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ON THE GENERALIZED MIXED SCHWARZ INEQUALITY

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Abstract. In this work, an extension of the generalized mixed Schwarz inequality is proved. A companion of the generalized mixed Schwarz inequality is established by merging both Cartesian and Polar decompositions of operators. Based on that some numerical radius inequalities are proved.

1. Introduction

Let $\mathscr{B}(\mathscr{H})$ be the Banach algebra of all bounded linear operators defined on a complex Hilbert space $(\mathscr{H};\langle\cdot,\cdot\rangle)$ with the identity operator $1_{\mathscr{H}}$ in $\mathscr{B}(\mathscr{H})$. A bounded linear operator A defined on $\mathscr{H}$ is selfadjoint if and only if $\langle Ax,x\rangle\in\mathbb{R}$ for all $x\in\mathscr{H}$. Consider the real vector space $\mathscr{B}(\mathscr{H})_{sa}$ of self-adjoint operators on $\mathscr{H}$ and its positive cone $\mathscr{B}(\mathscr{H})^{+}$ of positive operators on $\mathscr{H}$. A partial order is naturally equipped on $\mathscr{B}(\mathscr{H})_{sa}$ by defining $A\le B$ if and only if $B-A\in\mathscr{B}(\mathscr{H})^{+}$. We write $A>0$ to mean that A is a strictly positive operator, or equivalently, $A\ge 0$ and A is invertible. When $\mathscr{H}=\mathbb{C}^n$, we identify $\mathscr{B}(\mathscr{H})$ with the algebra $\mathfrak{M}_{n\times n}$ of n-by-n complex matrices. Then, $\mathfrak{M}^{+}_{n\times n}$ is just the cone of n-by-n positive semidefinite matrices.

The Schwarz inequality for positive operators reads that if A is a positive operator in $\mathscr{B}(\mathscr{H})$, then

$$|\langle Ax,y\rangle|^2\le\langle Ax,x\rangle\langle Ay,y\rangle \tag{1.1}$$

for any vectors $x,y\in\mathscr{H}$.

In 1951, Reid [16] proved an inequality which in some senses considered a variant of the Schwarz inequality. In fact, he proved that for all operators $A\in\mathscr{B}(\mathscr{H})$ such that A is positive and AB is selfadjoint then

$$|\langle ABx,y\rangle|\le\|B\|\langle Ax,x\rangle, \tag{1.2}$$

for all $x\in\mathscr{H}$. In [5], Halmos presented his stronger version of the Reid inequality (1.2) by replacing $r(B)$ instead of $\|B\|$.

In 1952, Kato [13] introduced a companion inequality of (1.1), called the mixed Schwarz inequality, which asserts

$$|\langle Ax,y\rangle|^2\le\left\langle|A|^{2\alpha}x,x\right\rangle\left\langle|A^*|^{2(1-\alpha)}y,y\right\rangle,\qquad 0\le\alpha\le 1. \tag{1.3}$$

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for every operators $A \in \mathscr{B}(\mathscr{H})$ and any vectors $x, y \in \mathscr{H}$, where $|A| = (A^*A)^{1/2}$.

In 1988, Kittaneh [12] proved a very interesting extension combining both the Halmos–Reid inequality (1.2) and the mixed Schwarz inequality (1.3). His result reads that

$$|\langle ABx, y\rangle| \le r(B)\, \|f(|A|)\,x\|\, \|g(|A^*|)\,y\| \tag{1.4}$$

for any vectors $x, y \in \mathscr{H}$, where $A, B \in \mathscr{B}(\mathscr{H})$ such that $|A|B = B^*|A|$ and f, g are nonnegative continuous functions defined on $[0, \infty)$ satisfying that $f(t)g(t) = t$ ($t \ge 0$). Clearly, choose $f(t) = t^{\alpha}$ and $g(t) = t^{1-\alpha}$ with $B = 1_{\mathscr{H}}$ we refer to (1.3). Moreover, choosing $\alpha = \frac{1}{2}$ some manipulations refer to the Halmos version of the Reid inequality.

In 2006, Lin and Dragomir [14] proved the following sequence of inequalities of Halmos–Ried's type:

$$\left.\begin{array}{l}
|\langle Tx, y\rangle|^2 \le r(T)\,\langle Tx, x\rangle\,\|y\|^2 \\[2ex]
|\langle TSx, Cy\rangle| \le r(S)\,r(C)\,\langle Tx, x\rangle^{1/2}\,\langle Ty, y\rangle^{1/2} \\[2ex]
|\langle TSx, Cy\rangle| \le r(S)\,r(C)\,\langle Tx, x\rangle \\[2ex]
|\langle Ax, By\rangle|^2 \le r(A)\,r(B)\,\|Ax\|\,\|By\|\,\|x\|\,\|y\|
\end{array}\right\} \tag{1.5}$$

where $A, B, C, T, S \in \mathscr{B}(\mathscr{H})$ such that T is non-negative operator, S and C are arbitrary operators, and TS, TC, A and B be selfadjoint operators, for all vectors $x, y \in \mathscr{H}$.

