants such that $0 \le \alpha, \beta < \pi$, and the nonlinear term $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in C\left([0,\pi] \times \mathbb{R}^2 \times \mathbb{R}\,;\, \mathbb{R}^2\right)$ satisfies the condition:

$$g(x, w, \lambda) = o(|w|) \;\text{ as }\; |w| \to \infty, \tag{1.4}$$

uniformly with respect to $x \in [0,\pi]$ and $\lambda \in \Lambda$ for every compact interval $\Lambda \subset \mathbb{R}$ (here $|\cdot|$ denotes a norm in $\mathbb{R}^2$).

It is obvious that Eq. (1.1) is equivalent to the system of two ordinary differential equations of first order

$$\begin{aligned} v'(x) - p(x)u(x) &= \lambda u(x) + g_1(x, u(x), v(x), \lambda), \\ u'(x) + r(x)v(x) &= -\lambda v(x) - g_2(x, u(x), v(x), \lambda). \end{aligned} \tag{1.5}$$

In studying the global bifurcation of solutions from zero and infinity of nonlinear eigenvalue problems for ordinary differential equations, the nodal properties of the solutions is played an important role for detailed analysis of the structure

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and behavior of continua of solutions. The oscillation properties of the eigenfunctions of the ordinary differential operators were investigated in [1-3, 6, 7, 9, 11-16, 18-21, 29]). For the first time, in a recent paper [3], the oscillatory properties of eigenvector-functions of the one-dimensional Dirac problem (1.1)-(1.3) with $g \equiv 0$ were studied, where the number of zeros of the components of eigenvector-functions were found.

If the nonlinear term g satisfies the condition

$$g(x, w, \lambda) = o(|w|) \text{ as } |w| \to 0, \tag{1.6}$$

uniformly with respect to $x \in [0, \pi]$ and $\lambda \in \Lambda$ for every compact interval $\Lambda \subset \mathbb{R}$, then we can consider bifurcation from zero problem. This problem for the Sturm-Liouville equation has been considered in [23] where the existence of two families of unbounded continua of solutions, corresponding to the usual nodal properties and emanating from the bifurcation points corresponding to the eigenvalues of linear problem are proved. Further, nonlinearizable Sturm-Liouville problems are studied in [8, 10, 25, 27] and the existence of two families of unbounded components of the solutions set, having usual nodal properties and emanating from bifurcation intervals of the line of trivial solutions are shown. The similar results for nonlinear eigenvalue problems for ordinary differential equations of fourth order were obtained in the papers [1, 2, 22].

If condition (1.4) hold, then we can consider bifurcation from infinity problem (or asymptotically bifurcation problem). Bifurcation from infinity problems for Sturm-Liouville equation have been investigated in the works [10, 24, 25, 28, 30]. In these papers the authors show the existence of two families of unbounded continua of solutions emanating from asymptotically bifurcation points and intervals, and having the usual nodal properties in near of these points and intervals. It should be noted that this problem for the nonlinear eigenvalue problems for ordinary differential equations of fourth order, for the first time, completely investigated in recent paper [3].

In [5] using the oscillation properties of the linear problem (1.1)-(1.3) with $g = 0$ obtained in [4] and the global bifurcation technique from [23], the bifurcation from zero problems for nonlinear Dirac equation was studied.

The purpose of this paper, is to study the global bifurcation of solutions from infinity for asymptotically linear Dirac problems (1.1)-(1.3).

2. Preliminaries

It is known (see [19, Ch. I, § 10, 11]) that the linear one dimensional Dirac eigenvalue problem

$$\begin{aligned} &\ell w(x) = \lambda w(x), \ 0 < x < \pi, \\ &U(w) = \begin{pmatrix} U_1(w) \\ U_2(w) \end{pmatrix} = 0, \end{aligned} \tag{2.1}$$

has a sequence $\{\lambda_k\}_{k=1}^{\infty}$ real and simple eigenvalues that take values from $-\infty$ to $+\infty$ and can be numbered in increasing order.

