

## PARA-COMPLEX STRUCTURES ON LINEAR COFRAME BUNDLE WITH SASAKIAN METRIC

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**Abstract.** By using a Riemannian metric on a differentiable manifold, the Sasakian metric is introduced on the linear coframe bundle of the Riemannian manifold. Geometric properties of Levi-Civita connection of Sasakian metric are investigated. Also, para-complex structures on the linear coframe bundle with Sasakian metric are constructed and some interesting properties of those structures are studied.

### 1. Introduction

Let  $M$  be an  $n$ -dimensional manifold of class  $C^\infty$ . The problem of extending differential-geometrical structures on  $M$  to its fiber bundles has been the subject of a number of papers. An account of these can be found in Yano and Ishihara [22] (see, also [2]). In [21], Sasaki by using a Riemannian metric  $g$  on a differentiable manifold  $M$ , constructed a Riemannian metric  $\tilde{g}$  on the tangent bundle  $T(M)$  of  $M$ . Then, some geometers such as Kowalski, Aso, Musso and Tricerri studied interesting geometric properties of this metric, that is called Sasaki metric (see [1], [8], [11]). Some properties and applications for the Riemannian metrics of the cotangent, linear frame, linear coframe and tensor bundles are given in [3, 5, 9, 10, 16-18]. Also noteworthy are the papers devoted to the study of various differential geometric structures, including submanifolds of Sasakian manifolds. V.A.Khan and M.A.Khan studied pseudo-slant submanifolds of Sasakian manifolds (see [7]). The main results related to the projective curvature tensor in Sasakian manifolds are due to U.K.Gautam, A.Haseeb and R.Prasad (see [6]).

Let  $M_{2k}$  be a  $2k$ -dimensional differentiable manifold endowed with an almost (para) complex structure  $\varphi$  and a pseudo-Riemannian metric  $g$  of signature  $(k, k)$  such that  $g(\varphi X, Y) = g(X, \varphi Y)$  for arbitrary vector fields  $X$  and  $Y$  on  $M_{2k}$ , i.e.  $g$  is pure with respect to  $\varphi$ . The metric  $g$  is called Nordenian metric. Nordenian metrics are also referred to as anti-Hermitian metrics or B-metrics. They present extensive application in mathematics as well as in theoretical physics. Many authors considered almost (para) complex Nordenian structures on the tangent, cotangent and tensor bundles [4, 12-15].

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This paper is devoted to the investigation of para-Nordenian structures in the linear coframe bundle with Sasakian metric. In 2 we briefly describe the definitions and results that are needed later, after which the Sasakian metric  ${}^Sg$  and the Levi-Civita connection  ${}^S\nabla$  of this metric in the linear coframe bundle  $F^*(M)$  over a Riemannian manifold  $(M, g)$  are studied in 3. The para-Nordenian structures on the linear coframe bundle  $F^*(M)$  with Sasakian metric are introduced in 4. In 5 we study the almost para-holomorphic vector fields on the linear coframe bundle  $F^*(M)$  with Sasakian metric. The integrabilities of almost para-Nordenian structures on the  $F^*(M)$  with Sasakian metric are investigated in 6.

## 2. Preliminaries

In this section we shall summarize briefly the basic definitions and results which be used later. Let  $(M, g)$  be an  $n$ -dimensional Riemannian manifold and  $F^*(M)$  its coframe bundle (see, [19,20]). The coframe bundle  $F^*(M)$  over  $M$  consists of all pairs  $(x, u^*)$ , where  $x$  is a point of  $M$  and  $u^*$  is a basis (coframe) for the cotangent space  $T_x^*M$ . We denote by  $\pi$  the natural projection of  $F^*(M)$  to  $M$  defined by  $\pi(x, u^*) = x$ . If  $(U; x^1, x^2, \dots, x^n)$  is a system of local coordinates in  $M$ , then a coframe  $u^* = (X^\alpha) = (X^1, X^2, \dots, X^n)$  for  $T_x^*M$  can be expressed uniquely in the form  $X^\alpha = X_i^\alpha(dx^i)_x$  and hence

$$(\pi^{-1}(U); x^1, x^2, \dots, x^n, X_1^1, X_2^1, \dots, X_n^n)$$

is a system of local coordinates in  $F^*(M)$  (see, [19]). Indices  $i, j, k, \dots, \alpha, \beta, \gamma, \dots$  have range in  $\{1, 2, \dots, n\}$ , while indices  $A, B, C, \dots$  have range in

$$\{1, \dots, n, n+1, \dots, n+n^2\}.$$

We put  $h_\alpha = \alpha \cdot n + h$ . Summation over repeated indices is always implied.

We denote by  $\mathfrak{S}_s^r(M)$  the set of all differentiable tensor fields of type  $(r, s)$  on  $M$ . Let  $V = V^i\partial_i$  and  $\omega = \omega_i dx^i$  be the local expressions in  $U \subset M$  of a vector and a covector (1-form) fields  $V \in \mathfrak{S}_0^1(M)$  and  $\omega \in \mathfrak{S}_1^0(M)$ , respectively. Then the complete and horizontal lifts  ${}^C V, {}^H V \in \mathfrak{S}_0^1(F^*(M))$  of  $V$  and the  $\beta$ -th vertical lifts  ${}^{V_\beta} \omega \in \mathfrak{S}_0^1(F^*(M))$  ( $\beta = 1, 2, \dots, n$ ) of  $\omega$  are given, respectively, by

$${}^C V = \begin{pmatrix} V^i \\ -X_j^\alpha \partial_i V^j \end{pmatrix}, \quad {}^H V = \begin{pmatrix} V^i \\ X_j^\alpha \Gamma_{ik}^j V^k \end{pmatrix}, \quad {}^{V_\beta} \omega = \begin{pmatrix} 0 \\ \delta_\beta^\alpha \omega_i \end{pmatrix} \quad (2.1)$$

with respect to the natural frame  $\{\partial_i, \partial_{i_\alpha}\} = \left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial X_i^\alpha} \right\}$ , (see [10] for more details).