For a bounded linear operator T on a Hilbert space $\mathscr{H}$, the numerical range $W(T)$ is the image of the unit sphere of $\mathscr{H}$ under the quadratic form $x \to \langle Tx, x\rangle$ associated with the operator. More precisely,

$$W(T) = \{\langle Tx, x\rangle : x \in \mathscr{H}, \|x\| = 1\}$$

Also, the numerical radius is defined to be

$$w(T) = \sup\{|\lambda| : \lambda \in W(T)\} = \sup_{\|x\|=1} |\langle Tx, x\rangle|\,.$$

The spectral radius of an operator T is defined to be

$$r(T) = \sup\{|\lambda| : \lambda \in \mathrm{sp}(T)\}\,.$$

We recall that, the usual operator norm of an operator T is defined to be

$$\|T\| = \sup\{\|Tx\| : x \in H, \|x\| = 1\}\,.$$

It is well known that $w(\cdot)$ defines an operator norm on $\mathscr{B}(\mathscr{H})$ which is equivalent to operator norm $\|\cdot\|$. Moreover, we have

$$\frac{1}{2}\|T\| \le w(T) \le \|T\| \tag{1.6}$$

for any $T \in \mathscr{B}(\mathscr{H})$ and this inequality is sharp.

In 2003, Kittaneh [9] refined the right-hand side of (1.6), where he proved that

$$w(T) \le \frac{1}{2}\left(\|T\| + \|T^2\|^{1/2}\right) \tag{1.7}$$

for any $T \in \mathscr{B}(\mathscr{H})$.

After that in 2005, the same author in [7] proved that

$$\frac{1}{4}\|A^*A + AA^*\| \le w^2(A) \le \frac{1}{2}\|A^*A + AA^*\|. \qquad (1.8)$$

The inequality is sharp. This inequality was also reformulated and generalized in [3] but in terms of Cartesian decomposition.

In 2007, Yamazaki [18] improved (1.7) by proving that

$$w(T) \le \frac{1}{2}\left(\|T\| + w\left(\widetilde{T}\right)\right) \le \frac{1}{2}\left(\|T\| + \left\|T^2\right\|^{1/2}\right) \qquad (1.9)$$

where $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$ and U is the unitary operator in the polar decomposition T of the form $T = U|T|$.

In 2008, Dragomir [2] used Buzano inequality to improve (1.1), where he proved that

$$w^2(T) \le \frac{1}{2}\left(\|T\| + w\left(T^2\right)\right) \qquad (1.10)$$

This result was also recently generalized by Sattari et al. in [17]. For more recent results about the numerical radius see [1], [8], [10], [15] and the recent monograph study [1]. For basic properties of numerical radius and other related topics the reader may refer to the classical books of Gustafson & Rao [4] and Horn & Johnson [6].

In this work, an extension of the Kittaneh inequality (1.4) and a generalization of Lin-Dragomir version of Halmos–Ried type inequalities are proved. Namely, we generalize the Kittaneh inequality (1.4) which already extend the mixed Schwarz inequality (1.3) to be in more general case. A generalization of the obtained result for several operators is also pointed out. A companion of the generalized mixed Schwarz inequality (or Kittaneh inequality) in which the Cartesian decomposition of operators is replaced by the polar decomposition is also given. As application, some numerical radius and norm inequalities are established.

2. The Result(s)

This section is divided into two parts; the first part is devoted to generalize the Kittaneh inequality (1.4) with other related consequences. In the second part, by merging the Cartesian and Polar decompositions of operators, we present a new type of mixed Schwarz inequality called "the Mixed hybrid Schwarz inequality ".

2.1. The mixed Schwarz inequality. Let us start with the following elementary result which is a simple consequence of (1.3).

Lemma 2.1. *Let $A, C, D \in \mathscr{B}(\mathscr{H})$ such that $|A|D = D^*|A|$ and $|A^*|C = C^*|A^*|$. If f and g are nonnegative continuous functions on $[0,\infty)$ satisfying $f(t)g(t) = t$ $(t \ge 0)$, then*

$$|\langle ADu, Cv\rangle|^2 \le \left\langle D^* f^2(|A|)\, Du, u\right\rangle \left\langle C^* g^2(|A^*|)\, Cv, v\right\rangle \qquad (2.1)$$

for all vectors $u, v \in \mathscr{H}$.

Proof. Since $x,y\in\mathscr{H}$ are arbitrary vectors, by the given assumptions there exists $u,v\in\mathscr{H}$ respectively; such that $x=Du$ and $y=Cv$, and this is true for any $x,y\in\mathscr{H}$. Therefore by setting $B=1_{\mathscr{H}}$ in (1.4). Then we have

$$\begin{aligned}|\langle ADu,Cv\rangle|^2&\le \|f(|A|)Du\|^2\|g(|A^*|)Cv\|^2\\&\le \langle f(|A|)Du,f(|A|)Du\rangle\langle g(|A^*|)Cv,g(|A^*|)Cv\rangle\\&\le \left\langle D^*f^2(|A|)Du,u\right\rangle\left\langle C^*g^2(|A^*|)Cv,v\right\rangle.\end{aligned}$$

For all vectors $u,v\in\mathscr{H}$, which proves the result. □

Remark 2.1. Let $A,C,D\in\mathscr{B}(\mathscr{H})$ be as in Lemma 2.1. In particular case, choosing $f(t)=t^\alpha$, $g(t)=t^{1-\alpha}$, $t\ge 0$ in (2.1) we get

$$|\langle ADu,Cv\rangle|^2\le\left\langle D^*|A|^{2\alpha}Du,u\right\rangle\left\langle C^*|A^*|^{2(1-\alpha)}Cv,v\right\rangle.$$

In special case for $\alpha=\frac{1}{2}$ we have

$$|\langle ABu,Cv\rangle|^2\le\langle D^*|A|Du,u\rangle\langle C^*|A^*|Cv,v\rangle$$

for all vectors $u,v\in\mathscr{H}$.

