Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 46, Number 1, 2020, Pages 79–93
https://doi.org/10.29228/proc.19

CHARACTERIZATION OF CONTINUOUS g-FRAMES VIA OPERATORS

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Abstract. In this paper we introduce and show some new notions and results on cg-frames of Hilbert spaces. We define cg-orthonormal bases for a Hilbert space H and verify their properties and relations with cg-frames. Actually, we present that every cg-frame can be represented as a composition of a cg-orthonormal basis and an operator under some conditions. Also, we find for any cg-frame an induced c-frame and study their properties and relations. Moreover, we show that every cg-frame can be written as addition of two Parseval cg-frames. In addition, we show each cg-frame as a sum of a cg-orthonormal basis and a cg-Riesz basis.

1. Introduction

Frames (discrete frames) in Hilbert spaces were introduced by Duffin and Schaeffer [7] in 1952 to study some deep problems in nonharmonic Fourier series. After the illustrious paper [6] by Daubechies, Grossmann and Meyer, frame theory popularized immensely.

A frame for a Hilbert space allows each vector in the space to be written as a linear combination of the elements in the frame, but linear independence between the frame elements is not required. Intuitively, a frame can be thought as a basis to which one has added more elements.

Generally, frames have been used in signal processing, image processing, data compression and sampling theory. Later, motivated by the theory of coherent states, this concept was generalized to families indexed by some locally compact space endowed with a Radon measure. This approach leads to the notion of continuous frames [2, 3, 11, 13]. Some results about continuous frames and their generalizations can be found in [8, 9, 10, 15].

In this paper, inspired by [16] and [12], we generalize some results to cg-frames. The paper is organized as follows. In Section 2, we introduce the concept of cg-orthonormal bases for Hilbert spaces and discuss about their characteristics and their relations with cg-frames and c-frames. Our aim in Section 3 is describing every continuous g-frame as a sum of two Parseval continuous g-frames. We also present that every continuous g-frame can be written as a linear combination of an cg-orthonormal basis and a cg-Riesz basis.

2010 *Mathematics Subject Classification.* 42C15, 46C05.

Key words and phrases. Hilbert space, c-frame, cg-frame, cg-orthonormal basis.

Throughout this paper, H is a separable Hilbert space, (Ω,μ) is a measure space with positive measure μ and $\{H_\omega\}_{\omega\in\Omega}$ is a family of separable Hilbert spaces.

We first review the definitions of continuous frames and continuous g-frames.

Definition 1.1. ([15]) Suppose that (Ω,μ) is a measure space with positive measure μ. A mapping $f:\Omega\longrightarrow H$ is called a *continuous frame*, or simply a *c-frame*, with respect to (Ω,μ) for H, if:
(i) for each $h\in H$, $\omega\longmapsto\langle h,f(\omega)\rangle$ is a measurable function,
(ii) there exist positive constants A and B such that

$$A\|h\|^2\leq\int_\Omega|\langle h,f(\omega)\rangle|^2d\mu(\omega)\leq B\|h\|^2,\quad h\in H. \tag{1.1}$$

The constants A,B are called *c-frame bounds.* If A,B can be chosen such that $A=B$, then f is called a *tight c-frame* and if $A=B=1$, it is called a Parseval c-frame. A mapping f is called *c-Bessel mapping* if the second inequality in (1.1) holds. In this case, B is called the *Bessel bound.*

Some operators associated to c-Bessel mappings can be useful to characterize them.

Proposition 1.1. ([15]) *Let (Ω,μ) be a measure space and $f:\Omega\longrightarrow H$ be a c-Bessel mapping for H. Then the operator $T_f:L^2(\Omega,\mu)\longrightarrow H$, weakly defined by*

$$\langle T_f\varphi,h\rangle=\int_\Omega\varphi(\omega)\langle f(\omega),h\rangle d\mu(\omega),\quad h\in H, \tag{1.2}$$

is well defined, linear, bounded, and its adjoint is given by

$$T_f^*:H\longrightarrow L^2(\Omega,\mu),\quad T_f^*h(\omega)=\langle h,f(\omega)\rangle,\ \ \omega\in\Omega. \tag{1.3}$$

The operator T_f is called *synthesis operator* and T_f^* is called the *analysis operator* of f.

If f is a c-Bessel mapping with respect to (Ω,μ) for H, then the operator $S_f:H\longrightarrow H$ defined by $S_f=T_fT_f^*$, is called *frame operator* of f. Thus

$$\langle S_fh,k\rangle=\int_\Omega\langle h,f(\omega)\rangle\langle f(\omega),k\rangle d\mu(\omega),\quad h,k\in H.$$

If f is a c-frame for H, then S is invertible.

The converse of above proposition holds when μ is σ-finite in the measure space (Ω,μ).

Theorem 1.1. ([15]) *Suppose that (Ω,μ) is a measure space where μ is σ-finite. Let $f:\Omega\longrightarrow H$ be a mapping such that for each $h\in H$, $\omega\longmapsto\langle h,f(\omega)\rangle$ is measurable. The mapping f is a c-frame with respect to (Ω,μ) for H if and only if the operator $T_f:L^2(\Omega,\mu)\longrightarrow H$ defined by (1.2), is a bounded and onto operator.*

Definition 1.2. Let $\varphi\in\Pi_{\omega\in\Omega}H_\omega$. We say that φ is *strongly measurable* if φ as a mapping of Ω to $\oplus_{\omega\in\Omega}H_\omega$ is measurable, where

$$\Pi_{\omega\in\Omega}H_\omega=\big\{f:\Omega\longrightarrow\cup_{\omega\in\Omega}H_\omega\ ;\ f(\omega)\in H_\omega\big\}.$$

Now, we review the definition of continuous g-frames.

Definition 1.3. We call $\{\Lambda_\omega \in B(H, H_\omega) : \omega \in \Omega\}$ a *continuous generalized frame*, or simply a *cg*-frame, for H with respect to $\{H_\omega\}_{\omega\in\Omega}$, if:
(i) for each $f \in H$, $\{\Lambda_\omega f\}_{\omega\in\Omega}$ is strongly measurable,
(ii) there are two positive constants A and B such that

$$A\|f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) \leq B\|f\|^2, \quad f \in H. \tag{1.4}$$

We call A and B the lower and upper *cg*-frame bounds, respectively. If A, B can be chosen such that $A = B$, then $\{\Lambda_\omega\}_{\omega\in\Omega}$ is called a *tight cg-frame* and if $A = B = 1$, it is called a Parseval *cg*-frame. A family $\{\Lambda_\omega\}_{\omega\in\Omega}$ is called *cg-Bessel family* if the second inequality in (1.4) holds.