The vertical lift of a smooth function  $f$  on  $M$  is a function  ${}^V f$  on  $F^*(M)$  defined by  ${}^V f = f \circ \pi$ .

Let  $(U, x^i)$  be a coordinate system in  $M$ . In  $U \in M$ , we put

$$X_{(i)} = \frac{\partial}{\partial x^i}, \theta^{(i)} = dx^i, i = 1, 2, \dots, n.$$

Taking account of (2.1), we easily see that the components of  ${}^H X_{(i)}$  and  ${}^{V_\alpha} \theta^{(i)}$  are respectively, given by

$$D_i = {}^H X_{(i)} = (A_i^H) = \begin{pmatrix} \delta_i^h \\ X_j^\alpha \Gamma_{ih}^j \end{pmatrix}, \quad (2.2)$$

$$D_{i_\alpha} = V_\alpha \theta^{(i)} = (A_{i_\alpha}^H) = \begin{pmatrix} 0 \\ \delta_\beta^\alpha \delta_h^i \end{pmatrix} \quad (2.3)$$

with respect to the natural frame  $\{\partial_i, \partial_{i_\alpha}\}$ . We call the set  $\{{}^H X_{(i)}, V_\alpha \theta^{(i)}\}$  the frame adapted to the Levi-Civita connection  $\nabla_g$ . On putting

$$D_i = {}^H X_{(i)}, \quad D_{i_\alpha} = V_\alpha \theta^{(i)},$$

we write the adapted frame as  $\{D_I\} = \{D_i, D_{i_\alpha}\}$ . From equations (2.2), (2.3), and (2.1) we see that  ${}^H V$  and  $V_\alpha \omega$  have respectively, components

$${}^H V = V^i D_i, \quad {}^H V = ({}^H V^I) = \begin{pmatrix} V^i \\ 0 \end{pmatrix}, \quad (2.4)$$

$$V_\alpha \omega = \sum_i \omega_i \delta_\alpha^\beta D_{i_\alpha}, \quad V_\alpha \omega = (V_\alpha \omega^I) = \begin{pmatrix} 0 \\ \delta_\alpha^\beta \omega_i \end{pmatrix} \quad (2.5)$$

with respect to the adapted frame  $\{D_I\}$ , where  $V^i$  and  $\omega_i$  being local components of  $V \in \mathfrak{S}_0^1(M)$  and  $\omega \in \mathfrak{S}_1^0(M)$ , respectively.

For each  $x \in M$ , the scalar product  $g^{-1} = (g^{ij})$  is defined on the cotangent space  $T_x^*M$  by  $g^{-1}(\omega, \theta) = g^{ij} \omega_i \theta_j$  for all  $\omega, \theta \in \mathfrak{S}_1^0(M)$ .

The bracket operation of vertical and horizontal vector fields is given by the formulas

$$\begin{aligned} [{}^{V_\beta} \omega, V_\gamma \theta] &= 0, \quad [{}^H X, V_\beta \omega] = V_\beta (\nabla_X \omega), \\ [{}^H X, {}^H Y] &= {}^H [X, Y] + \sum_{\sigma=1}^n V_\sigma (X^\sigma \circ R(X, Y)) \end{aligned} \quad (2.6)$$

for all  $X, Y \in \mathfrak{S}_0^1(M)$  and  $\omega, \theta \in \mathfrak{S}_1^0(M)$ , where  $R$  denotes the curvature tensor field of the linear connection  $\nabla$ .

### 3. The Sasakian metric on the coframe bundle

We define a Riemannian metric  ${}^S g$  on the coframe bundle  $F^*(M)$  by the following three equations

$${}^S g({}^H X, {}^H Y) = V(g(X, Y)) = g(X, Y) \circ \pi, \quad (3.1)$$

$${}^S g({}^H X, V_\beta \omega) = 0, \quad (3.2)$$

$${}^S g(V_\beta \omega, V_\gamma \theta) = \delta^{\beta\gamma} V(g^{-1}(\omega, \theta)) = \delta^{\beta\gamma} (g^{-1}(\omega, \theta) \circ \pi) \quad (3.3)$$

for any  $X, Y \in \mathfrak{S}_0^1(M)$  and  $\omega, \theta \in \mathfrak{S}_1^0(M)$  (see, [5]). We call the metric  ${}^S g$  the Sasakian metric on the coframe bundle  $F^*(M)$  over the Riemannian manifold  $(M, g)$ . Since any tensor field of type  $(0, 2)$  on the  $F^*(M)$  is completely determined by its action on vector fields of type  ${}^H X$  and  $V_\beta \omega$  it follows that  ${}^S g$  is completely determined by its eqs (3.1), (3.2) and (3.3). The metric  ${}^S g$  is a Riemannian metric on  $F^*(M)$  uniquely determined by the metric  $g$ .