For each fixed $\lambda \in \mathbb{C}$ there exists a unique solution $w(x,\lambda) = \begin{pmatrix} u(x,\lambda) \\ v(x,\lambda) \end{pmatrix}$ of equation

$$\ell w(x) = \lambda w(x),\ 0 < x < \pi,$$

satisfying the initial condition

$$u(0,\lambda) = \cos\alpha, \quad v(0,\lambda) = -\sin\alpha\,; \tag{2.2}$$

moreover, for each fixed $x \in [0,\,\pi]$ the functions $u(x,\lambda)$ and $v(x,\lambda)$ are entire functions of the argument λ (see [4] and [19, Ch. 1, § 1]).

We introduce the Prüfer angular variable $\theta(x,\lambda) = \tan^{-1}(v(x,\lambda)/u(x,\lambda))$ (see [6, Ch. 8, § 3]), or more precisely,

$$\theta(x,\lambda) = \arg\{u(x,\lambda) + iv(x,\lambda)\}. \tag{2.3}$$

Taking account of relation (2.2), we define the initial value as

$$\theta(0,\lambda) = -\,\alpha. \tag{2.4}$$

For the remaining values of x and λ, the function $\theta(x,\,\lambda)$ is given by (2.3) modulo 2π, since the functions $u(x,\lambda)$ and $v(x,\lambda)$ cannot vanish simultaneously. This multiple of 2π is to be fixed so that $\theta(x,\,\lambda)$ satisfies (2.4) and is continuous in x and λ. Since the (x,λ)-region, namely, $0 \le x \le \pi$, $-\infty < \lambda < +\infty$, is simply-connected, this defines $\theta(x,\,\lambda)$ uniquely.

The problem (2.1) has the following oscillation properties.

Theorem 2.1. [4, Theorem 3.1] *The eigenvalues λ_k, $k \in \mathbb{Z}$, of the problem* (2.1) *can be numbered in ascending order on the real axis*

$$\ldots < \lambda_{-k} < \ldots < \lambda_{-1} < \lambda_0 < \lambda_1 < \ldots < \lambda_k < \ldots,$$

so that the corresponding angular function $\theta(x,\lambda_k)$ at $x = \pi$ satisfy the condition

$$\theta(\pi,\lambda_k) = -\beta + k\pi. \tag{2.5}$$

The eigenvector-functions $w_k(x) = w(x,\lambda_k) = \begin{pmatrix} u(x,\lambda_k) \\ v(x,\lambda_k) \end{pmatrix} = \begin{pmatrix} u_k(x) \\ v_k(x) \end{pmatrix}$, $k \in \mathbb{Z}$, have, with a suitable interpretation, the following oscillation properties: if $k > 0$ and $k = 0, \alpha \ge \beta$ (except the cases $\alpha = \beta = 0$ and $\alpha = \beta = \pi/2$), then

$$\begin{pmatrix} s(u_k) \\ s(v_k) \end{pmatrix} = \begin{pmatrix} k - 1 + \chi(\alpha - \pi/2) + \chi(\pi/2 - \beta) \\ k - 1 + \mathrm{sgn}\alpha \end{pmatrix}, \tag{2.6}$$

and if $k < 0$ and $k = 0$, $\alpha < \beta$, then

$$\begin{pmatrix} s(u_k) \\ s(v_k) \end{pmatrix} = \begin{pmatrix} |\,k| - 1 + \chi(\pi/2-\alpha) + \chi(\beta - \pi/2) \\ |\,k| - 1 + \mathrm{sgn}\beta \end{pmatrix}, \tag{2.7}$$

where $s(g)$ the number of zeros of the function $g \in C([0,\pi]\,;\mathbb{R})$ in the interval $(0,\pi)$ and

$$\chi\,(x) = \begin{cases} 0, & if\ x \le 0, \\ 1, & if\ x > 0. \end{cases}$$

Let $E = C\left([0,\pi];\,\mathbb{R}^2\right) \cap \{w\,:\,U(w) = 0\}$ to be the Banach space with the usual norm $||w|| = \max\limits_{x\in[0,\pi]} |u(x)| + \max\limits_{x\in[0,\pi]} |v(x)|$. Let S be the subset of E given by

$$S = \{w \in E :\ |u(x) + |v(x)|\ > 0,\, \forall x \in [0,\pi]\}$$

with metric inherited from E.