Now, let the space $\big(\oplus_{\omega\in\Omega} H_\omega, \mu\big)_{L^2} \subseteq \Pi_{\omega\in\Omega} H_\omega$ be defined as follows:

$$\big(\oplus_{\omega\in\Omega} H_\omega, \mu\big)_{L^2} = \Big\{\varphi |\ \varphi \textit{ is strongly measurable}, \int_\Omega \|\varphi(\omega)\|^2 d\mu(\omega) < \infty\Big\}.$$

The space $\big(\oplus_{\omega\in\Omega} H_\omega, \mu\big)_{L^2}$ is a Hilbert space with inner product

$$\langle \varphi, \psi\rangle = \int_\Omega \langle \varphi(\omega), \psi(\omega)\rangle d\mu(\omega).$$

Proposition 1.2. ([1]) *Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a cg-Bessel family for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ with Bessel bound B. Then the mapping T of $\big(\oplus_{\omega\in\Omega} H_\omega, \mu\big)_{L^2}$ to H defined by*

$$\langle T\varphi, h\rangle = \int_\Omega \langle \Lambda_\omega^* \varphi(\omega), h\rangle d\mu(\omega), \quad \varphi \in \big(\oplus_{\omega\in\Omega} H_\omega, \mu\big)_{L^2},\ h \in H, \tag{1.5}$$

is linear and bounded with $\|T\| \leq \sqrt{B}$. Furthermore for each $h \in H$ and $\omega \in \Omega$,

$$T^*(h)(\omega) = \Lambda_\omega h. \tag{1.6}$$

The operators T and T^* are called *synthesis* and *analysis* operators of *cg*-Bessel family $\{\Lambda_\omega\}_{\omega\in\Omega}$, respectively.

Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a *cg*-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ with frame bounds A, B. The operator $S : H \longrightarrow H$ defined by

$$\langle Sf, g\rangle = \int_\Omega \langle f, \Lambda_\omega^* \Lambda_\omega g\rangle d\mu(\omega), \quad f, g \in H, \tag{1.7}$$

is called the *frame operator* of $\{\Lambda_\omega\}_{\omega\in\Omega}$ which is a positive and invertible operator.

Now, we state a known result that is helpful in proving some results.

Proposition 1.3. ([4]) *Let $K : H \longrightarrow H$ be a bounded linear operator. Then the following hold.*
(i) $K = \alpha(U_1 + U_2 + U_3)$, where each $U_j,\ j = 1, 2, 3,$ is a unitary operator and α is a constant.
(ii) If K is onto, then it can be written as a linear combination of two unitary operators if and only if K is invertible.

2. cg-Orthonormal bases

Similar to the continuous frames, we want to generalize orthonormal bases. Indeed, our purpose here is to define a mapping $f:\Omega\longrightarrow H$ that has similar properties to an orthonormal basis of H.

Definition 2.1. Suppose that (Ω,μ) is a measure space. A mapping $f:\Omega\longrightarrow H$ is called a *c-orthonormal basis* with respect to (Ω,μ) for H, if:
(i) for each $h\in H$, $\omega\longmapsto\langle h,f(\omega)\rangle$ is measurable,
(ii) for almost all $\nu\in\Omega$,

$$\int_\Omega \langle f(\omega),f(\nu)\rangle d\mu(\omega)=1,$$

(iii) for each $h\in H$, $\int_\Omega |\langle h,f(\omega)\rangle|^2 d\mu(\omega)=\|h\|^2$.

Now we define the generalization of orthonormal basis in case of operators.

Definition 2.2. Assume (Ω,μ) is a measure space. A family of operators $\Lambda=\{\Lambda_\omega\in B(H,H_\omega):\omega\in\Omega\}$ is called a *continuous g-orthonormal basis* or simply a *cg-orthonormal basis*, for H with respect to $\{H_\omega\}_{\omega\in\Omega}$, whenever:
(i) for each $h\in H$, $\{\Lambda_\omega h\}_{\omega\in\Omega}$ is strongly measurable,
(ii) for almost all $\nu\in\Omega$,

$$\int_\Omega \langle \Lambda_\omega^* f_\omega,\Lambda_\nu^* g_\nu\rangle d\mu(\omega)=\langle f_\nu,g_\nu\rangle,\ \ \{f_\omega\}_{\omega\in\Omega}\in(\oplus_{\omega\in\Omega}H_\omega,\mu)_{L^2},\ g_\nu\in H_\nu,$$

(iii) for each $h\in H$, $\int_\Omega \|\Lambda_\omega h\|^2 d\mu(\omega)=\|h\|^2$.

If only conditions (i) and (ii) hold, $\Lambda=\{\Lambda_\omega\in B(H,H_\omega);\omega\in\Omega\}_{\omega\in\Omega}$ is called a *cg-orthonormal system* for H with respect to $\{H_\omega\}_{\omega\in\Omega}$.

Example 2.1. Suppose that $\Omega=\{a,b,c\}$, $\Sigma=\big\{\emptyset,\{a,b\},\{c\},\Omega\big\}$ and $\mu:\Sigma\longrightarrow[0,\infty]$ is a measure such that $\mu(\emptyset)=0$, $\mu(\{a,b\})=1$, $\mu(\{c\})=1$ and $\mu(\Omega)=2$. Let H be a 2 dimensional Hilbert space with an orthonormal basis $\{e_1,e_2\}$. We define

$$f:\Omega\longrightarrow H$$

by $f=e_1\chi_{\{a,b\}}+e_2\chi_{\{c\}}$. So for each $h\in H$,

$$\langle f(\omega),h\rangle=\langle e_1,h\rangle\chi_{\{a,b\}}(\omega)+\langle e_2,h\rangle\chi_{\{c\}}(\omega),\quad \omega\in\Omega,$$

hence $\omega\longmapsto\langle h,f(\omega)\rangle$ is measurable. Now, for each $\omega\in\Omega$, we define

$$\Lambda_\omega:H\longrightarrow\mathbb{C}$$

$$\Lambda_\omega(h)=\langle h,f(\omega)\rangle.$$

Actually, we consider for each $\omega\in\Omega$, $H_\omega=\mathbb{C}$. By an easy calculation, we have

$$\Lambda_\omega^*(z)=f(\omega)z,\ \ z\in\mathbb{C}.$$

For any $\nu\in\Omega$, $x_\nu\in\mathbb{C}$ and any $\{z_\omega\}_{\omega\in\Omega}\in(\oplus_{\omega\in\Omega}H_\omega,\mu)_{L^2}=L^2(\Omega,\mu)$, due to the Example 4.2 in [16], we have

$$\int_\Omega \langle \Lambda_\omega^* z_\omega,\Lambda_\nu^* x_\nu\rangle d\mu(\omega)=\int_\Omega z_\omega\overline{x_\nu}\langle f(\omega),f(\nu)\rangle d\mu(\omega)=z_\nu\overline{x_\nu}.$$

Also for each $h \in H$,

$$\int_{\Omega} \|\Lambda_\omega h\|^2 d\mu(\omega) = \int_{\Omega} |\langle h, f(\omega)\rangle|^2 d\mu(\omega) = \|h\|^2.$$

Therefore $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a cg-orthonormal basis for H with respect to $\{H_\omega\}_{\omega\in\Omega}$, where for each $\omega \in \Omega$, $H_\omega = \mathbb{C}$.