From equations (3.1), (3.2) and (3.3) it follows that

$$\begin{aligned} {}^S g_{ij} &= {}^S g(D_i, D_j) = V(g(\partial_i, \partial_j)) = g_{ij}, \\ {}^S g_{i_\alpha j} &= {}^S g(D_{i_\alpha}, D_j) = 0, \\ {}^S g_{i_\alpha j_\beta} &= {}^S g(D_{i_\alpha}, D_{j_\beta}) = \delta^{\alpha\beta} V(g^{-1}(dx^i, dx^j)) = \delta^{\alpha\beta} g^{ij}, \end{aligned}$$

i.e.  ${}^Sg$  has components in the form

$${}^Sg = \begin{pmatrix} g_{ij} & 0 \\ 0 & \delta^{\alpha\beta} g^{ij} \end{pmatrix} \quad (3.4)$$

with respect to the adapted frame  $\{D_I\}$ .

Let us consider local 1-forms  $\tilde{\eta}^I$  in  $\pi^{-1}(U)$  defined by

$$\tilde{\eta}^I = \bar{A}^I{}_J dx^J,$$

where

$$A^{-1} = (\bar{A}^I{}_J) = \begin{pmatrix} \bar{A}^i{}_j & \bar{A}^i{}_{j\beta} \\ \bar{A}^{\alpha}{}_j & \bar{A}^{\alpha}{}_{j\beta} \end{pmatrix} = \begin{pmatrix} \delta_j^i & 0 \\ -X_m^\alpha \Gamma_{ij}^m & \delta_\beta^\alpha \delta_i^j \end{pmatrix}. \quad (3.5)$$

The matrix (3.8) is the inverse of the matrix

$$A = (A_K{}^J) = \begin{pmatrix} A_k^j & A_{k\gamma}^j \\ A_k^{j\beta} & A_{k\gamma}^{j\beta} \end{pmatrix} = \begin{pmatrix} \delta_k^j & 0 \\ X_m^\beta \Gamma_{jk}^m & \delta_\gamma^\beta \delta_j^k \end{pmatrix} \quad (3.6)$$

of the transformation  $D_K = A_K{}^J \partial_J$  ( see (2.2) and (2.3)). It is easy to establish that the set  $\{\tilde{\eta}^I\}$  is the coframe dual to the adapted frame  $\{D_K\}$ , i.e.

$$\tilde{\eta}^I(D_K) = \bar{A}^I{}_J A_K{}^J = \delta_K^I.$$

Since the adapted frame is non-holonomic, we put

$$[D_I, D_J] = \Omega_{IJ}{}^K D_K$$

from which we have

$$\Omega_{IJ}{}^K = (D_I A_J^L - D_J A_I^L) \bar{A}_L^K.$$

According to (2.2), (2.3), (3.5) and (3.6), the components of non-holonomic object  $\Omega_{IJ}{}^K$  are given by

$$\begin{cases} \Omega_{ij\beta}{}^{k\gamma} = -\Omega_{j\beta i}{}^{k\gamma} = -\delta_\beta^\gamma \Gamma_{ik}^j, \\ \Omega_{ij}{}^{k\gamma} = X_m^\gamma R_{ijk}^m, \end{cases} \quad (3.7)$$

all the others being zero, where  $R_{ijk}^m$  local components of the curvature tensor field  $R$  of  $\nabla_g$ .

Let  ${}^S\nabla$  be the Levi-Civita connection determined by the Sasakian metric  ${}^Sg$  on the coframe bundle  $F^*(M)$ . We put

$${}^S\nabla_{D_I} D_J = {}^S\Gamma_{IJ}^K D_K.$$

From the equation

$${}^S\nabla_X Y - {}^S\nabla_Y X = [X, Y], \forall X, Y \in \mathfrak{S}_0^1(F^*(M))$$

we have

$${}^S\Gamma_{IJ}^K - {}^S\Gamma_{JI}^K = \Omega_{IJ}{}^K. \quad (3.8)$$

The equation

$$({}^S\nabla_X {}^Sg)(Y, Z) = 0$$

has form

$$D_L {}^Sg_{IJ} - {}^S\Gamma_{LI}^K {}^Sg_{KJ} - {}^S\Gamma_{LJ}^K {}^Sg_{IK} = 0 \quad (3.9)$$

with respect to the adapted frame  $\{D_K\}$ . By using (3.8) and (3.9), we obtain:

$$\begin{aligned} {}^S\Gamma_{IJ}^K &= \frac{1}{2} {}^Sg^{KL} (D_I {}^Sg_{LJ} + D_J {}^Sg_{IL} - D_L {}^Sg_{IJ}) + \\ &+ \frac{1}{2} (\Omega_{IJ}{}^K + \Omega_{IK}{}^J + \Omega_{JK}{}^I), \end{aligned} \quad (3.10)$$

where  $\Omega^K_{IJ} = Sg^{KLS}g_{PJ}\Omega_{LI}^P$  and

$$(Sg)^{-1} = (Sg^{KJ}) = \begin{pmatrix} g^{kj} & 0 \\ 0 & \delta_{\gamma\beta}g_{kj} \end{pmatrix}. \tag{3.11}$$

Taking account (3.4), (3.7) and (3.11), we obtain from (3.10)

$$\begin{cases} S\Gamma_{ij}^k = \Gamma_{ij}^k, & S\Gamma_{i\alpha j\beta}^k = S\Gamma_{i\alpha j}^{k\gamma} = S\Gamma_{i\alpha j\beta}^{k\gamma} = 0, \\ S\Gamma_{ij\beta}^k = \frac{1}{2}X_m^\beta R_{i\cdot}^{kjm}, & S\Gamma_{i\alpha j}^k = \frac{1}{2}X_m^\alpha R_{\cdot j}^{kim}, \\ S\Gamma_{ij}^{k\gamma} = \frac{1}{2}X_m^\gamma R_{ijk}^m, & S\Gamma_{ij\beta}^{k\gamma} = -\delta_\gamma^\beta \Gamma_{ik}^j. \end{cases} \tag{3.12}$$

Let  $\tilde{X}, \tilde{Y} \in \mathfrak{S}_0^1(F^*(M))$  and  $\tilde{X} = \tilde{X}^I D_I, \tilde{Y} = \tilde{Y}^J D_J$ . Then the covariant derivative  ${}^S\nabla_{\tilde{Y}}\tilde{X}$  along  $\tilde{Y}$  has components in the form

$${}^S\nabla_{\tilde{Y}}\tilde{X}^I = \tilde{Y}^J D_J \tilde{X}^I + S\Gamma_{JK}^I \tilde{X}^K \tilde{Y}^J \tag{3.13}$$

with respect to the adapted frame  $\{D_I\}$ .