For each $w = \begin{pmatrix} u \\ v \end{pmatrix} \in S$ we define $\theta(w, \cdot)$ to be continuous function on $[0, \pi]$ satisfying

$$\theta(w, x) = \arctan \frac{v(x)}{u(x)},\ \theta(w, 0) = -\alpha.$$

By virtue of (2.4) and (2.5) we have

$$\theta(w_k, 0) = -\alpha,\ \theta(w_k, \pi) = -\beta + k\pi,\ \ k \in \mathbb{Z}. \tag{2.8}$$

Let S_k^ν, $k \in \mathbb{Z}$, $\nu \in \{+, -\}$, denote the set of functions $w \in S$ satisfying the following conditions:

(i) $\theta(w, \pi) = -\beta + k\pi$;

(ii) if $k > 0$ or $k = 0, \alpha \geq \beta$ (except the cases $\alpha = \beta = 0$ and $\alpha = \beta = \pi/2$), then for fixed w, as x increases from 0 to π, the function θ cannot tend to a multiple of $\pi/2$ from above, and as x decreases, the function θ cannot tend to a multiple of $\pi/2$ from below; if $k < 0$ or $k = 0, \alpha < \beta$, then for fixed w, as x increases, the function θ cannot tend to a multiple of $\pi/2$ from below, and as x decreases, the function θ cannot tend to a multiple of $\pi/2$ from above;

(iii) the function $\nu u(x)$ is positive in a deleted neighborhood of $x = 0$.

Let $S_k^- = -S_k^+$ and $S_k = S_k^- \cup S_k^+$. By (2.8) it follows Theorem 2.1 and [4, Theorem 2.1] that $w_k \in S_k,\ k \in \mathbb{Z}$, i.e. the sets S_k^-, S_k^+ and S_k are nonempty. Moreover, if $w(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \in S_k$, $k \in \mathbb{Z}$, then the number of zeros of functions $u(x)$ and $v(x)$ are determined by (2.6) and (2.7) respectively and there functions have only nodal zeros in $(0, \pi)$.

From now on ν will denote an element of $\{+, -\}$ that is, either $\nu = +$ or $\nu = -$.

It follows from the definition of the sets S_k^ν, $k \in \mathbb{Z}$, $\nu \in \{+, -\}$, that, there sets are disjoint and open in E. Furthermore, if $w \in \partial S_k^\nu$, then there exists a point $\tau \in [0, \pi]$ such that $|w(\tau)| = 0$, i.e. $u(\tau) = v(\tau) = 0$.

Lemma 2.1. [5, Lemma 2.8] *If* $(\lambda, w) \in \mathbb{R} \times E$ *is a solution of problem* (1.1)-(1.3) *under condition* (1.6) *and* $w \in \partial S_k^\nu$, *then* $w \equiv 0$.

Theorem 2.2. [5, Theorem 3.1] *Suppose that* (1.6) *holds. Then for each integer* k *and each* ν, *there exists a continuum of solutions* C_k^ν *of problem* (1.1)-(1.3) *in* $(\mathbb{R} \times S_k^\nu) \cup \{(\lambda_k, 0)\}$ *which meets* $(\lambda_k, 0)$ *and* ∞ *in* $\mathbb{R} \times E$.

3. Reducing problem (1.1)-(1.3) to an operator equation

As $\lambda = 0$ is not an eigenvalue of the linear problem (2.1), the problem (1.1)-(1.3) can be converted to the equivalent integro-differential equation

$$w(x) = \lambda \int_0^\pi K(x, t) w(t) dt + \int_0^\pi K(x, t) g(t, w(t), \lambda) dt, \tag{3.1}$$

where $K(x, t) = K(x, t, 0)$ is the appropriate Green's matrix (see [19, Ch. 1, formula (13.8)]).