We present some equal conditions for cg-orthonormal bases.

Theorem 2.1. *Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a cg-orthonormal system for H with respect to $\{H_\omega\}_{\omega\in\Omega}$. Also assume that for each $h \in H$, $\int_\Omega \|\Lambda_\omega h\|^2 d\mu(\omega) < \infty$. Then the following conditions are equivalent:*
(i) $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a cg-orthonormal basis for H with respect to $\{H_\omega\}_{\omega\in\Omega}$.
(ii) For each $h, k \in H$,

$$\langle h, k\rangle = \int_{\Omega} \langle \Lambda_\omega h, \Lambda_\omega k\rangle d\mu(\omega).$$

(iii) If $\Lambda_\omega h = 0$, a.e. $[\mu]$, then $h = 0$.
(v) For each zero measure set $\Omega_0 \subseteq \Omega$, $H = \overline{span}\{\Lambda_\omega^*(H_\omega)\}_{\omega\in\Omega\setminus\Omega_0}$.

Proof. $(i) \Leftrightarrow (ii)$ Since $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a Parseval cg-frame for H, so its frame operator $S_\Lambda = I$. Hence (ii) is obvious. The converse side clearly holds.
$(ii) \Rightarrow (iii)$ If $\Lambda_\omega h = 0$, *a.e.* $[\mu]$, then for every $k \in H$,

$$\langle h, k\rangle = \int_{\Omega} \langle \Lambda_\omega h, \Lambda_\omega k\rangle d\mu(\omega) = 0.$$

Therefore $h = 0$.
$(iii) \Rightarrow (v)$ Suppose that $\Omega_0 \subseteq \Omega$ and $h \perp \overline{span}\{\Lambda_\omega^*(H_\omega)\}_{\omega\in\Omega\setminus\Omega_0}$, so for almost all $\omega \in \Omega$, $\langle \Lambda_\omega^*\Lambda_\omega h, h\rangle = 0$. Then $\|\Lambda_\omega h\|^2 = 0$, *a.e.* $[\mu]$, which implies $h = 0$.
$(v) \Rightarrow (ii)$ Let $k \in H$. Assume that

$$\mathcal{H}_k = \Big\{h \in H : \langle h, k\rangle = \int_{\Omega} \langle \Lambda_\omega h, \Lambda_\omega k\rangle d\mu(\omega)\Big\}.$$

$\mathcal{H}_k$ is a subspace of H. Also, it is closed, since if $\lim_{n\to\infty} h_n = h$, where h_n's belong to $\mathcal{H}_k$, then

$$\langle h, k\rangle = \lim_{n\to\infty} \langle h_n, k\rangle = \lim_{n\to\infty} \int_{\Omega} \langle \Lambda_\omega h_n, \Lambda_\omega k\rangle d\mu(\omega).$$

According to assumption,

$$\int_{\Omega} |\langle \Lambda_\omega h, \Lambda_\omega k\rangle| d\mu(\omega) \leq \Big(\int_{\Omega} \|\Lambda_\omega h\|^2 d\mu(\omega)\Big)^{\frac{1}{2}} \Big(\int_{\Omega} \|\Lambda_\omega g\|^2 d\mu(\omega)\Big)^{\frac{1}{2}} < \infty.$$

So by Lebesgue's Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_{\Omega} \langle \Lambda_\omega h_n, \Lambda_\omega k\rangle d\mu(\omega) = \int_{\Omega} \langle \Lambda_\omega h, \Lambda_\omega k\rangle d\mu(\omega),$$

which means $h \in \mathcal{H}_k$. For almost all $\nu \in \Omega$ and each $f \in H$, we have

$$\begin{aligned}\int_{\Omega} \langle \Lambda_\omega \Lambda_\nu^* \Lambda_\nu f, \Lambda_\omega k\rangle d\mu(\omega) &= \int_{\Omega} \langle \Lambda_\nu^* \Lambda_\nu f, \Lambda_\omega^* \Lambda_\omega k\rangle d\mu(\omega) \\ &= \langle \Lambda_\nu f, \Lambda_\nu k\rangle = \langle \Lambda_\nu^* \Lambda_\nu f, k\rangle,\end{aligned}$$

therefore $\Lambda_\nu^*\Lambda_\nu f \in \mathcal{H}_k$. Assume $f \perp \mathcal{H}_k$, then for almost all $\nu \in \Omega$,

$$0 = \langle \Lambda_\nu^*\Lambda_\nu f, f\rangle = \|\Lambda_\nu f\|^2,$$

which gives $\Lambda_\nu f = 0$. For almost all $\nu \in \Omega$ and any $g_\nu \in H_\nu$,

$$\langle f, \Lambda_\nu^* g_\nu\rangle = \langle \Lambda_\nu f, g_\nu\rangle = 0.$$

So $f \perp \overline{span}\{\Lambda_\omega^*(H_\omega)\}_{\omega\in\Omega\setminus\Omega_0}$, where Ω_0 is a zero measure subset of Ω. By condition (υ), $\overline{span}\{\Lambda_\omega^*(H_\omega)\}_{\omega\in\Omega\setminus\Omega_0} = H$. Thus $f = 0$ and $\mathcal{H}_k = H$. Therefore

$$\langle h, k\rangle = \int_\Omega \langle \Lambda_\omega h, \Lambda_\omega k\rangle d\mu(\omega), \quad h, k \in H.$$

□

In the following of this section, suppose that there exists a *cg*-orthonormal basis for H.

Proposition 2.1. *Suppose that $\{\Theta_\omega\}_{\omega\in\Omega}$ is a cg-orthonormal basis for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ and $\{\Lambda_\omega \in B(H, H_\omega) : \omega \in \Omega\}$ is a family such that for each $h \in H$, $\{\Lambda_\omega h\}_{\omega\in\Omega}$ is strongly measurable. Then $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a Parseval cg-frame for H if and only if there exists a unique isometry $V \in B(H)$ such that $\Lambda_\omega = \Theta_\omega V$, a.e. $[\mu]$.*

Proof. Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a Parseval *cg*-frame for H. We define the operator V weakly by

$$\langle Vf, h\rangle = \int_\Omega \langle \Theta_\omega^*\Lambda_\omega f, h\rangle d\mu(\omega), \quad f, h \in H.$$

For each $f, h \in H$, we have

$$|\langle Vf, h\rangle| \le \Big(\int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega)\Big)^{\frac{1}{2}} \Big(\int_\Omega \|\Theta_\omega h\|^2 d\mu(\omega)\Big)^{\frac{1}{2}} \le \|f\|\|h\|.$$