Now, we are ready to present our generalization of (1.4) and the above inequality (2.1) for any bounded linear operators.

Corollary 2.1. *Let $A,B,C\in\mathscr{B}(\mathscr{H})$ such that $|A|B=B^*|A|$ and $|A^*|C=C^*|A^*|$. If f and g are nonnegative continuous functions on $[0,\infty)$ satisfying $f(t)g(t)=t$ $(t\ge 0)$, then*

$$|\langle ABx,Cu\rangle|\le r(B)\,r(C)\,\|f(|A|)x\|\,\|g(|A^*|)u\| \tag{2.2}$$

for all vectors $x,u\in\mathscr{H}$.

Proof. It's enough to show that the inequality

$$\begin{aligned}|\langle ABx,Cu\rangle|^{2^n}&\le\left\langle f^2(|A|)B^{2^n}x,x\right\rangle\left\langle f^2(|A|)x,x\right\rangle^{2^{n-1}-1}\\&\quad\times\left\langle g^2(|A^*|)C^{2^n}u,u\right\rangle\left\langle g^2(|A^*|)u,u\right\rangle^{2^{n-1}-1},\end{aligned} \tag{2.3}$$

is valid for all positive integer n. The proof is going likewise the proof of Theorem 5 in [12] taking into account Lemma 2.1. □

Corollary 2.2. *Let $A,C\in\mathscr{B}(\mathscr{H})$ such that $|A^*|C=C^*|A^*|$. If f and g as in Corollary 2.1, then*

$$|\langle Ax,Cu\rangle|\le r(C)\,\|f(|A|)x\|\,\|g(|A^*|)u\|$$

for any vectors $x,u\in\mathscr{H}$.

Proof. Setting $B=1_{\mathscr{H}}$ in (2.2) we get the required result. □

Corollary 2.3. *Let $A,B,C\in\mathscr{B}(\mathscr{H})$ such that $|A|B=B^*|A|$ and $|A^*|C=C^*|A^*|$. Then*

$$|\langle ABx,Cu\rangle|^2\le r^2(B)\,r^2(C)\left\langle|A|^{2\alpha}x,x\right\rangle\left\langle|A^*|^{2(1-\alpha)}u,u\right\rangle \tag{2.4}$$

for any vectors $x,u\in\mathscr{H}$. In particular we have

$$|\langle Bx,Cu\rangle|\le r(B)\,r(C)\,\langle x,x\rangle\,\langle u,u\rangle.$$

Proof. Setting $f(t) = t^{\alpha}$ and $g(t) = t^{1-\alpha}$, $0 \le \alpha \le 1$, $t \ge 0$ in Corollary 2.1. The particular case follows by setting $A = 1_{\mathscr{H}}$ in (2.4). □

A more general mixed Schwarz inequality can be stated as follows:

Corollary 2.4. *Let $A, D, B_1, B_2, C_1, C_2 \in \mathscr{B}(\mathscr{H})$ such that*

$$|A|B_1 = B_1^*|A| \quad \text{and} \quad |A^*|C_1 = C_1^*|A^*|,$$
$$|D|B_2 = B_2^*|D| \quad \text{and} \quad |D^*|C_2 = C_2^*|D^*|.$$

If f and g are nonnegative continuous functions on $[0,\infty)$ satisfying $f(t)g(t) = t$ $(t \ge 0)$, then

$$\begin{aligned} &|\langle (C_1^* A B_1 + C_2^* D B_2)\, x, u\rangle| \\ &\le r(B_1)\, r(C_1)\, \|f(|A|)\, x\|\, \|g(|A^*|)\, u\| \\ &\qquad + r(B_2)\, r(C_2)\, \|f(|D|)\, x\|\, \|g(|D^*|)\, u\| \\ &\le \max\{r(B_1)\, r(C_1), r(B_2)\, r(C_2)\} \cdot \left(\|f(|A|)\, x\|^p + \|f(|D|)\, x\|^p\right)^{1/p} \\ &\qquad \times \left(\|g(|A^*|)\, u\|^q + \|g(|D^*|)\, u\|^q\right)^{1/q} \end{aligned} \tag{2.5}$$

for any vectors $x, u \in \mathscr{H}$ and all $p > 1$ with $q = \frac{p}{p-1}$.

Proof. Since

$$\begin{aligned} |\langle (C_1^* A B_1 + C_2^* D B_2)\, x, u\rangle| &= |\langle C_1^* A B_1 x, u\rangle + \langle C_2^* D B_2 x, u\rangle| \\ &\le |\langle C_1^* A B_1 x, u\rangle| + |\langle C_2^* D B_2 x, u\rangle|\,. \end{aligned}$$

So that the first inequality follows from (2.2). The second inequality follows by applying the Hölder inequality. □

Corollary 2.5. *Let $A, D, B_1, B_2, C_1, C_2 \in \mathscr{B}(\mathscr{H})$ such that*

$$|A|B_1 = B_1^*|A| \quad \text{and} \quad |A^*|C_1 = C_1^*|A^*|,$$
$$|D|B_2 = B_2^*|D| \quad \text{and} \quad |D^*|C_2 = C_2^*|D^*|.$$

If f and g are nonnegative continuous functions on $[0,\infty)$ satisfying $f(t)g(t) = t$ $(t \ge 0)$, then

$$\begin{aligned} &|\langle (C_1^* A B_1 + C_2^* D B_2)\, x, u\rangle|^2 \\ &\qquad \le r^2(B_1)\, r^2(C_1) \left\langle |A|^{2\alpha} x, x\right\rangle \left\langle |A^*|^{2(1-\alpha)} u, u\right\rangle \\ &\qquad\qquad + r^2(B_2)\, r^2(C_2) \left\langle |D|^{2\alpha} x, x\right\rangle \left\langle |D^*|^{2(1-\alpha)} u, u\right\rangle \end{aligned} \tag{2.6}$$

for any vectors $x, u \in \mathscr{H}$.