Using (2.4)-(2.6), (3.1)-(3.3), (3.12) and (3.13), we have

**Theorem 3.1.** *Let  $M$  be a Riemannian manifold with metric  $g$  and  ${}^S\nabla$  be the Levi-Civita connection of the coframe bundle  $F^*(M)$  equipped with Sasakian metric  ${}^Sg$ . Then  ${}^S\nabla$  satisfies*

$$\begin{aligned} i) & {}^S\nabla_{V_{\alpha\omega}} V_\beta \theta = 0, \\ ii) & {}^S\nabla_{V_{\alpha\omega}} H Y = \frac{1}{2} H \left( R(\tilde{X}^\alpha, \tilde{\omega}) Y \right), \\ iii) & {}^S\nabla_{H X} V_\beta \theta = V_\beta (\nabla_X \theta) + \frac{1}{2} H \left( R(\tilde{X}^\beta, \tilde{\theta}) X \right), \\ iv) & {}^S\nabla_{H X} H Y = H (\nabla_X Y) + \frac{1}{2} \sum_{\sigma=1}^n V_\sigma (X^\sigma \circ R(X, Y)) \end{aligned}$$

for all  $X, Y \in \mathfrak{S}_0^1(M)$  and  $\omega, \theta \in \mathfrak{S}_1^0(M)$ , where  $\tilde{\omega} = g^{-1} \circ \omega \in \mathfrak{S}_0^1(M)$ ,  $\tilde{\theta} = g^{-1} \circ \theta \in \mathfrak{S}_0^1(M)$ ,  $\tilde{X}^\alpha = g^{-1} \circ X^\alpha \in \mathfrak{S}_0^1(M)$ .

We note that the analogue of Theorem 3.1 in the case of cotangent bundle is proved in [18].

#### 4. Para-Nordenian structures on $(F^*(M), {}^Sg)$

An almost para-complex manifold is an almost product manifold  $(M_n, \varphi)$ ,  $\varphi^2 = I$ , such that the two eigenbundles  $T^+(M_n)$  and  $T^-(M_n)$  associated to the two eigenvalues  $+1$  and  $-1$  of  $\varphi$ , respectively, have the same rank. The dimension of an almost paracomplex manifold is necessarily even.

A tensor field  $t \in \mathfrak{S}_q^0(M_{2n})$  is said to be a pure with respect to the para-complex structure  $\varphi$ , if

$$t(\varphi X_1, X_2, \dots, X_q) = t(X_1, \varphi X_2, \dots, X_q) = t(X_1, X_2, \dots, \varphi X_q)$$

for any  $X_1, X_2, \dots, X_q \in \mathfrak{S}_0^1(M_{2n})$ .

We define the following operator  $\phi_\varphi$  associated with  $\varphi$  and apply to the pure tensor field  $t$  :

$$(\phi_\varphi t)(Y, X_1, X_2, \dots, X_q) = (\varphi Y)(t(X_1, X_2, \dots, X_q))$$

$$-Y(t(\varphi X_1, X_2, \dots, X_q)) + t((L_{X_1}\varphi)Y, X_2, \dots, X_q) \\ + \dots + t(X_1, X_2, \dots, (L_{X_q}\varphi)Y).$$

We note that  $\phi_\varphi t \in \mathfrak{S}_q^0(M_{2n})$ . If  $\phi_\varphi t = 0$  then  $t$  is said to be an almost para-holomorphic (see [5, 15, 16, 18]).

**Definition 4.1.** In a manifold with almost para-complex structure  $\varphi$ , a vector field  $X$  is called an almost para-holomorphic vector field if  $L_X\varphi = 0$ .

A Riemannian manifold  $(M_{2n}, g)$  with an almost para-complex structure  $\varphi$  is said to be almost para-Nordenian if the Riemannian metric  $g$  is pure with respect to  $\varphi$ . It is well known that the almost para-Nordenian manifold is para-Kahler ( $\nabla_g\varphi = 0$ ) if and only if  $g$  is para-holomorphic ( $\phi_\varphi g = 0$ ) (see [18]).