So V is well-defined and bounded. For almost all $\nu \in \Omega$ and each $f \in H$, $h_\nu \in H_\nu$,

$$\langle \Theta_\nu Vf, h_\nu\rangle = \langle Vf, \Theta_\nu^* h_\nu\rangle = \int_\Omega \langle \Theta_\omega^*\Lambda_\omega f, \Theta_\nu^* h_\nu\rangle d\mu(\omega) = \langle \Lambda_\nu f, h_\nu\rangle,$$

since $\{\Theta_\omega\}_{\omega\in\Omega}$ is a *cg*-orthonormal basis. Thus $\Lambda_\omega = \Theta_\omega V$, *a.e.* $[\mu]$.
For each $f \in H$,

$$\begin{aligned}\|Vf\|^2 = \langle Vf, Vf\rangle &= \int_\Omega \langle \Theta_\omega^*\Lambda_\omega f, Vf\rangle d\mu(\omega) = \int_\Omega \langle \Lambda_\omega f, \Theta_\omega Vf\rangle d\mu(\omega)\\ &= \int_\Omega \langle \Lambda_\omega f, \Lambda_\omega f\rangle d\mu(\omega) = \|f\|^2.\end{aligned}$$

Therefore V is an isometry.

Now, let V_1 and V_2 be two isometries such that $\Lambda_\omega = \Theta_\omega V_1$, *a.e.* $[\mu]$ and $\Lambda_\omega = \Theta_\omega V_2$, *a.e.* $[\mu]$. Then for each $f \in H$, $\Theta_\omega((V_1 - V_2)f) = 0$, *a.e.* $[\mu]$, which implies

$$0 = \int_\Omega \|\Theta_\omega(V_1 - V_2)f\|^2 d\mu(\omega) = \|(V_1 - V_2)f\|^2,$$

so $V_1 f = V_2 f$.

Conversely, let $V \in B(H)$ be a unique isometry such that $\Lambda_\omega = \Theta_\omega V$, *a.e.* $[\mu]$. For any $f \in H$,

$$\int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) = \int_\Omega \|\Theta_\omega V f\|^2 d\mu(\omega) = \|Vf\|^2 = \|f\|^2.$$

Hence $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a Parseval cg-frame for H. □

Theorem 2.2. *Assume that $\{\Theta_\omega\}_{\omega\in\Omega}$ is a cg-orthonormal basis for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ and $\{\Lambda_\omega \in B(H, H_\omega) : \omega \in \Omega\}$ is a family such that for each $h \in H$, $\{\Lambda_\omega h\}_{\omega\in\Omega}$ is strongly measurable. Then $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a cg-frame for H with bounds A and B if and only if there exists a unique $V \in B(H)$ such that $\Lambda_\omega = \Theta_\omega V$, a.e. $[\mu]$ and $AI \le V^*V \le BI$.*

Proof. Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a cg-frame for H with bounds A, B. Similar to the proof of Proposition 2.1, the operator V weakly defined by

$$\langle Vf, h\rangle = \int_\Omega \langle \Theta_\omega^* \Lambda_\omega f, h\rangle d\mu(\omega), \quad f, h \in H,$$

is a one-to-one and bounded operator such that $\Lambda_\omega = \Theta_\omega V$, *a.e.* $[\mu]$. Also for each $f \in H$,

$$\begin{aligned}\|Vf\|^2 = \langle Vf, Vf\rangle = \int_\Omega \langle \Theta_\omega^* \Lambda_\omega, Vf\rangle d\mu(\omega) &= \int_\Omega \langle \Lambda_\omega f, \Theta_\omega V f\rangle d\mu(\omega)\\ &= \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega).\end{aligned}$$

Therefore

$$A\langle f, f\rangle \le \langle V^*Vf, f\rangle \le B\langle f, f\rangle,$$

which implies $AI \le VV^* \le BI$.
The opposite implication is similar to the Proposition 2.1. □

Theorem 2.3. *Suppose that $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a cg-orthonormal basis for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ and $V \in B(H)$. Then $\{\Lambda_\omega V\}_{\omega\in\Omega}$ is a cg-orthonormal basis for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ if and only if V is unitary.*

Proof. Assume that $\{\Lambda_\omega V\}_{\omega\in\Omega}$ is a cg-orthonormal basis for H. Since $\{\Lambda_\omega\}_{\omega\in\Omega}$ is also a cg-orthonormal basis for H, for each $f \in H$, we have

$$\|Vf\|^2 = \int_\Omega \|\Lambda_\omega V f\|^2 d\mu(\omega) = \|f\|^2.$$

Hence V is an isometry and $V^*V = I$. Considering $\Theta_\omega = \Lambda_\omega V$, $\omega \in \Omega$, in Theorem 2.1, there exists a unique isometry $U \in B(H)$ such that $\Lambda_\omega = \Lambda_\omega VU$, *a.e.* $[\mu]$. Let T_Λ and $T_{\Lambda VU}$ be the synthesis operators of Parseval cg-frames $\{\Lambda_\omega\}_{\omega\in\Omega}$ and $\{\Lambda_\omega VU\}_{\omega\in\Omega}$, respectively. Then $T_\Lambda = (VU)^* T_\Lambda = U^*V^*T_\Lambda$. We deduce $T_\Lambda T_\Lambda^* = U^*V^*T_\Lambda T_\Lambda^*$ or $S_\Lambda = U^*V^*S_\Lambda$, where S_Λ is the frame operator of $\{\Lambda_\omega\}_{\omega\in\Omega}$. Since $S_\Lambda = I$, so $I = U^*V^*$ or equivalently $VU = I$. This implies that V is onto. Also V is one-to-one, so V is invertible and $V^{-1} = V^*$, which means V is a unitary.

For the reverse implication, suppose that V is a unitary operator. Now, we show that $\{\Lambda_\omega V\}_{\omega\in\Omega}$ is a cg-orthonormal basis for H. For almost all $\nu \in \Omega$, each

$g_\nu \in H_\nu$ and each $\{f_\omega\}_{\omega\in\Omega} \in (\oplus_{\omega\in\Omega} H_\omega, \mu)_{L^2}$, we have

$$\int_\Omega \langle(\Lambda_\omega V)^* f_\omega, (\Lambda_\nu V)^* g_\nu\rangle d\mu(\omega) = \int_\Omega \langle V^*\Lambda_\omega^* f_\omega, V^*\Lambda_\nu^* g_\nu\rangle d\mu(\omega)$$
$$= \int_\Omega \langle \Lambda_\omega^* f_\omega, \Lambda_\nu^* g_\nu\rangle d\mu(\omega) = \langle f_\nu, g_\nu\rangle.$$

Also for each $f \in H$,

$$\int_\Omega \|\Lambda_\omega V f\|^2 d\mu(\omega) = \|Vf\|^2 = \|f\|^2.$$

Therefore $\{\Lambda_\omega V\}_{\omega\in\Omega}$ is a cg-orthonormal basis for H with respect to $\{H_\omega\}_{\omega\in\Omega}$. □

Concerning to cg-Riesz bases which are defined in [14], we have next result.