Proof. Setting $f(t) = t^{\alpha}$ and $g(t) = t^{1-\alpha}$, $0 \le \alpha \le 1$, $t \ge 0$ in Corollary 2.4. □

In fact, one may establish a generalization of Corollary 2.4 to several operators, by letting $A_i, B_i, C_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, n)$ such that

$$|A_i|B_i = B_i^*|A_i| \quad \text{and} \quad |A_i^*|C_i = C_i^*|A_i^*|.$$

If f,g are as above, proceeding as in the presented proof above, then we have

$$\begin{aligned}&\left|\left\langle\left(\sum_{i=1}^{n}C_i^*A_iB_i\right)x,u\right\rangle\right|\\&\le\sum_{i=1}^{n}r\left(B_i\right)r\left(C_i\right)\left\|f\left(\left|A_i\right|\right)x\right\|\left\|g\left(\left|A_i^*\right|\right)u\right\| \qquad (2.7)\\&\le\max_{1\le i\le n}\left\{r\left(B_i\right)r\left(C_i\right)\right\}\cdot\left(\sum_{i=1}^{n}\left\|f\left(\left|A_i\right|\right)x\right\|^p\right)^{1/p}\left(\sum_{i=1}^{n}\left\|g\left(\left|A_i^*\right|\right)u\right\|^q\right)^{1/q}.\end{aligned}$$

For all $x,u\in\mathscr{H}$, which follows by the properties of 'max' and Hölder inequality, where p,q are conjugate exponents, i.e., $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Thus, one may has the following norm inequality

$$\begin{aligned}&\left\|\sum_{i=1}^{n}C_i^*A_iB_i\right\|\\&\qquad\le\max_{1\le i\le n}\left\{r\left(B_i\right)r\left(C_i\right)\right\}\cdot\left(\sum_{i=1}^{n}\left\|f\left(\left|A_i\right|\right)\right\|^p\right)^{1/p}\left(\sum_{i=1}^{n}\left\|g\left(\left|A_i^*\right|\right)\right\|^q\right)^{1/q}. \qquad (2.8)\end{aligned}$$

For instance, consider $f(t)=t^{\alpha}$ and $g(t)=t^{1-\alpha}$, one has from (2.8) that

$$\left\|\sum_{i=1}^{n}C_i^*A_iB_i\right\|\le\max_{1\le i\le n}\left\{r\left(B_i\right)r\left(C_i\right)\right\}\cdot\left(\sum_{i=1}^{n}\left\|\left|A_i\right|^{\alpha}\right\|^p\right)^{1/p}\left(\sum_{i=1}^{n}\left\|\left|A_i^*\right|^{1-\alpha}\right\|^q\right)^{1/q}.$$

Also, if $C_i=B_i=1_{\mathscr{H}}$ for all $i=1,\cdots,n$, then the last inequality reduces to

$$\left\|\sum_{i=1}^{n}A_i\right\|\le\left(\sum_{i=1}^{n}\left\|\left|A_i\right|^{\alpha}\right\|^p\right)^{1/p}\left(\sum_{i=1}^{n}\left\|\left|A_i^*\right|^{1-\alpha}\right\|^q\right)^{1/q}.$$

2.2. The mixed hybrid Schwarz inequality. Merging both Cartesian and Polar decompositions of operators will produce a new hybrid Mixed Schwarz inequality including both decompositions. The next result provides a new extension of the mixed Schwarz inequality (1.3) and their generalizations (1.4) and (2.2).

Theorem 2.1. *Let $A\in\mathscr{B}\left(\mathscr{H}\right)$ with the Cartesian decomposition $A=P+iQ$. If f and g are as in Corollary 2.1, then*

$$\left|\left\langle Ax,y\right\rangle\right|\le\left\{\left\|f\left(\left|P\right|\right)x\right\|\left\|g\left(\left|P\right|\right)y\right\|+\left\|f\left(\left|Q\right|\right)x\right\|\left\|g\left(\left|Q\right|\right)y\right\|\right\} \qquad (2.9)$$

for all $x,y\in\mathscr{H}$.

Proof. Let $P+iQ$ be the Cartesian decomposition of A. Setting $B=C=1_{\mathscr{H}}$ in (2.2), then

$$\begin{aligned}\left|\left\langle Ax,y\right\rangle\right|&=\left(\left\langle Px,y\right\rangle^2+\left\langle Qx,y\right\rangle^2\right)^{1/2}\\&\le\left|\left\langle Px,y\right\rangle\right|+\left|\left\langle Qx,y\right\rangle\right|\\&\le\left\{\left\|f\left(\left|P\right|\right)x\right\|\left\|g\left(\left|P^*\right|\right)y\right\|+\left\|f\left(\left|Q\right|\right)x\right\|\left\|g\left(\left|Q^*\right|\right)y\right\|\right\}\end{aligned}$$

for all $x, y \in \mathscr{H}$, where the last inequality follows form (2.2). So that, the required result follows since P and Q are selfadjoint operators. ☐

Corollary 2.6. *Let $A \in \mathscr{B}(\mathscr{H})$ with the Cartesian decomposition $A = P + iQ$. Then,*

$$|\langle Ax, y\rangle| \le \left\{ \| |P|^{\alpha} x\| \left\| |P|^{(1-\alpha)} y \right\| + \| |Q|^{\alpha} x\| \left\| |Q|^{(1-\alpha)} y \right\| \right\} \tag{2.10}$$

for all $x, y \in \mathscr{H}$.