Let  $(F^*(M), Sg)$  be the linear coframe bundle with the Sasakian metric  $Sg$ . Define a tensor field  $F_\alpha$  of type  $(1, 1)$  on  $F^*(M)$  for each  $\alpha = 1, 2, \dots, n$ , by

$$F_\alpha({}^H X) = V_\alpha \tilde{X}, \quad F_\alpha({}^{V_\beta} \omega) = \delta_\alpha^\beta {}^H \tilde{\omega} \quad (4.1)$$

for any  $X \in \mathfrak{S}_0^1(M)$  and  $\omega \in \mathfrak{S}_1^0(M)$ , where  $\tilde{X} = g \circ X \in \mathfrak{S}_1^0(M)$ ,  $\tilde{\omega} = g^{-1} \circ \omega \in \mathfrak{S}_0^1(M)$  and the horizontal lifts are considered with respect to the Levi-Civita connection of  $g$ . Each  $F_\alpha$  satisfies the condition

$$F_\alpha^2 = I.$$

Indeed, by virtue of (4.1), we have

$$F_\alpha^2({}^H X) = F_\alpha(F_\alpha({}^H X)) = F_\alpha(V_\alpha \tilde{X}) = \delta_\alpha^\alpha {}^H \tilde{X} = {}^H X, \\ F_\alpha^2({}^{V_\beta} \omega) = F_\alpha(F_\alpha({}^{V_\beta} \omega)) = F_\alpha(\delta_\alpha^\beta {}^H \tilde{\omega}) = \delta_\alpha^\beta {}^{V_\alpha} \tilde{\omega} = {}^{V_\beta} \omega$$

for any  $X \in \mathfrak{S}_0^1(M)$  and  $\omega \in \mathfrak{S}_1^0(M)$ , which implies  $F_\alpha^2 = I$  for each  $\alpha = 1, 2, \dots, n$ .

The following theorem holds.

**Theorem 4.1.** *The triple  $(F^*(M), Sg, F_\alpha)$ , for each  $\alpha = 1, 2, \dots, n$ , is an almost para-Nordenian manifold.*

*Proof.* If we put

$$A(\tilde{X}, \tilde{Y}) = Sg(F_\alpha \tilde{X}, \tilde{Y}) - Sg(\tilde{X}, F_\alpha \tilde{Y})$$

for any  $\tilde{X}, \tilde{Y} \in \mathfrak{S}_0^1(F^*(M))$ . Then from (3.1)-(3.3) and (4.1), we have

$$A({}^H X, {}^H Y) = Sg(F_\alpha({}^H X), {}^H Y) - Sg({}^H X, F_\alpha({}^H Y)) \\ = Sg(V_\alpha \tilde{X}, {}^H Y) - Sg({}^H X, V_\alpha \tilde{Y}) = 0, \\ A({}^H X, {}^{V_\beta} \omega) = Sg(F_\alpha({}^H X), {}^{V_\beta} \omega) - Sg({}^H X, F_\alpha({}^{V_\beta} \omega)) \\ = Sg(V_\alpha \tilde{X}, {}^{V_\beta} \omega) - Sg({}^H X, \delta_\alpha^\beta {}^H \tilde{\omega}) \\ \delta^{\alpha\beta} g^{-1}(gX, \omega) - g(X, g^{-1}\omega) \delta_\alpha^\beta = 0, \\ A({}^{V_\beta} \omega, {}^H Y) = -A({}^H Y, {}^{V_\beta} \omega) = 0, \\ A({}^{V_\beta} \omega, {}^{V_\gamma} \theta) = Sg(F_\alpha({}^{V_\beta} \omega), {}^{V_\gamma} \theta) - Sg({}^{V_\beta} \omega, F_\alpha({}^{V_\gamma} \theta)) \\ = Sg(\delta_\alpha^\beta {}^H \tilde{\omega}, {}^{V_\gamma} \theta) - Sg({}^{V_\beta} \omega, \delta_\alpha^\gamma {}^H \tilde{\theta}) \\ = \delta_\alpha^\beta Sg({}^H \tilde{\omega}, {}^{V_\gamma} \theta) - \delta_\alpha^\gamma Sg({}^{V_\beta} \omega, {}^H \tilde{\theta}) = 0,$$

i.e.  $Sg$  is pure with respect to  $F_\alpha$ , for each  $\alpha = 1, 2, \dots, n$ . Thus Theorem 4.1 is proved.

Let us consider the covariant derivative of  $F_\alpha$ , for each  $\alpha = 1, 2, \dots, n$ , with respect to the Levi-Civita connection  ${}^S\nabla$  of metric  ${}^Sg$ . Taking into account (i) – (iv) of Theorem 3.1, we obtain

$$\begin{aligned} & ({}^S\nabla_{H_X}F_\alpha)({}^HY) = {}^H\nabla_{H_X}F_\alpha({}^HY) - F_\alpha({}^S\nabla_{H_X}{}^HY) \\ & = {}^S\nabla_{H_X}V_\alpha\tilde{Y} - F_\alpha({}^H(\nabla_XY) + \frac{1}{2}\sum_{\beta=1}^nV_\beta(X^\beta \circ R(X, Y))) \\ & = V_\alpha(\nabla_X\tilde{Y}) + \frac{1}{2}(R(\tilde{X}^\alpha, \tilde{Y})X) - F_\alpha({}^H(\nabla_XY) + \frac{1}{2}\sum_{\beta=1}^nV_\beta(X^\beta \circ R(X, Y))) \\ & = \frac{1}{2}{}^H(R(\tilde{X}^\alpha, Y)X) - \frac{1}{2}\delta_\alpha^\beta \sum_{\beta=1}^n {}^H(g^{-1}X^\beta \circ R(X, Y)) \end{aligned} \tag{4.2}$$

$$\begin{aligned} & = \frac{1}{2}{}^H(g^{-1}X^\alpha(R(\quad, Y)X - R(X, Y))), \\ & ({}^S\nabla_{V_{\beta\omega}}F_\alpha)({}^HY) = {}^H\nabla_{V_{\beta\omega}}F_\alpha({}^HY) - F_\alpha({}^S\nabla_{V_{\beta\omega}}{}^HY) \end{aligned} \tag{4.3}$$