Theorem 2.4. *Let (Ω, μ) be a measure space where μ is σ-finite. Suppose that $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a cg-Riesz basis for H and $V \in B(H)$. Then $\{\Lambda_\omega V\}_{\omega\in\Omega}$ is a cg-Riesz basis for H if and only if V is invertible.*

Proof. Let $\{\Lambda_\omega V\}_{\omega\in\Omega}$ be a cg-Riesz basis for H. By definition of a cg-Riesz basis, the operator $T_{\Lambda V}$ weakly defined by

$$\langle T_{\Lambda V}\varphi, h\rangle = \int_\Omega \langle(\Lambda_\omega V)^*\varphi(\omega), h\rangle d\mu(\omega), \quad \varphi \in (\oplus_{\omega\in\Omega} H_\omega, \mu)_{L^2}, h \in H,$$

is well-defined and there exist positive constants A and B such that

$$A\|\varphi\| \le \|T_{\Lambda V}\varphi\| \le B\|\varphi\|, \quad \varphi \in (\oplus_{\omega\in\Omega} H_\omega, \mu)_{L^2}.$$

An easy calculation shows $T_{\Lambda V} = V^* T_\Lambda$, where T_Λ is defined similarly for $\{\Lambda_\omega\}_{\omega\in\Omega}$. By Lemma 3.2 (i) in [14], $T_{\Lambda V}$ and T_Λ both are invertible. So $V^* = T_{\Lambda V} T_\Lambda^{-1}$ is invertible and V is invertible.
Conversely, let $V \in B(H)$ be invertible. If $\Lambda_\omega V f = 0$, *a.e.* $[\mu]$, then $Vf = 0$ and it implies $f = 0$. The operator $T_{\Lambda V}$ given by

$$\langle T_{\Lambda V}\varphi, h\rangle = \int_\Omega \langle(\Lambda_\omega V)^*\varphi(\omega), h\rangle d\mu(\omega), \quad \varphi \in (\oplus_{\omega\in\Omega} H_\omega, \mu)_{L^2}, h \in H,$$

is well-defined, bounded and $T_{\Lambda V} = V^* T_\Lambda$, where T_Λ is defined similar to $T_{\Lambda V}$. Also, there are positive constants A and B such that

$$A\|\varphi\| \le \|T_\Lambda\varphi\| \le B\|\varphi\|, \quad \varphi \in (\oplus_{\omega\in\Omega} H_\omega, \mu)_{L^2}.$$

For each $\varphi \in (\oplus_{\omega\in\Omega} H_\omega, \mu)_{L^2}$, $\|T_{\Lambda V}\varphi\| = \|V^* T_\Lambda\varphi\| \le B\|V^*\|\|\varphi\|$, and

$$\|T_{\Lambda V}\varphi\| = \|V^* T_\Lambda\varphi\| \ge \frac{1}{\|V^{-1}\|}\|T_\Lambda\varphi\| \ge \frac{A}{\|V^{-1}\|}\|\varphi\|.$$

This shows that $\{\Lambda_\omega V\}_{\omega\in\Omega}$ is a cg-Riesz basis for H. □

Now, we define cg-complete families as follows:

Definition 2.3. A family $\{\Lambda_\omega \in B(H, H_\omega) : \omega \in \Omega\}$ is called a cg-complete family for H with respect to $\{H_\omega\}_{\omega\in\Omega}$, if:

$$\{h : \ \Lambda_\omega h = 0, \ a.e. \ [\mu]\} = \{0\}.$$

Lemma 2.1. *$\{\Lambda_\omega\}_{\omega\in\Omega}$ is a cg-complete family for H and $V \in B(H)$. Then $\{\Lambda_\omega V\}_{\omega\in\Omega}$ is a cg-complete family for H if and only if V is one-to-one.*

Proof. Let $\{\Lambda_\omega V\}_{\omega\in\Omega}$ be a *cg*-complete family. If $Vh = 0$, then

$$\Lambda_\omega V h = 0,\ \omega \in \Omega,$$

so $h = 0$ and V is one-to-one.
Now, suppose V is one-to-one and $\Lambda_\omega V h = 0$, *a.e.* $[\mu]$. Since $\{\Lambda_\omega\}_{\omega\in\Omega}$ is *cg*-complete, $Vh = 0$. Hence $h = 0$, which implies $\{\Lambda_\omega V\}_{\omega\in\Omega}$ is *cg*-complete. □

Proposition 2.2. *Let (Ω,μ) be a measure space where μ is σ-finite and $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a cg-Bessel family for H. Then $\{\Lambda_\omega\}_{\omega\in\Omega}$ is cg-complete if and only if $\overline{R(T_\Lambda)} = H$, where T_Λ is the synthesis operator of $\{\Lambda_\omega\}_{\omega\in\Omega}$.*

Proof. Assume that $\{\Lambda_\omega\}_{\omega\in\Omega}$ is *cg*-complete. To show that $\overline{R(T_\Lambda)} = H$, it is enough to prove that if $f \in H$ and $f \perp R(T_\Lambda)$, then $f = 0$. Let $f \in H$ and $f \perp R(T_\Lambda)$, so for each $F \in (\oplus_{\omega\in\Omega} H_\omega, \mu)_{L^2}$,

$$0 = \langle T_\Lambda F, f\rangle = \int_\Omega \langle \Lambda_\omega^* F(\omega), f\rangle d\mu(\omega).$$

Since (Ω,μ) is σ-finite, there exists a family $\{\Omega_n\}_{n=1}^\infty$ of disjoint measurable subsets of Ω, such that $\Omega = \bigcup_{n=1}^\infty \Omega_n$ and $\mu(\Omega_n) < \infty$, $n \geq 1$. For each $n \geq 1$, set

$$F_n(\omega) = \begin{cases} \Lambda_\omega f, & \omega \in \Omega_n \\ 0, & otherwise \end{cases},$$

then

$$\langle T_\Lambda F_n, f\rangle = \int_\Omega \langle F_n(\omega), \Lambda_\omega f\rangle d\mu(\omega) = \|\Lambda_\omega f\|^2 \mu(\Omega_n) = 0.$$