Proof. Setting $f(t) = t^{\alpha}$ and $g(t) = t^{1-\alpha}$, $0 \le \alpha \le 1$, $t \ge 0$ in Theorem 2.1 we get (2.10). ☐

The Cartesian companion decomposition of the Kato's inequality (1.3) can be deduced as follows:

Corollary 2.7. *Let $A \in \mathscr{B}(\mathscr{H})$ with the Cartesian decomposition $A = P + iQ$. Then,*

$$|\langle Ax, y\rangle| \le \left\langle |P|^{2\alpha} x, x \right\rangle^{1/2} \left\langle |P|^{2(1-\alpha)} y, y \right\rangle^{1/2} + \left\langle |Q|^{2\alpha} x, x \right\rangle^{1/2} \left\langle |Q|^{2(1-\alpha)} y, y \right\rangle^{1/2} \tag{2.11}$$

for all $x, y \in \mathscr{H}$ and any $0 \le \alpha \le 1$.

Proof. Setting $f(t) = t^{\alpha}$ and $g(t) = t^{1-\alpha}$, $0 \le \alpha \le 1$, $t \ge 0$ in Corollary 2.6, then we have

$$\begin{aligned} |\langle Ax, y\rangle| &\le \| |P|^{\alpha} x\| \left\| |P|^{1-\alpha} y \right\| + \| |Q|^{\alpha} x\| \left\| |Q|^{1-\alpha} y \right\| \\ &= \left\langle |P|^{2\alpha} x, x \right\rangle^{1/2} \left\langle |P|^{2(1-\alpha)} y, y \right\rangle^{1/2} + \left\langle |Q|^{2\alpha} x, x \right\rangle^{1/2} \left\langle |Q|^{2(1-\alpha)} y, y \right\rangle^{1/2} \end{aligned}$$

for all $x, y \in \mathscr{H}$, which proves the required result. ☐

Remark 2.2. Some Weyl type inequalities can be deduced by following the same approach considered in [12]. In fact by making use of (2.2) and (2.9) instead of (1.4) in [12], a general Weyl type inequality can be deduced. Similarly, some inequalities for the p-Schatten norm can be pointed out following the same pattern in [12]. We shall omit the details.

3. Numerical Radius inequalities

In order to prove our results in this section we need some of the following well-known facts.

Lemma 3.1. *The Power-Young inequality reads that*

$$ab \le \frac{a^{\alpha}}{\alpha} + \frac{b^{\beta}}{\beta} \le \left(\frac{a^{p\alpha}}{\alpha} + \frac{b^{p\beta}}{\beta} \right)^{\frac{1}{p}} \tag{3.1}$$

for all $a, b \ge 0$ and $\alpha, \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ and all $p \ge 1$.

Lemma 3.2. (The McCarty inequality). *Let $A \in \mathscr{B}(\mathscr{H})^{+}$, then*

$$\langle Ax, x\rangle^{p} \le \langle A^{p} x, x\rangle, \qquad p \ge 1 \tag{3.2}$$

for any unit vector $x \in \mathscr{H}$

Lemma 3.3. *If $A, B \in \mathscr{B}(\mathscr{H})$, then*

$$r(AB) \le \frac{1}{4}\left(\|AB\| + \|BA\| + \sqrt{\left(\|AB\| - \|BA\|\right)^2 + 4m(A,B)}\right), \tag{3.3}$$

where $m(A,B) := \min\{\|A\|\,\|BAB\|, \|B\|\,\|ABA\|\}$.

In some of our results we need the following two fundamental norm estimates, which are:

$$\|A+B\| \le \frac{1}{2}\left(\|A\| + \|B\| + \sqrt{\left(\|A\| - \|B\|\right)^2 + 4\left\|A^{1/2}B^{1/2}\right\|^2}\right), \tag{3.4}$$

and

$$\left\|A^{1/2}B^{1/2}\right\| \le \|AB\|^{1/2}. \tag{3.5}$$

Both estimates are valid for all positive operators $A, B \in \mathscr{B}(\mathscr{H})$. Also, it should be noted that (3.4) is sharper than the triangle inequality as pointed out by Kittaneh in [11].

3.1. Inequalities using the mixed Schwarz inequality. Depending on the obtained results in Section 2.1, in this part we provide some numerical radius inequalities. Let us start with the following result.

Theorem 3.1. *Let $A, B, C \in \mathscr{B}(\mathscr{H})$ such that $|A|B = B^*|A|$ and $|A^*|C = C^*|A^*|$. If f and g are nonnegative continuous functions on $[0,\infty)$ satisfying $f(t)g(t) = t$ $(t \ge 0)$, then*

$$\begin{aligned} w(C^*AB) &\le \frac{1}{2} r(B)\, r(C) \cdot \left\|f^2(|A|) + g^2(|A^*|)\right\| \\ &\le \frac{1}{16}\left(\|B\| + \left\|B^2\right\|^{1/2}\right)\left(\|C\| + \left\|C^2\right\|^{1/2}\right) \\ &\qquad \times \Big\{\left\|f^2(|A|)\right\| + \left\|g^2(|A^*|)\right\| \\ &\qquad + \sqrt{\left(\|f^2(|A|)\| - \|g^2(|A^*|)\|\right)^2 + 4\|f(|A|)\,g(|A^*|)\|^2}\Big\}. \end{aligned} \tag{3.6}$$