$$\begin{aligned} & = {}^S\nabla_{V_{\beta\omega}}V_\alpha\tilde{Y} - \frac{1}{2}F_\alpha{}^H(R(\tilde{X}^\beta, \tilde{\omega})Y) = -\frac{1}{2}F_\alpha{}^H(X^\beta g^{-1} \circ R(\quad, \tilde{\omega})Y) \\ & = -\frac{1}{2}V_\alpha(X^\beta \circ R(\quad, \tilde{\omega})Y), \end{aligned}$$

$$\begin{aligned} & ({}^S\nabla_{H_X}F_\alpha)({}^{V\gamma}\theta) = {}^H\nabla_{H_X}F_\alpha({}^{V\gamma}\theta) - F_\alpha({}^S\nabla_{H_X}{}^{V\gamma}\theta) \\ & = {}^S\nabla_{H_X}\delta_\alpha^\gamma{}^H\tilde{\theta} - F_\alpha({}^{V\gamma}(\nabla_X\theta) + \frac{1}{2}{}^H(R(\tilde{X}^\gamma, \tilde{\theta})X)) \\ & = \delta_\alpha^\gamma{}^H(\nabla_X\tilde{\theta}) + \delta_\alpha^\gamma\frac{1}{2}\sum_{\beta=1}^nV_\beta(X^\beta \circ R(X, \tilde{\theta})) - \delta_\alpha^\gamma{}^H(g^{-1} \circ \nabla_X\theta) \\ & - \frac{1}{2}V_\alpha(g \circ R(\tilde{X}^\gamma, \tilde{\theta})X) = \delta_\alpha^\gamma\frac{1}{2}\sum_{\beta=1}^nV_\beta(X^\beta \circ R(X, \tilde{\theta})) \end{aligned} \tag{4.4}$$

$$\begin{aligned} & - \frac{1}{2}V_\alpha(g \circ R(\tilde{X}^\gamma, \tilde{\theta})X), \\ & ({}^S\nabla_{V_{\beta\omega}}F_\alpha)({}^{V\gamma}\theta) = {}^H\nabla_{V_{\beta\omega}}F_\alpha({}^{V\gamma}\theta) - F_\alpha({}^S\nabla_{V_{\beta\omega}}{}^{V\gamma}\theta) \\ & = {}^S\nabla_{V_{\beta\omega}}(\delta_\alpha^\gamma{}^H\tilde{\theta}) = \delta_\alpha^\gamma{}^S\nabla_{V_{\beta\omega}}{}^H\tilde{\theta} = \delta_\alpha^\gamma\frac{1}{2}{}^H(R(\tilde{X}^\beta, \tilde{\omega})\tilde{\theta}) \end{aligned} \tag{4.5}$$

$$= \delta_\alpha^\gamma\frac{1}{2}{}^H(X^\beta g^{-1} \circ R(\quad, \tilde{\omega})\tilde{\theta}).$$

From (4.2)-(4.5), we get

**Theorem 4.2.** *The linear coframe bundle  $F^*(M)$  of a Riemannian manifold  $(M, g)$  is para-Kahlerian (para-holomorphic Nordenian) with respect to the metric  ${}^Sg$  and almost para-complex structure  $F_\alpha$ , for each  $\alpha = 1, 2, \dots, n$ , if and only if the Riemannian manifold  $(M, g)$  is flat.*

### 5. Para-holomorphic vector-fields on $(F^*(M), {}^Sg)$

Let  $(M, g)$  be a Riemannian manifold, and let  $F^*(M)$  be its linear coframe bundle with Sasakian metric  ${}^Sg$  and with the almost para-Nordenian structures  $F_\alpha$ ,  $\alpha = 1, 2, \dots, n$ . A vector field  $\tilde{X} \in \mathfrak{S}_0^1(F^*(M))$  with respect to which the almost para-Nordenian structure  $F_\alpha$  has a vanishing Lie derivative ( $L_{\tilde{X}}F_\alpha = 0$ ) is said to be almost para-holomorphic (see, [18]). It is well known that [5]

$$[{}^C X, {}^H Y] = {}^H [X, Y] + \sum_{\alpha=1}^n V_\alpha (X^\alpha \circ (L_X \nabla) Y), \quad (5.1)$$

$$[{}^C V, {}^{V_\gamma} \omega] = {}^{V_\gamma} (L_V \omega) \quad (5.2)$$

for any  $X, Y \in \mathfrak{S}_0^1(M)$  and  $\omega \in \mathfrak{S}_1^0(M)$ , where

$$(L_X \nabla) Y = \nabla_Y \nabla X + R(X, Y)$$

and

$$(L_X \nabla)(Y, Z) = L_X(\nabla_Y X) - \nabla_Y(L_X Z) - \nabla_{[X, Y]} Z.$$

A vector field  $X \in \mathfrak{S}_0^1(M)$  is called a Killing vector field (or infinitesimal isometry) if  $L_X g = 0$  and  $X$  called an infinitesimal affine transformation if  $L_V \nabla g = 0$ . A Killing vector field is necessarily an infinitesimal affine transformation, i.e. we have  $L_V \nabla g = 0$  as a consequence of  $L_X g = 0$ . Now we consider the Lie derivative of  $F_\alpha$  ( $\alpha = 1, 2, \dots, n$ ) with respect to the complete lift  ${}^C X$ . Taking account of (4.1), (5.1) and (5.2), we find