Let

$$Lw(x) = \int_0^\pi K(x,t)w(t)dt, \tag{3.2}$$

$$G(\lambda, w(x)) = \int_0^\pi K(x,t)g(t,w(t),\lambda)dt. \tag{3.3}$$

The Green matrix $K(x,t)$ is continuous in $[0,\pi;0,\pi]$ everywhere except on the diagonal $x=t$, where it has a jump $K(x,x+0)-K(x,x-0)=B$. Then $L: E \to E$ is compact and linear, and all all characteristic values of L are real and simple (which coincide with the eigenvalues of the linear problem (2.1)). The operator G can be represented as a composition of a operator L and the superposition operator $\mathbf{g}(\lambda, w)(x)) = g(x, w(x), \lambda)$. Since $g(x,w,\lambda) \in C\left([0,\pi]\times\mathbb{R}^2\times\mathbb{R}\,;\,\mathbb{R}^2\right)$, then the operator $\mathbf{g}$ maps $\mathbb{R}\times E$ to $C\left([0,\pi];\,\mathbb{R}^2\right)$. Hence the operator $G: \mathbb{R}\times E \to E$ is continuous.

By virtue of (3.1)-(3.3) problem (1.1)-(1.3) can be written in the following equivalent form

$$w = \lambda Lw + G(\lambda, w). \tag{3.4}$$

Lemma 3.1. *The operator $G: \mathbb{R}\times E \to E$ is satisfies the following condition*

$$G(\lambda, w) = o(||w||) \quad \text{as} \quad ||w|| \to \infty, \tag{3.5}$$

uniformly on bounded λ intervals.

Proof. Let $\Lambda \subset \mathbb{R}$ be a bounded interval and fix $\varepsilon > 0$. It follows from (1.4) that there exists $\Delta_\varepsilon > 0$ such that

$$|g(x,w,\lambda)| < \varepsilon|w|,\ x\in[0,\pi], w\in\mathbb{R}^2,\ |w| > \Delta_\varepsilon,\ \lambda\in\Lambda. \tag{3.6}$$

Moreover, since $g \in C\left([0,\pi]\times\mathbb{R}^2\times\mathbb{R}\,;\,\mathbb{R}^2\right)$ it follows that there exists $K > 0$ such that

$$|g(x,w,\lambda)| \le M,\ x\in[0,\pi], w\in\mathbb{R}^2,\ |w| \le \Delta_\varepsilon,\ \lambda\in\Lambda. \tag{3.7}$$

Choosing $\Delta_{1,\varepsilon} > \Delta_\varepsilon$ so large that $\frac{M}{\Delta_{1,\varepsilon}} < \varepsilon$ and choosing $w \in E$ so that $||w|| \ge \Delta_{1,\varepsilon}$. Then, by virtue of (3.6) and (3.7), we have the following estimate:

$$\begin{gathered} ||G(\lambda,w)|| = \\ \max_{x\in[0,\pi]}\left|\int_0^\pi \{K_{11}(x,t)g_1(t,w(t),\lambda) + K_{12}(x,t)g_2(t,w(t),\lambda)\}\,dt\right| + \\ \max_{x\in[0,\pi]}\left|\int_0^\pi \{K_{21}(x,t)g_1(t,w(t),\lambda) + K_{22}(x,t)g_2(t,w(t),\lambda)\}\,dt\right| \le \\ c\left\{\int_{|w(x)|\le\Delta_\varepsilon} |g(t,w(t),\lambda)|\,dt + \int_{|w(x)|\,>\Delta_\varepsilon} |g(t,w(t),\lambda)|dt\right\} \le \\ c\pi\left\{M + \varepsilon||w||\right\} \le c\pi\left\{\varepsilon\,\Delta_{1,\varepsilon} + \varepsilon||w||\right\} < 2c\pi\varepsilon||w||, \end{gathered} \tag{3.8}$$

where $K_{i,j}(x,t)$, $i,\ j=1,\ 2$, are components of the Green matrix $K(x,t)$ and

$$c=\max\left\{\sup_{(x,t)\in\Pi}K_{11}(x,t),\ \sup_{(x,t)\in\Pi}K_{12}(x,t),\ \sup_{(x,t)\in\Pi}K_{21}(x,t),\ \sup_{(x,t)\in\Pi}K_{22}(x,t)\right\},$$

$$\Pi=[0,\pi;0,\pi].$$

Thus it follows from (3.8) that for any $\varepsilon>0$ there exists $\Delta_{1,\varepsilon}>0$ such that

$$\frac{||G(\lambda,w)||}{||w||}<2c\pi\varepsilon \text{ for any } \lambda\in\Lambda \text{ and } ||w||>\Delta_{1,\varepsilon},$$

which implies (3.5). The proof of this lemma is complete.