In particular, we have

$$\begin{aligned} w(C^*C) &\le \frac{1}{2} r(C) \cdot \left\|f^2(|C|) + g^2(|C^*|)\right\| \\ &\le \frac{1}{8}\left(\|C\| + \left\|C^2\right\|^{1/2}\right)\Big\{\left\|f^2(|C|)\right\| + \left\|g^2(|C^*|)\right\| \\ &\qquad + \sqrt{\left(\|f^2(|C|)\| - \|g^2(|C^*|)\|\right)^2 + 4\|f(|C|)\,g(|C^*|)\|^2}\Big\}. \end{aligned} \tag{3.7}$$

Proof. Setting $y = x$ in (2.2), we get

$$\begin{aligned}
|\langle C^*ABx, x\rangle| &\le r(B)\, r(C)\, \|f(|A|)\, x\|\, \|g(|A^*|)\, x\| \\
&= r(B)\, r(C) \left\langle f^2(|A|)\, x, x\right\rangle^{1/2} \left\langle g^2(|A^*|)\, x, x\right\rangle^{1/2} \\
&\le \frac{1}{2} r(B)\, r(C) \left(\left\langle f^2(|A|)\, x, x\right\rangle + \left\langle g^2(|A^*|)\, x, x\right\rangle\right) \\
&= \frac{1}{2} r(B)\, r(C) \left\langle \left(f^2(|A|) + g^2(|A^*|)\right) x, x\right\rangle \\
&= \frac{1}{2} r(B)\, r(C) \left\|\left(f^2(|A|) + g^2(|A^*|)\right)\right\|.
\end{aligned}$$

Thus, by taking the supremum over $x \in \mathscr{H}$ we get the first inequality in (3.6). The second inequality in (3.6) follows by employing (3.3) on the first inequality and use (3.4). The inequality (3.7) follows from (3.6) by setting $B = I$ and $A = C$. □

Corollary 3.1. *Let $A, B, C \in \mathscr{B}(\mathscr{H})$ such that $|A|B = B^*|A|$ and $|A^*|C = C^*|A^*|$. Then*

$$\begin{aligned}
w(C^*AB) &\le \frac{1}{2} r(B)\, r(C) \cdot \left\| |A|^{2\alpha} + |A^*|^{2(1-\alpha)} \right\| \qquad (3.8)\\
&\le \frac{1}{16} \left(\|B\| + \left\|B^2\right\|^{1/2}\right) \left(\|C\| + \left\|C^2\right\|^{1/2}\right) \\
&\quad \times \left\{ \left\| |A|^{2\alpha} \right\| + \left\| |A^*|^{2(1-\alpha)} \right\| \right. \\
&\quad \left. + \sqrt{\left(\left\| |A|^{2\alpha} \right\| - \left\| |A^*|^{2(1-\alpha)} \right\|\right)^2 + 4 \left\| |A|^{\alpha} |A^*|^{1-\alpha} \right\|^2} \right\}.
\end{aligned}$$

Proof. Setting $f(t) = t^{\alpha}$ and $g(t) = t^{1-\alpha}$, $0 \le \alpha \le 1$, $t \ge 0$ in Theorem 3.1 we get (3.8). □

Remark 3.1. Setting $\alpha = \frac{1}{2}$ in (3.8) and then employing (3.5) we get

$$w(C^*AB) \le \frac{1}{8} \left(\|B\| + \left\|B^2\right\|^{1/2}\right) \left(\|C\| + \left\|C^2\right\|^{1/2}\right) \left(\|A\| + \left\|A^2\right\|^{1/2}\right) \qquad (3.9)$$

Remark 3.2. Letting $A = C$ and $B = 1_{\mathscr{H}}$ in (3.8). Then

$$\begin{aligned}
w(C^*C) &\le \frac{1}{2} r(C) \cdot \left\| |C|^{2\alpha} + |C^*|^{2(1-\alpha)} \right\| \\
&\le \frac{1}{8} \left(\|C\| + \left\|C^2\right\|^{1/2}\right) \left\{ \left\| |C|^{2\alpha} \right\| + \left\| |C^*|^{2(1-\alpha)} \right\| \right. \\
&\quad \left. + \sqrt{\left(\left\| |C|^{2\alpha} \right\| - \left\| |C^*|^{2(1-\alpha)} \right\|\right)^2 + 4 \left\| |C|^{\alpha} |C^*|^{1-\alpha} \right\|^2} \right\}.
\end{aligned}$$

Moreover, setting $\alpha = \frac{1}{2}$ in the above inequality and use (3.5) with the fact that $\||C|\| = \||C^*|\| = \|C\|$. So that we get

$$w(C^*C) \le \frac{1}{4} \left(\|C\| + \left\|C^2\right\|^{1/2}\right)^2.$$

Corollary 3.2. *Let $A, B, C \in \mathscr{B}(\mathscr{H})$ such that $|A|C = C^*|A|$ and $|A^*|C = C^*|A^*|$. If f and g are nonnegative continuous functions on $[0,\infty)$ satisfying $f(t)g(t) = t$ $(t \geq 0)$, then*

$$\begin{aligned} w\left(C^*AC\right) &\leq \frac{1}{2} r^2\left(C\right) \cdot \left\|f^2\left(|A|\right) + g^2\left(|A^*|\right)\right\| \\ &\leq \frac{1}{16}\left(\|C\| + \left\|C^2\right\|^{1/2}\right)^2 \left\{\left\|f^2\left(|A|\right)\right\| + \left\|g^2\left(|A^*|\right)\right\| \right. \\ &\qquad \left. + \sqrt{\left(\left\|f^2\left(|A|\right)\right\| - \left\|g^2\left(|A^*|\right)\right\|\right)^2 + 4\left\|f\left(|A|\right) g\left(|A^*|\right)\right\|^2}\right\}. \end{aligned}$$