$$\begin{aligned} (L_{{}^C X} F_\alpha)({}^{V_\gamma} \theta) &= L_{{}^C X}(F_\alpha({}^{V_\gamma} \theta) - F_\alpha(L_X {}^{V_\gamma} \theta)) \\ &= \delta_\alpha^\gamma L_{{}^C X} {}^H \tilde{\theta} - \delta_\alpha^\gamma (g^{-1} \circ (L_X \theta)) = \delta_\alpha^\gamma \left[ {}^H [X, \tilde{\theta}] + \sum_{\beta=1}^n V_\beta (X^\beta \circ L_X \nabla \tilde{\theta}) \right. \\ &\quad \left. - {}^H (g^{-1} \circ (L_X \theta)) \right] = \delta_\alpha^\gamma \left[ {}^H (L_X (g^{-1} \circ \theta) - g^{-1} \circ (L_X \theta)) \right. \\ &\quad \left. + \sum_{\beta=1}^n V_\beta (X^\beta \circ L_X \nabla \tilde{\theta}) \right], \end{aligned} \quad (5.3)$$

$$\begin{aligned} (L_{{}^C X} F_\alpha)({}^H Y) &= L_{{}^C X}(F_\alpha({}^H Y) - F_\alpha(L_X {}^H Y)) \\ &= L_{{}^C X} V_\alpha \tilde{Y} - F_\alpha({}^H [X, Y] + \sum_{\beta=1}^n V_\beta (X^\beta \circ L_X \nabla Y)) \\ &= L_{{}^C X} V_\alpha \tilde{Y} - V_\alpha (g \circ [X, Y]) + \sum_{\beta=1}^n \delta_\alpha^\beta V_\beta (g^{-1} X^\beta \circ L_X \nabla Y) \\ &= V_\alpha (L_X (g \circ Y) - g \circ L_X Y) + \sum_{\beta=1}^n \delta_\alpha^\beta V_\beta (g^{-1} X^\beta \circ L_X \nabla Y). \end{aligned} \quad (5.4)$$

Let now  $X$  be a Killing vector field ( $L_X g = 0$ ). Then by virtue of  $L_X \nabla = 0$ , from (5.3) and (5.4), we have  $L_{{}^C X} F_\alpha = 0$ ,  $\alpha = 1, 2, \dots, n$ , i.e.  ${}^C X$  is para-holomorphic with respect to each  $F_\alpha$ . Hence, we have

**Theorem 5.1.** *Let  $(M, g)$  be a Riemannian manifold and let  $(F^*(M), {}^Sg, F_\alpha)$  be para-Nordenian manifold for each  $\alpha = 1, 2, \dots, n$ . Then the complete lift  ${}^C X$  of vector field  $X \in \mathfrak{X}_0^1(M)$  to  $F^*(M)$  is almost para-holomorphic vector field with respect to the each almost para-Nordenian structure  $(F_\alpha, {}^Sg)$ , if  $X$  is a Killing vector field on a Riemannian manifold  $(M, g)$ .*

## 6. Integrability of para-Nordenian structures on $(F^*(M), {}^Sg)$

In this section, we study the integrability of almost para-Nordenian structure  $F_\alpha$  for each  $\alpha = 1, 2, \dots, n$ , on  $(F^*(M), {}^Sg)$ . We assume that  $\nabla$  is a Levi-Civita connection of a Riemannian metric  $g$ . Denoting by  $N_{F_\alpha}$  the Nijenhuis tensor of  $F_\alpha$ , we have

$$N_{F_\alpha}(\tilde{X}, \tilde{Y}) = [F_\alpha \tilde{X}, F_\alpha \tilde{Y}] - F_\alpha[F_\alpha \tilde{X}, \tilde{Y}] - F_\alpha[\tilde{X}, F_\alpha \tilde{Y}] + [\tilde{X}, \tilde{Y}] \quad (6.1)$$

for all  $\tilde{X}, \tilde{Y} \in \mathfrak{X}_0^1(F^*(M))$ . Then taking account of (2.6) and (6.1), we obtain