Thus $\Lambda_\omega f = 0$, *a.e.* $[\mu]$, which implies $f = 0$. So $\overline{R(T_\Lambda)} = H$.
For the opposite implication, suppose that $\overline{R(T_\Lambda)} = H$ and there exists a $f \neq 0$ such that

$$\Lambda_\omega f = 0,\ a.e.\ [\mu].$$

There exists a sequence $\{F_n\}_{n=1}^\infty \subseteq (\oplus_{\omega\in\Omega} H_\omega, \mu)_{L^2}$ such that $\lim_{n\to\infty} T_\Lambda F_n = f$. Then

$$\begin{aligned}
\|f\|^2 = \langle f, f\rangle &= \langle \lim_{n\to\infty} T_\Lambda F_n, f\rangle = \lim_{n\to\infty} \langle T_\Lambda F_n, f\rangle \\
&= \lim_{n\to\infty} \int_\Omega \langle \Lambda_\omega^* F_n(\omega), f\rangle d\mu(\omega) \\
&= \lim_{n\to\infty} \int_\Omega \langle F_n(\omega), \Lambda_\omega f\rangle d\mu(\omega) = 0,
\end{aligned}$$

which is a contradiction. □

Remark 2.1. Let (Ω,μ) be a measure space and consider the family $\{\Lambda_\omega \in B(H, H_\omega) : \omega \in \Omega\}$. Also, suppose that $\{e_{\omega,k}\}_{\omega\in\Omega, k\in\mathbb{K}_\omega}$ is an orthonormal basis for Hilbert space $\oplus_{\omega\in\Omega} H_\omega$ such that for each $\omega \in \Omega$, $\{e_{\omega,k}\}_{k\in\mathbb{K}_\omega}$ is an orthonormal basis of H_ω and for each $h \in H$, the mapping

$$\Omega \times \mathbb{K} \longrightarrow \mathbb{C}$$
$$(\omega, k) \longmapsto \langle h, e_{\omega,k}\rangle$$

is measurable, where $\mathbb{K} = \bigcup_{\omega\in\Omega} \mathbb{K}_\omega$.

The mapping $h \longmapsto \langle \Lambda_\omega h, e_{\omega,k}\rangle$, $\omega \in \Omega$, $k \in \mathbb{K}_\omega$, defines a bounded linear functional on H. So there exist some $u_{\omega,k} \in H$ such that

$$\langle h, u_{\omega,k}\rangle = \langle \Lambda_\omega h, e_{\omega,k}\rangle, \quad h \in H,\ \omega \in \Omega,\ k \in \mathbb{K}_\omega.$$

Therefore $\Lambda_\omega h = \sum_{k\in\mathbb{K}_\omega} \langle h, u_{\omega,k}\rangle e_{\omega,k}$, $h \in H$. Since

$$\sum_{k\in\mathbb{K}_\omega} |\langle h, u_{\omega,k}\rangle e_{\omega,k}|^2 = \|\Lambda_\omega h\|^2 \le \|\Lambda_\omega\|^2 \|h\|^2,$$

so for each $\omega \in \Omega$, $\{u_{\omega,k}\}_{k\in\mathbb{K}_\omega}$ is a c-Bessel family for H_ω. Also

$$u_{\omega,k} = \Lambda^* e_{\omega,k}, \quad \omega \in \Omega,\ k \in \mathbb{K}_\omega. \tag{2.1}$$

The family $\{u_{\omega,k}\}_{\omega\in\Omega,k\in\mathbb{K}_\omega}$ is called the family induced by $\{\Lambda_\omega\}_{\omega\in\Omega}$ with respect to $\{e_{\omega,k}\}_{\omega\in\Omega,k\in\mathbb{K}_\omega}$.

Consider the mapping $u(\omega, k) : \Omega \times \mathbb{K} \longrightarrow H$ defined by

$$u(\omega,k) = \begin{cases} u_{\omega,k}\ , & k \in \mathbb{K}_\omega \\ 0\ , & otherwise \end{cases},$$

where $\mathbb{K} = \bigcup_{\omega\in\Omega} \mathbb{K}_\omega$.

For each $h \in H$, $(\omega, k) \longmapsto \langle u_{\omega,k}, h\rangle$ is measurable and

$$\begin{aligned}\int_\Omega \|\Lambda_\omega h\|^2 d\mu(\omega) &= \int_\Omega \sum_{k\in\mathbb{K}_\omega} |\langle h, u_{\omega,k}\rangle|^2 d\mu(\omega)\\ &= \int_\Omega \Big(\int_{\mathbb{K}} |\langle h, u_{\omega,k}\rangle|^2 dl(k)\Big) d\mu(\omega), \end{aligned} \tag{2.2}$$

where $l : \mathbb{K} \longrightarrow \mathbb{K}$ is the counting measure on $\mathbb{K}$. If $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a cg-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$, then u is a c-frame for H with respect to $(\Omega\times\mathbb{K}, \mu \times\ l)$ and with the same bounds of $\{\Lambda_\omega\}_{\omega\in\Omega}$.

The converse of above statement is true, too; if $\{u_{\omega,k}\}_{\omega\in\Omega,k\in\mathbb{K}_\omega}$ is a c-frame for H, then $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a cg-frame for H with the same bounds of $\{u_{\omega,k}\}_{k\in\mathbb{K}_\omega}$.

Theorem 2.5. *Let (Ω, μ) be a measure space where μ is σ-finite. Consider the family $\{\Lambda_\omega \in B(H, H_\omega); \omega \in \Omega\}$ and let $\{u_{\omega,k}\}_{\omega\in\Omega,k\in\mathbb{K}_\omega}$ be defined as in (2.1). Then $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a cg-frame (respectively cg-Bessel family, tight cg-frame, cg-Riesz basis, cg-orthonormal basis) for H if and only if $\{u_{\omega,k}\}_{\omega\in\Omega,k\in\mathbb{K}_\omega}$ is a c-frame (respectively c-Bessel family, tight c-frame, c-Riesz basis, c-orthonormal basis) for H.*

Proof. We see from (2.2) that

$$\int_\Omega \|\Lambda_\omega h\|^2 d\mu(\omega) = \int_\Omega \Big(\int_{\mathbb{K}} |\langle h, u_{\omega,k}\rangle|^2 dl(k)\Big) d\mu(\omega), \quad h \in H.$$

Hence $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a cg-frame (respectively cg-Bessel family, tight cg-frame) for H if and only if $\{u_{\omega,k}\}_{\omega\in\Omega,k\in\mathbb{K}_\omega}$ is a c-frame (respectively c-Bessel family, tight c-frame).

Now, assume that $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a cg-Riesz basis for H. So there are constants $A, B > 0$ such that the operator $T_\Lambda : (\oplus_{\omega\in\Omega} H_\omega, \mu)_{L^2} \longrightarrow H$ defined by

$$\langle T_\Lambda F, h\rangle = \int_\Omega \langle \Lambda^*_\omega F(\omega), h\rangle d\mu(\omega), \quad F \in (\oplus_{\omega\in\Omega} H_\omega, \mu)_{L^2}, h \in H,$$

satisfies in

$$A\|F\| \leq \|T_\Lambda F\| \leq B\|F\|, \;\; F \in (\oplus_{\omega\in\Omega} H_\omega, \mu)_{L^2}.$$

Consider the operator $\mathfrak{T} : L^2(\Omega \times \mathbb{K}) \longrightarrow H$ which is defined by

$$\begin{aligned}\langle \mathfrak{T}\varphi, h\rangle &= \int_\Omega \int_{\mathbb{K}} \varphi(\omega, k)\langle u_{\omega,k}, h\rangle dl(k) d\mu(\omega)\\ &= \int_\Omega \sum_{k\in\mathbb{K}_\omega} \varphi(\omega, k)\langle u_{\omega,k}, h\rangle d\mu(\omega), \quad \varphi \in L^2(\Omega \times \mathbb{K}), h \in H.\end{aligned}$$