Proof. Setting $B = C$ in Theorem 3.1. □

A generalization of Theorem 3.1 to higher order power is given as follows:

Theorem 3.2. *Let $A, B \in \mathscr{B}(\mathscr{H})(\Omega)$ such that $|A|B = B^*|A|$. If f, g be non-negative continuous functions on $[0,\infty)$ satisfying $f(t)g(t) = t$, $(t \geq 0)$, then*

$$w^p\left(C^*AB\right) \leq r^p\left(B\right) r^p\left(C\right) \cdot \left\|\frac{1}{\alpha} f^{\alpha p}\left(|A|\right) + \frac{1}{\beta} g^{\beta p}\left(|A^*|\right)\right\| \tag{3.10}$$

for all $p \geq 1$, $\alpha \geq \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ and $\beta p \geq 2$. Moreover we have

$$\begin{aligned} w^p\left(C^*AB\right) &\leq \frac{1}{2^{p+2}} \cdot \gamma \cdot \left(\|B\| + \left\|B^2\right\|^{1/2}\right)^p \left(\|C\| + \left\|C^2\right\|^{1/2}\right)^p \\ &\qquad \times \left\{\left\|f^{\alpha p}\left(|A|\right)\right\| + \left\|g^{\beta p}\left(|A^*|\right)\right\| \right. \\ &\qquad \left. + \sqrt{\left[\left|f^{\alpha p}\left(|A|\right)\right| - \left\|g^{\beta p}\left(|A^*|\right)\right\|\right]^2 + 4\left\|f^{p\alpha}\left(|A|\right) g^{p\beta}\left(|A^*|\right)\right\|^2}\right\}, \end{aligned} \tag{3.11}$$

where $\gamma = \max\{\frac{1}{\alpha}, \frac{1}{\beta}\}$.

Proof. Setting $u = x$ in the generalized mixed Schwarz inequality (2.2), we have

$$\begin{aligned} &\left|\left\langle C^*ABx, x\right\rangle\right|^p \\ &\leq r^p\left(B\right) r^p\left(C\right) \left\|f\left(|A|\right)x\right\|^p \left\|g\left(|A^*|\right)x\right\|^p \\ &= r^p\left(B\right) r^p\left(C\right) \left\langle f^2\left(|A|\right)x, x\right\rangle^{\frac{p}{2}} \left\langle g^2\left(|A^*|\right)x, x\right\rangle^{\frac{p}{2}} \\ &\leq r^p\left(B\right) r^p\left(C\right) \left[\frac{1}{\alpha}\left\langle f^2\left(|A|\right)x, x\right\rangle^{\frac{\alpha p}{2}} + \frac{1}{\beta}\left\langle g^2\left(|A^*|\right)x, x\right\rangle^{\frac{\beta p}{2}}\right] \qquad \text{(by (3.1))} \\ &\leq r^p\left(B\right) r^p\left(C\right) \left[\frac{1}{\alpha}\left\langle f^{\alpha p}\left(|A|\right)x, x\right\rangle + \frac{1}{\beta}\left\langle g^{\beta p}\left(|A^*|\right)x, x\right\rangle\right] \qquad \text{(by (3.2))} \\ &= r^p\left(B\right) r^p\left(C\right) \left\langle \left[\frac{1}{\alpha} f^{\alpha p}\left(|A|\right) + \frac{1}{\beta} g^{\beta p}\left(|A^*|\right)\right] x, x\right\rangle. \end{aligned}$$

Taking the supremum over $x \in \mathscr{H}$, we obtain the inequality in (3.10). To obtain the second inequality, by (3.4) we have

$$\begin{aligned}
&\left\|\frac{1}{\alpha}f^{\alpha p}\left(|A|\right)+\frac{1}{\beta}g^{\beta p}\left(|A^*|\right)\right\| \\
&\le \max\{\frac{1}{\alpha},\frac{1}{\beta}\}\cdot\left\|f^{\alpha p}\left(|A|\right)+g^{\beta p}\left(|A^*|\right)\right\| \\
&\le \frac{1}{2}\gamma\left(\left\|f^{\alpha p}\left(|A|\right)\right\|+\left\|g^{\beta p}\left(|A^*|\right)\right\|\right. \\
&\qquad\left.+\sqrt{\left[\left|f^{\alpha p}\left(|A|\right)\right|-\left\|g^{\beta p}\left(|A^*|\right)\right\|\right]^2+4\left\|f^{p\alpha}\left(|A|\right)g^{p\beta}\left(|A^*|\right)\right\|^2}\right).
\end{aligned}$$

Now, employing (3.3) with $B = 1_{\mathscr{H}}$ and then substituting all in (3.10) we get (3.11).