Let $\bar{B}_r=\{w\in E:||w||\le r\}$.

Lemma 3.2. *The operator*

$$\tilde{G}:(\lambda,w)\to||w||^2\,G\left(\lambda,\frac{w}{||w||^2}\right)$$

is compact.

Proof. Let $\varepsilon>0$ be fixed. Note that $||\frac{w}{||w||^2}||=\frac{1}{||w||}\ge\Delta_{1,\varepsilon}$ if $||w||\le\frac{1}{\Delta_{1,\varepsilon}}$. Then, by virtue of (3.8), we have

$$||\tilde{G}(\lambda,w)||=||w||^2\left\|G\left(\lambda,\frac{w}{||w||^2}\right)\right\|\le2c\pi\varepsilon||w|| \text{ for any } (\lambda,w)\in\Lambda\times\bar{B}_{\Delta_{1,\varepsilon}}.$$

Consequently, the set $\tilde{G}(\Lambda,\bar{B}_{\Delta_{1,\varepsilon}})$ is bounded in E. By (3.1) $\tilde{w}=\tilde{G}(\lambda,w)$ satisfies the equation

$$\ell\tilde{w}(x)=\lambda w(x)+G(x,w(x),\lambda),\ 0<x<\pi.$$

Solving this equation for $\tilde{w}'$ we get uniform bounds for first derivatives of $\tilde{w}$ in $\Lambda\times\bar{B}_{\Delta_{1,\varepsilon}}$. Thus operator $\tilde{G}$ is compact by virtue of Arzelà-Ascoli theorem. The proof of this lemma is complete.

Remark 3.1. We extend $\tilde{G}$ to $w=0$ by setting $\tilde{G}(\lambda,0)=0$. Then $\tilde{G}:\mathbb{R}\times E\to E$ is continuous.

Remark 3.2. If $(\lambda,w)\in\mathbb{R}\times E$ is a solution of problem (1.1)-(1.3) under condition (1.4) and $w\in\partial S_k^\nu$, then it does not follow that $w\equiv0$.

4. Asymptotically global bifurcation of problem (1.1)-(1.3)

In this section we consider problem (1.1)-(1.3) under condition (1.4).

Theorem 4.1 *For each for each integer k and each ν there exists a continua D_k^ν of solutions of problem* (1.1)-(1.3) *which meets (λ_k,∞) and has the following properties:*

(i) *there exists a neighborhood V_k of (λ_k,∞) in $\mathbb{R}\times E$ such that $V_k\cap D_k^\nu\subset\mathbb{R}\times S_k^\nu$;*

(ii) *either D_k^ν meets (λ'_k,∞) respect to the set $\mathbb{R}\times S_{k'}^{\nu'}$ for some $(k',\nu')\ne(k,\nu)$, or D_k^ν meets $(\lambda,0)$ for some $\lambda\in\mathbb{R}$, or D_k^ν has an unbounded projection on $\mathbb{R}$.*

Proof. Recall that the problem (1.1)-(1.3) is equivalent to the problem (3.4). Since the characteristic values of L are real and simple, it follows from [17, Ch. 4, § 3, Theorem 3.1] that for each $k\in\mathbb{Z}$ the point (λ_k,∞) is an asymptotic bifurcation point of problem (1.1)-(1.3) and this point corresponds to the continuous branch D_k of solutions going to infinity.

Let $\mathcal{F}$ denote the set of solutions of problem (1.1)-(1.3) (or (3.4)). Assume that $(\lambda, w) \in \mathcal{F}$ with $||w|| \neq 0$. Setting $\tilde{w} = \frac{w}{||w||^2}$ and dividing (3.4) by $||w||^2$ yields the equation

$$\tilde{w} = \lambda L\tilde{w} + \tilde{G}(\lambda, \tilde{w}). \tag{4.1}$$

Since $G(\lambda, w) = o(||w||)$ as $||w|| \to \infty$ by (3.5) (see Lemma 2.2) uniformly on bounded λ intervals, it follows that

$$\tilde{G}(\lambda, \tilde{w}) = o(||\tilde{w}||) \;\; as \;\; ||\tilde{w}|| \to 0, \tag{4.2}$$

uniformly on bounded λ intervals.