To show that $\{u_{\omega,k}\}_{k\in\mathbb{K}_\omega}$ is a c-Riesz basis for H, it is enough to show that $\mathfrak{T}$ is one-to-one (by Theorem 2.1 in [16]). If $\mathfrak{T}\varphi = 0$, then for each $h \in H$,

$$\begin{aligned}0 = \langle \mathfrak{T}\varphi, h\rangle &= \int_\Omega \sum_{k\in\mathbb{K}_\omega} \varphi(\omega, k)\langle \Lambda^*_\omega e_{\omega,k}, h\rangle d\mu(\omega)\\ &= \int_\Omega \langle \sum_{k\in\mathbb{K}_\omega} \varphi(\omega, k) e_{\omega,k}, \Lambda_\omega h\rangle d\mu(\omega)\\ &= \int_\Omega \langle \Lambda^*_\omega \psi, h\rangle d\mu(\omega) = \langle T_\Lambda \psi, h\rangle = 0,\end{aligned}$$

where $\psi(\omega) = \sum_{k\in\mathbb{K}_\omega} \varphi(\omega, k)e_{\omega,k}, \; \omega \in \Omega$. So $T_\Lambda\psi = 0$. Since T_Λ is bounded below, $\psi = 0$. But $\|\psi\| = \|\varphi\|$, so $\varphi = 0$. Hence $\mathfrak{T}$ is one-to-one and it implies $\{u_{\omega,k}\}_{k\in\mathbb{K}_\omega}$ is a c-Riesz basis for H.
Now, let $\{u_{\omega,k}\}_{\omega\in\Omega,k\in\mathbb{K}_\omega}$ be a c-Riesz basis for H. By Theorem 3.3 in [14], it suffices to show that T_Λ is one-to-one. If $T_\Lambda\phi = 0$, then for each $h \in H$,

$$\begin{aligned}0 = \langle T_\Lambda\phi, h\rangle &= \int_\Omega \langle \Lambda^*_\omega \phi(\omega), h\rangle d\mu(\omega) = \int_\Omega \langle \Lambda^*_\omega (\sum_{k\in\mathbb{K}_\omega} \langle \phi(\omega), e_{\omega,k}\rangle e_{\omega,k}), h\rangle d\mu(\omega)\\ &= \int_\Omega \sum_{k\in\mathbb{K}_\omega} \langle \phi(\omega), e_{\omega,k}\rangle \langle \Lambda^*_\omega e_{\omega,k}, h\rangle d\mu(\omega) = (*),\end{aligned}$$

set $\varphi(\omega, k) = \langle \phi(\omega), e_{\omega,k}\rangle$, $\omega \in \Omega, k \in \mathbb{K}_\omega$, then

$$\int_\Omega \sum_{k\in\mathbb{K}_\omega} |\langle \phi(\omega), e_{\omega,k}\rangle|^2 d\mu(\omega) = \int_\Omega \|\phi(\omega)\|^2 d\mu(\omega) = \|\phi\|^2.$$

So $\varphi \in L^2(\Omega, \mathbb{K})$ and $\|\varphi\| = \|\phi\|$. Also

$$(*) = \int_\Omega \int_{\mathbb{K}} \varphi(\omega, k)\langle \Lambda^*_\omega e_{\omega,k}, h\rangle dl(k) d\mu(\omega).$$

So for each $h \in H$, $0 = \langle \mathfrak{T}\varphi, h\rangle = 0$, hence $\mathfrak{T}\varphi = 0$. Since $\{u_{\omega,k}\}_{\omega\in\Omega,k\in\mathbb{K}_\omega}$ is a c-Riesz basis for H, then $\mathfrak{T}$ is invertible, which implies $\varphi = 0$ and $\phi = 0$. Thus T_Λ is one-to-one.
Now, suppose that $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a cg-orthonormal basis for H. For almost all $\nu \in \Omega$

and all $m \in \mathbb{K}$,

$$\begin{aligned}\int_{\Omega}\int_{\mathbb{K}} \langle u_{\omega,k}, u_{\nu,m}\rangle dl(k)d\mu(\omega) &= \int_{\Omega}\int_{\mathbb{K}} \langle \Lambda_{\omega}^* e_{\omega,k}, \Lambda_{\nu}^* e_{\nu,m}\rangle dl(k)d\mu(\omega) \\ &= \int_{\mathbb{K}}\int_{\Omega} \langle \Lambda_{\omega}^* e_{\omega,k}, \Lambda_{\nu}^* e_{\nu,m}\rangle d\mu(\omega)dl(k) \\ &= \int_{\mathbb{K}} \langle e_{\nu,k}, e_{\nu,m}\rangle dl(k) = 1.\end{aligned}$$

Also, for each $h \in H$,

$$\int_{\Omega} \|\Lambda_{\omega} h\|^2 d\mu(\omega) = \int_{\Omega}\int_{\mathbb{K}} |\langle h, u_{\omega,k}\rangle|^2 dl(k)d\mu(\omega) = \|h\|^2.$$

So $\{u_{\omega,k}\}_{\omega\in\Omega,k\in\mathbb{K}_{\omega}}$ is a c-orthonormal basis.

The opposite implications are similar. □

3. cg-Orthonormal bases and cg-frames

At first, we present some result on cg-frames which are constructed by composing with operators.

Proposition 3.1. *Let $\{\Lambda_{\omega}\}_{\omega\in\Omega}$ be a cg-frame for H with bounds A and B and $V \in B(H)$. Then $\{\Lambda_{\omega} V\}_{\omega\in\Omega}$ is a cg-frame for H if and only if there exists a positive constat α such that*

$$\|Vf\|^2 \geq \alpha\|f\|^2, \quad f \in H.$$

Proof. Suppose $\{\Lambda_{\omega} V\}_{\omega\in\Omega}$ is a cg-frame for H with bounds C and D. For each $f \in H$,

$$C\|f\|^2 \leq \int_{\Omega} \|\Lambda_{\omega} Vf\|^2 d\mu(\omega) \leq B\|Vf\|^2,$$

so $\|Vf\|^2 \geq \frac{C}{B}\|f\|^2$. Set $\alpha = \frac{C}{B}$, then the proof is done.
Conversely, let α be such that

$$\|Vf\|^2 \geq \alpha\|f\|^2, \quad f \in H.$$

For each $f \in H$,

$$\int_{\Omega} \|\Lambda_{\omega} Vf\|^2 d\mu(\omega) \leq B\|Vf\|^2 \leq B\|V\|^2\|f\|^2,$$

and

$$\int_{\Omega} \|\Lambda_{\omega} Vf\|^2 d\mu(\omega) \geq A\|Vf\|^2 \geq A\alpha\|f\|^2.$$