□

Remark 3.3. Letting $u = x$ in (2.7), then by taking the supremum over $x \in \mathscr{H}$ with $\|x\| = 1$ so that we get

$$\begin{aligned}
&w\left(\sum_{i=1}^{n}C_i^*A_iB_i\right) \qquad (3.12)\\
&\le \sum_{i=1}^{n} r\left(B_i\right)r\left(C_i\right)\left\|f\left(|A_i|\right)\right\|\left\|g\left(|A_i^*|\right)\right\| \\
&\le \max_{1\le i\le n}\left\{r\left(B_i\right)r\left(C_i\right)\right\}\cdot\left(\sum_{i=1}^{n}\left\|f\left(|A_i|\right)\right\|^p\right)^{1/p}\left(\sum_{i=1}^{n}\left\|g\left(|A_i^*|\right)\right\|^q\right)^{1/q}.
\end{aligned}$$

Following the same approach applied in the proof of Theorems 3.1 and 3.2, one can state other bounds for the second inequality above. Several special cases can also be obtained as in the Corollaries 3.1 and 3.2. Of course the same inequalities still valid for norms instead of numerical radius.

3.2. Inequalities using the mixed hybrid Schwarz inequality.

Theorem 3.3. *Let $A \in \mathscr{B}(\mathscr{H})$ with the Cartesian decomposition $A = P + iQ$. If f and g are as in Corollary 2.1. Then*

$$w\left(A\right) \le \left\|f^p\left(|P|\right)+f^p\left(|Q|\right)\right\|^{1/p}\left\|g^q\left(|P|\right)+g^q\left(|Q|\right)\right\|^{1/q} \qquad (3.13)$$

for all $p, q \ge 2$ with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Letting $y = x$ in (2.8), then we have

$$\begin{aligned}
&\left|\langle Ax, y\rangle\right| \\
&\le \left\{\left\|f\left(|P|\right)x\right\|\left\|g\left(|P|\right)y\right\| + \left\|f\left(|Q|\right)x\right\|\left\|g\left(|Q|\right)y\right\|\right\} \\
&\le \left(\left\|f\left(|P|\right)x\right\|^p + \left\|f\left(|Q|\right)x\right\|^p\right)^{1/p} \\
&\qquad\quad \times \left(\left\|g\left(|P|\right)y\right\|^q + \left\|g\left(|Q|\right)y\right\|^q\right)^{1/q} \qquad \text{(by Hölder inequaity)} \\
&\le \left(\left\langle f^2\left(|P|\right)x, x\right\rangle^{p/2} + \left\langle f^2\left(|Q|\right)x, x\right\rangle^{p/2}\right)^{1/p} \\
&\qquad \times \left(\left\langle g^2\left(|P|\right)x, x\right\rangle^{q/2} + \left\langle g^2\left(|Q|\right)x, x\right\rangle^{q/2}\right)^{1/q} \\
&\le \left(\left\langle f^p\left(|P|\right)x, x\right\rangle + \left\langle f^p\left(|Q|\right)x, x\right\rangle\right)^{1/p} \\
&\qquad \times \left(\left\langle g^q\left(|P|\right)x, x\right\rangle + \left\langle g^q\left(|Q|\right)x, x\right\rangle\right)^{1/q} \qquad \text{(by (3.2))} \\
&\le \left\langle\left[f^p\left(|P|\right) + f^p\left(|Q|\right)\right]x, x\right\rangle^{1/p}\left\langle\left[g^q\left(|P|\right) + g^q\left(|Q|\right)\right]x, x\right\rangle^{1/q}
\end{aligned}$$

for all $p, q \ge 2$ with $\frac{1}{p} + \frac{1}{q} = 1$. Taking the supremum over all unit vector $x \in \mathscr{H}$ we get the desired result.

□

However, we can still have a little more manipulation; by employing (3.4) for the above two norms we get

$$\begin{aligned}
&\left\|\left(f^p\left(|P|\right) + f^p\left(|Q|\right)\right)\right\| \\
&\le \frac{1}{2}\left(\left\|f^p\left(|P|\right)\right\| + \left\|f^p\left(|Q|\right)\right\|\right. \qquad (3.14) \\
&\qquad \left. + \sqrt{\left(\left\|f^p\left(|P|\right)\right\| - \left\|f^p\left(|Q|\right)\right\|\right)^2 + 4\left\|f^{p/2}\left(|P|\right)f^{p/2}\left(|Q|\right)\right\|^2}\right),
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\left(g^q\left(|P|\right) + g^q\left(|Q|\right)\right)\right\| \\
&\le \frac{1}{2}\left(\left\|g^q\left(|P|\right)\right\| + \left\|g^q\left(|Q|\right)\right\|\right. \qquad (3.15) \\
&\qquad \left. + \sqrt{\left(\left\|g^q\left(|P|\right)\right\| - \left\|g^q\left(|Q|\right)\right\|\right)^2 + 4\left\|g^{q/2}\left(|P|\right)g^{q/2}\left(|Q|\right)\right\|^2}\right).
\end{aligned}$$

Substituting (3.14) and (3.15) in (3.13) we get another refinement of (3.13).

Remark 3.4. In an interesting case, one may consider $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$, $\alpha \in [0, 1]$ and $p = q = 2$. If we wish for $\alpha = \frac{1}{2}$, after some manipulations and making of use (3.5) we get

$$w\left(A\right) \le \frac{1}{2}\cdot\left(\left\||P|\right\| + \left\||Q|\right\| + \sqrt{\left(\left\||P|\right\| - \left\||Q|\right\|\right)^2 + 4\left\||P|\,|Q|\right\|}\right).$$

It should be noted that the authors in [3] have shown that $w\left(A\right) \le \left\||P|\right\| + \left\||Q|\right\|$, it is not hard to show that our estimate is better than the previous one.

Remark 3.5. Following the same approach considered in the proof of Theorem 3.2, one may state another bound of (3.13).

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