Note that by virtue of (3.5) and (4.2) the transformation

$$(\lambda, w) \to T(\lambda, w) = (\lambda, \tilde{w})$$

turns a "bifurcation at infinity" problem (3.4) into a "bifurcation from zero" problem (4.1).

Let $\tilde{\mathcal{F}} \subset \mathbb{R} \times E$ be the set of nontrivial solutions of problem (4.1). It follows from [23, Theorem 1.3] that for each $k \in \mathbb{Z}$ there exists a continuum $\tilde{D}_k$ of $\tilde{\mathcal{F}}$ such that $(\lambda_k, 0) \in \tilde{\mathcal{F}}$ and either (i) $\tilde{D}_k$ meets infinity in $\mathbb{R} \times E$, or (ii) $\tilde{D}_k$ meets $(\lambda_s, 0)$, where $s \neq k$. But in view of Remark 3.2, the statement of Theorem 2.2 does not hold for problem (4.1). While it follows from the proof of [23, Theorem 2.3] that $\tilde{D}_k$ is decompose into two subcontinua $\tilde{D}_k^+$ and $\tilde{D}_k^-$ which meet $(\lambda_k, 0)$ and have the following properties:

(a) there exists a neighborhood $\tilde{V}_k$ of $(\lambda_k, 0)$ in $\mathbb{R}\times E$ such that $\tilde{V}_k \cap \tilde{D}_k^\nu \subset \mathbb{R}\times S_k^\nu$ for each $\nu \in \{+, -\}$;

(b) either $\tilde{D}_k^\nu \cap \tilde{D}_{k'}^{\nu'} \neq \emptyset$ for some $(k', \nu') \neq (k, \nu)$, or $\tilde{D}_k^\nu$ meet infinity in $\mathbb{R} \times E$.

By construction we have $T(\mathcal{F}) = \tilde{\mathcal{F}}$ and $T(D_k) = \tilde{D}_k$. Let D_k^ν is the inverse image $T^{-1}(\tilde{D}_k^\nu)$ of $\tilde{D}_k^\nu$ under the transformation T. Then we have $D_k = D_k^+ \cup D_k^-$ and from the properties (a) and (b) of the sets $\tilde{D}_k^\nu$, $k \in \mathbb{Z}$, $\nu \in \{+, -\}$, it follows that D_k^ν has by properties (i)-(ii) (the second and third parts of alternative (ii) correspond to the various ways in which $\tilde{D}_k^\nu$ can be unbounded). The proof of this theorem is complete.

Remark 4.1. As can be seen from Theorem 4.1 for bifurcation from infinity, unlike the Theorem 2.2 for bifurcation from zero, it need not be the case that

$$D_k^\nu \subset (\mathbb{R} \times S_k^\nu) \cup \{(\lambda_k, \infty)\}.$$

It should be noted that if conditions (1.4) and (1.6) both hold then we can strengthen Theorems 2.2 and 4.1 in the following way.

Theorem 4.2 *If conditions* (1.4) *and* (1.6) *both hold, then for each integer k and each* ν *we have* $D_k^\nu \subset (\mathbb{R} \times S_k^\nu) \cup \{(\lambda_k, \infty)\}$, *and alternative (i) of Theorem 4.1 cannot hold. Moreover, if* D_k^ν *meets* $(\lambda, 0)$ *for some* $\lambda \in \mathbb{R}$, *then* $\lambda = \lambda_k$. *Similarly, if* C_k^ν *meets* (λ, ∞) *for some* $\lambda \in \mathbb{R}$, *then* $\lambda = \lambda_k$.