Hence $\{\Lambda_{\omega} V\}_{\omega\in\Omega}$ is a cg-frame for H. □

Corollary 3.1. *Let $\{\Lambda_{\omega}\}_{\omega\in\Omega}$ be a cg-frame for H and $V \in B(H)$. Then $\{\Lambda_{\omega} V\}_{\omega\in\Omega}$ is a cg-frame for H if and only if V^* is onto.*

Proof. By Lemma 2.4.1 (iii) in [5], it is obvious. □

Corollary 3.2. *Let M be a close subspace of H and $\{\Lambda_{\omega}\}_{\omega\in\Omega}$ be a cg-frame for H and $V \in B(H, M)$. Then $\{\Lambda_{\omega} V^*\}_{\omega\in\Omega}$ is a cg-frame for M if and only if there exists a positive constat α such that*

$$\|V^* f\|^2 \geq \alpha\|f\|^2, \quad f \in M.$$

Corollary 3.3. *Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a tight cg-frame for H with frame bound A and $V \in B(H)$. Then $\{\Lambda_\omega V\}_{\omega\in\Omega}$ is a tight cg-frame for H with frame bound α if and only if*

$$\|Vf\|^2 = \frac{\alpha}{A}\|f\|^2, \quad f \in H.$$

Proposition 3.2. *Suppose that $\{\Theta_\omega\}_{\omega\in\Omega}$ is a cg-orthonormal basis for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ and $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a cg-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$. Then there exists a bounded and one-to-one operator V on H such that $\Lambda_\omega = \Theta_\omega V$, a.e. $[\mu]$. Furthermore, V is invertible if $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a cg-Riesz basis for H and V is unitary if $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a cg-orthonormal basis for H.*

Proof. By the proof of Theorem 2.2, the first part is obvious. If $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a *cg*-Riesz basis for H, then by definition of V in the proof of Theorem 2.2, $V = T_\Theta T_\Lambda^*$. Theorem 3.3 in [14] implies that T_Λ^* is onto, So V is onto and consequently V is invertible. If $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a *cg*-orthonormal for H, then Theorem 2.3 implies the result. □

Proposition 3.3. *Suppose that $\{\Theta_\omega\}_{\omega\in\Omega}$ is a cg-orthonormal basis for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ and $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a cg-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$. Then there exist cg-orthonormal bases $\{\Psi_\omega\}_{\omega\in\Omega}$, $\{\Gamma_\omega\}_{\omega\in\Omega}$ and $\{\Phi_\omega\}_{\omega\in\Omega}$ for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ and a constant α such that*

$$\Lambda_\omega = \alpha(\Psi_\omega + \Gamma_\omega + \Phi_\omega), \ \textit{a.e. } [\mu].$$

Proof. Due to Proposition 3.2 and Proposition 1.3, we have an operator $V \in B(H)$ so that $V = \alpha(U_1 + U_2 + U_3)$, where each U_j, $j = 1, 2, 3$, is a unitary operator and α is a constant. Then $\Lambda_\omega = \Theta_\omega V = \alpha(\Theta_\omega U_1 + \Theta_\omega U_2 + \Theta_\omega U_3)$, *a.e.* $[\mu]$. It is obvious that every $\{\Theta_\omega U_j\}_{\omega\in\Omega}$, $j = 1, 2, 3$, is a *cg*-orthonormal basis for H. Assuming $\Psi_\omega = \Theta_\omega U_1$, *a.e.* $[\mu]$, $\Gamma_\omega = \Theta_\omega U_2$, *a.e.* $[\mu]$ and $\Phi_\omega = \Theta_\omega U_3$, *a.e.* $[\mu]$, the proof is completed. □

Proposition 3.4. *Consider $\{\Theta_\omega\}_{\omega\in\Omega}$ as a cg-orthonormal basis for H with respect to $\{H_\omega\}_{\omega\in\Omega}$. If $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a cg-Riesz basis for H, then $\{\Lambda_\omega\}_{\omega\in\Omega}$ is sum of two cg-orthonormal bases for H with respect to $\{H_\omega\}_{\omega\in\Omega}$.*

Proof. Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a *cg*-Riesz basis for H. Proposition 3.2 implies that there exists an invertible $V \in B(H)$ such that $\Lambda_\omega = \Theta_\omega V$, *a.e.* $[\mu]$. Via Proposition 1.3, V can be written as $V = aU_1 + bU_2$, where U_1 and U_2 are unitary operators. The rest of proof is similar to the proof of Proposition 3.3. □

Composing of a *cg*-orthonormal basis and an isometry, gives us a Parseval *cg*-frame.

Proposition 3.5. *If $V \in B(H)$ is an isometry and $\{\Theta_\omega\}_{\omega\in\Omega}$ is a cg-orthonormal basis for H, then $\{\Theta_\omega\Lambda\}_{\omega\in\Omega}$ is a Parseval cg-frame for H.*

Proof. A straightforward calculation gives the proof. □

Every bounded operator V on H has a representation in the form $V = U|V|$ (called the *polar decomposition* of V), where U is a partial isometry, $|V|$ is a positive operator defined by $|V| = \sqrt{V^*V}$ and $kerU = kerV$.
Also, every positive operator P on H with $\|P\| \leq 1$ can be written in the form

$P = \frac{1}{2}(W + W^*)$, where $W = P + i\sqrt{1 - P^2}$ is unitary.

Next theorem shows that we can represent a *cg*-frame by some Parseval *cg*-frames.

Theorem 3.1. *Suppose that* $\{\Theta_\omega\}_{\omega\in\Omega}$ *is a cg-orthonormal basis for* H. *Every cg-frame for* H *can be written as a linear combination of two Parseval cg-frames.*

Proof. By Proposition 3.2, there exists a bounded and one-to-one operator $V \in B(H)$ such that $\Lambda_\omega = \Theta_\omega V$, *a.e.* $[\mu]$. By above note, V can be written as $V = \frac{1}{2}(UW + UW^*)$, where U is an isometry and W is unitary. So UW and UW^* are isometries. Proposition 3.5 implies that $\{\Theta_\omega UW\}_{\omega\in\Omega}$ and $\{\Theta_\omega UW^*\}_{\omega\in\Omega}$ are Parseval *cg*-frames for H. □

Now, we can show each *cg*-frame as a combination of a *cg*-orthonormal basis and a *cg*-Riesz basis of H.

Theorem 3.2. *Assuming* $\{\Theta_\omega\}_{\omega\in\Omega}$ *as a cg-orthonormal basis for* H, *Every cg-frame for* H *is sum of a cg-orthonormal basis for* H *and a cg-Riesz basis for* H.

Proof. The proof is similar to the proof of Theorem 4.2 in [16].

□

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Received: May 30, 2019; Revised: July 24, 2019; Accepted: January 16, 2020