$$\begin{aligned} N_{F_\alpha}({}^H X, {}^H Y) &= [F_\alpha {}^H X, F_\alpha {}^H Y] - F_\alpha[F_\alpha {}^H X, {}^H Y] \\ &\quad - F_\alpha[{}^H X, F_\alpha {}^H Y] + [{}^H X, {}^H Y] = [V_\alpha \tilde{X}, V_\alpha \tilde{Y}] - F_\alpha[V_\alpha \tilde{X}, {}^H Y] \\ &\quad - F_\alpha[{}^H X, V_\alpha \tilde{Y}] + {}^H[X, Y] + \sum_{\beta=1}^n V_\beta(X^\beta \circ R(X, Y)) \\ &= F_\alpha V_\alpha(\nabla_Y \tilde{X}) - F_\alpha V_\alpha(\nabla_X \tilde{Y}) + {}^H[X, Y] + \sum_{\beta=1}^n V_\beta(X^\beta \circ R(X, Y)) \\ &= \delta_\alpha^{\alpha H}(\nabla_Y X - \nabla_X Y) + {}^H[X, Y] + \sum_{\beta=1}^n V_\beta(X^\beta \circ R(X, Y)) \\ &= \sum_{\beta=1}^n V_\beta(X^\beta \circ R(X, Y)), \\ N_{F_\alpha}({}^H X, V_\beta \omega) &= [F_\alpha {}^H X, F_\alpha V_\beta \omega] - F_\alpha[F_\alpha {}^H X, V_\beta \omega] \\ &\quad - F_\alpha[{}^H X, F_\alpha V_\beta \omega] + [{}^H X, V_\beta \omega] = [V_\alpha \tilde{X}, \delta_\alpha^{\beta H} \tilde{\omega}] - F_\alpha[V_\alpha \tilde{X}, V_\beta \omega] \\ &\quad - F_\alpha[{}^H X, \delta_\alpha^{\beta H} \tilde{\omega}] + V_\beta(\nabla_X \omega) = \delta_\alpha^\beta [V_\alpha \tilde{X}, {}^H \tilde{\omega}] + V_\beta(\nabla_X \omega) \\ &\quad - \delta_\alpha^\beta F_\alpha({}^H[X, \tilde{\omega}] + \sum_{\gamma=1}^n V_\gamma(X^\gamma \circ R(X, \tilde{\omega}))) = -\delta_\alpha^\beta V_\alpha(\nabla_{\tilde{\omega}} \tilde{X}) \\ &\quad + V_\beta(\nabla_X \omega) - \delta_\alpha^\beta V_\alpha(g \circ [X, \tilde{\omega}]) - \delta_\alpha^\beta \sum_{\gamma=1}^n \delta_\alpha^{\gamma H}(g^{-1} X^\gamma \circ R(X, \tilde{\omega})) \\ &= -V_\beta(g \circ \nabla_{\tilde{\omega}} X) + V_\beta(g \circ \nabla_X \tilde{\omega}) - V_\beta(g \circ [X, \tilde{\omega}]) \\ &\quad - \delta_\alpha^\beta \sum_{\gamma=1}^n \delta_\alpha^{\gamma H}(g^{-1} X^\gamma \circ R(X, \tilde{\omega})) = V_\beta(g \circ [X, \tilde{\omega}]) - V_\beta(g \circ [X, \tilde{\omega}]) \\ &= -\delta_\alpha^\beta \sum_{\gamma=1}^n \delta_\alpha^{\gamma H}(g^{-1} X^\gamma \circ R(X, \tilde{\omega})), \\ N_{F_\alpha}(V_\beta \omega, V_\gamma \theta) &= [F_\alpha V_\beta \omega, F_\alpha V_\gamma \theta] - F_\alpha[F_\alpha V_\beta \omega, V_\gamma \theta] \end{aligned}$$

$$\begin{aligned}
& -F_\alpha[V^\beta\omega, F_\alpha V^\gamma\theta] + [V^\beta\omega, V^\gamma\theta] = [\delta_\alpha^\beta H\tilde{\omega}, \delta_\alpha^\gamma H\tilde{\theta}] \\
& -F_\alpha[\delta_\alpha^\beta H\tilde{\omega}, V^\gamma\theta] - F_\alpha[V^\beta\omega, \delta_\alpha^\gamma H\tilde{\theta}] = \delta_\alpha^\beta \delta_\alpha^\gamma [H\tilde{\omega}, H\tilde{\theta}] \\
& -\delta_\alpha^\beta F_\alpha[H\tilde{\omega}, V^\gamma\theta] - \delta_\alpha^\gamma F_\alpha[V^\beta\omega, H\tilde{\theta}] = \delta_\alpha^\beta \delta_\alpha^\gamma [\tilde{\omega}, \tilde{\theta}] \\
& + \delta_\alpha^\beta \delta_\alpha^\gamma \sum_{\sigma=1}^n V_\sigma(X^\sigma \circ R(\tilde{\omega}, \tilde{\theta})) - \delta_\alpha^\beta F_\alpha V_\gamma(\nabla_{\tilde{\omega}}\theta) \\
& + \delta_\alpha^\gamma F_\alpha V_\beta(\nabla_{\tilde{\theta}}\omega) = \delta_\alpha^\beta \delta_\alpha^\gamma H[\tilde{\omega}, \tilde{\theta}] + \delta_\alpha^\beta \delta_\alpha^\gamma \sum_{\sigma=1}^n V_\sigma(X^\sigma \circ R(\tilde{\omega}, \tilde{\theta})) \\
& - \delta_\alpha^\beta \delta_\alpha^\gamma H(g^{-1} \circ \nabla_{\tilde{\omega}}\theta) + \delta_\alpha^\gamma \delta_\alpha^\beta H(g^{-1} \circ \nabla_{\tilde{\theta}}\omega) = \delta_\alpha^\beta \delta_\alpha^\gamma H[\tilde{\omega}, \tilde{\theta}] \\
& + \delta_\alpha^\beta \delta_\alpha^\gamma \sum_{\sigma=1}^n V_\sigma(X^\sigma \circ R(\tilde{\omega}, \tilde{\theta})) - \delta_\alpha^\beta \delta_\alpha^\gamma H(\nabla_{\tilde{\omega}}\tilde{\theta}) \\
& + \delta_\alpha^\gamma \delta_\alpha^\beta H(\nabla_{\tilde{\theta}}\tilde{\omega}) = \delta_\alpha^\beta \delta_\alpha^\gamma \sum_{\sigma=1}^n V_\sigma(X^\sigma \circ R(\tilde{\omega}, \tilde{\theta}))
\end{aligned}$$

for all  $X, Y \in \mathfrak{S}_0^1(M)$ ,  $\omega, \theta \in \mathfrak{S}_1^0(M)$  and  $1 \leq \alpha \leq n$ .

From the above equations we conclude that  $N_{F_\alpha} = 0$  if and only if  $R(X, Y) = 0$  for all  $X, Y \in \mathfrak{S}_0^1(M)$ . Therefore, we have

**Theorem 6.1.** *Let  $(M, g)$  be a Riemannian manifold and  $F^*(M)$  be its linear coframe bundle equipped with Sasakian metric  ${}^Sg$ . Then the almost para-Nordenian manifold  $(F^*(M), {}^Sg, F_\alpha)$ , for each  $\alpha = 1, 2, \dots, n$ , is para-Nordenian if and only if  $R = 0$ .*

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