Proof. If condition (1.6) holds, then it follows from Lemma 2.1 that $\mathcal{F} \cap (\mathbb{R} \times \partial S_k^\nu) = \emptyset$. Consequently, the sets $\mathcal{F} \cap (\mathbb{R} \times S_k^\nu)$ and $\mathcal{F} \backslash (\mathbb{R} \times S_k^\nu)$ are mutually separated in $\mathbb{R} \times E$ (see, for example, [31]). Thus by virtue of [31, Corollary 26.6] any component of the set $\mathcal{F}$ must be a subset of $\mathcal{F} \cap (\mathbb{R} \times S_k^\nu)$ or $\mathcal{F} \backslash (\mathbb{R} \times S_k^\nu)$. Since $D_k^\nu \backslash \{(\lambda_k, \infty)\}$ is a component of $\mathcal{F}$ which intersect $\mathbb{R} \times S_k^\nu$, it follows that this component must be a subset of $\mathbb{R}\times S_k^\nu$, i.e. $D_k^\nu \subset (\mathbb{R}\times S_k^\nu)\cup\{(\lambda_k,\infty)\}$. This shows that alternative (i) of Theorem 4.1 cannot be satisfied.

Now let D_k^ν meets $(\lambda, 0)$ for some $\lambda \in \mathbb{R}$. Then there exists a sequence $\{(\lambda_{k,n}, w_{k,n})\}_{n=1}^\infty \subset D_k^\nu$ such that $\lambda_{k,n} \to \hat{\lambda}$ and $||w_{k,n}|| \to 0$ as $n \to \infty$. Setting $\hat{w}_{k,n} = \frac{w_{k,n}}{||w_{k,n}||}$ and dividing the equality

$$w_{k,n} = \lambda_{k,n} L w_{k,n} + G(\lambda_{k,n}, w_{k,n})$$

by $||w_{k,n}||$ we have

$$\hat{w}_{k,n} = \lambda_{k,n} L \hat{w}_{k,n} + \frac{G(\lambda_{k,n}, w_{k,n})}{||w_{k,n}||}. \tag{4.3}$$

Since the condition (1.6) holds it follows from [1, formula (2.23)] that G also satisfies the following condition

$$G(\lambda, w) = o\,(||w||) \quad \text{as} \quad ||w|| \to 0, \tag{4.4}$$

uniformly on bounded λ intervals. Due to the compactness of the operator L and relation (4.4), we can assume that $\hat{w}_{k,n} \to \hat{w}$ in E as $n \to \infty$. Then, taking into account (4.4) and passing to the limit (as $n \to \infty$) in (4.3), we obtain

$$\hat{w} = \hat{\lambda} L \hat{w}.$$

Since $||\hat{w}|| = 1$ and $\hat{w} \in \overline{S_k^\nu} = S_k^\nu \cup \partial S_k^\nu$ it follows from Lemma 2.1 that $\hat{w} \in S_k^\nu$. Then by virtue of Theorem 2.1 we have $\hat{\lambda} = \lambda_k$.

It is similarly proved that, if C_k^ν meets (λ, ∞) for some $\lambda \in \mathbb{R}$, then $\lambda = \lambda_k$. The proof of this theorem is complete.

References

[1] Z.S. Aliyev, Some global results for nonlinear fourth order eigenvalue problems, *Cent. Eur. J. Math.* **12** (2014), no.12, 1811-1828.

[2] Z.S. Aliev, On the global bifurcation of solutions of some nonlinear eigenvalue problems for ordinary differential equations of fourth order, *Sb. Math.* **207** (2016), no. 12, 1625-1649.

[3] Z.S. Aliev, N.A. Mustafayeva, Bifurcation of solutions from infinity for certain nonlinear eigenvalue problems of fourth-order ordinary differential equations, *Electron. J. Differ. Equ.* **2018** (2018), no. 98, 1-19.

[4] Z.S. Aliyev, H.Sh. Rzayeva, Oscillation properties for the equation of the relativistic quantum theory, *Appl. Math. Comput.*, **271** (2015), 308-316.

[5] Z.S. Aliev, H.Sh. Rzayeva, Global bifurcation for nonlinear Dirac problems. *Electron. J. Qual. Theory Differ. Equ.* (46) (2016), 1-14.

[6] F. V. Atkinson, *Discrete and Continuous Boundary Problems*, Academic Press, New York, London, 1964.

[7] D.O. Banks, G.J. Kurowski, A Prüfer transformation for the equation of a vibrating beam subject to axial forces, *J. Differential Equations* **24** (1977), 57-74.

[8] H. Berestycki, On some nonlinear Sturm-Liouville problems, *J. Differential Equations* **26** (1977), 375-390.

[9] R. Courant and D. Hilbert, Methods of Mathematical Physics, I, Interscience, New York, 1953.

[10] G. Dai, Global bifurcation from intervals for Sturm-Liouville problems which are not linearizable, *Electron. J. Qual. Theory Differ. Equ.* (65) (2013), 1-7.

[11] U. Elias, Eigenvalue problems for the equation $Ly + \lambda p(x)y = 0$, *J. Differential Equations* **29** (1978), no. 1, 28-57.
[12] F.R. Gantmacher and M. G. Krein, Oszillationsmatrizen, Oszillationskerne und kleine Schwingungen mechanischer Systeme, Akademie-Verlag, Berlin, 1960.
[13] S.N. Janczewsky, Oscillation theorems for the differential boundary value problems of the fourth order, *Ann. Math.* **29** (1928), no. 2, 521-542.
[14] S. Karlin, Total Positivity, Vol. 1, Stanford University Press, Stanford, Calif., 1968.
[15] O.D. Kellogg, The oscillations of functions of an orthogonal set, *Amer. J Math.* **38** (1916), 1-5.
[16] O.D. Kellogg, Orthogonal function sets arising from integral equations, *Amer. J. Math.* **40** (1918), 145-154.
[17] M. A. Krasnoselskii, *Topological methods in the theory of nonlinear integral equations*, Macmillan, New York, 1965.
[18] A.Yu. Levin, G.D. Stepanov, One dimensional boundary value problems with operators not reducing the number of changes of sign, *Siber. Math. J.* **17** (1976), no. 3-4, 466-482, 612-625.
[19] B. M. Levitan, I. S. Sargsjan, *Introduction to Spectral theory; Selfadjoint ordinary differential operators* , Transl. Math. Monogr. 39, Amer. Math. Soc., Providence, R.I., 1975.
[20] H. Prüfer, Neue Herleitung der Sturm-Liouvilleschen Reihenentwicklung stetiger Funktionen, *Math. Ann.* **95** (1926), 499-518
[21] J. Przybycin, The connection between number and form of bifurcation points and properties of the nonlinear perturbation of Berestycki type, *Ann. Pol. Math.* **50**(1989), no. 2, 129-136.
[22] J. Przybycin, Some applications of bifurcation theory to ordinary differential equations of the fourth order, *Ann. Pol. Math.* **53** (1991), no. 2, 153-160.
[23] P. H. Rabinowitz, Some global results for nonlinear eigenvalue problems, *J. Funct. Anal.* **7** (1971), 487-513.
[24] P.H. Rabinowitz, On bifurcation from infinity, *J. Differential Equations* **14** (1973), 462-475.
[25] B. P. Rynne, Bifurcation from zero or infinity in Sturm-Liouville problems which are not linearizable, *J. Math. Anal. Appl.* **228** (1998), 141-156.
[26] B.P. Rynne, Infinitely many solutions of superlinear fourth order boundary value problems, *Topol. Methods Nonlinear Anal.* **19** (2002), no. 2, 303-312.
[27] K. Schmitt, H. L. Smith, On eigenvalue problems for nondifferentiable mappings, *J. Differential Equations* **33** (1979), 294-319.
[28] C.A. Stuart, Solutions of large norm for non-linear Sturm-Liouville problems, *Quart. J. of Math. (Oxford)* **24** (1973), no. 2, 129-139.
[29] C. Sturm, Mémoire sur les Équations différentielles linéaires du second ordre, *J. Math. Pures Appl.* **1** (1836), 106-186.
[30] J.F. Toland, Asymptotic linearity and non-linear eigenvalue problems, *Quart. J. Math. (Oxford)* **24** (1973), no. 2, 241-250.
[31] S. Willard, *General Topology*, Addison-Wesley, Reading, MA, 1970.

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