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ON A BOUNDARY VALUE PROBLEM FOR A MIXED TYPE FRACTIONAL DIFFERENTIAL EQUATIONS WITH PARAMETERS

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Abstract. In this paper, we consider a boundary value problem for a mixed type partial differential equation with Hilfer operator of fractional integro-differentiation in a positive rectangular domain and with spectral parameter in a negative rectangular domain. The mixed differential equation depends from another positive small parameter in mixed derivatives. The considering mixed type differential equation brings to a spectral problem for a second order differential equation with respect to the second variable. Regarding the first variable, this equation is an ordinary fractional differential equation in the positive part of the considering segment, and is a second-order ordinary differential equation with spectral parameter in the negative part of this segment. Using the spectral method of separation of variables, the solution of the boundary value problem is constructed in the form of a Fourier series. Theorems on the existence and uniqueness of the problem are proved for regular values of the spectral parameter. It is proved the stability of solution with respect to boundary function and with respect to small positive parameter given in mixed derivatives. For irregular values of the spectral parameter, an infinite number of solutions in the form of a Fourier series are constructed.

1. Problem statement

In a rectangular domain $\Omega = \{(t, x): \ -a < t < b, \ 0 < x < l\}$ we consider the fractional partial differential equation of mixed type

$$0 = \begin{cases} \left(D^{\alpha,\gamma} - \nu D^{\alpha,\gamma} \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x^2}\right) U(t, x), & (t, x) \in \Omega_1, \\ \left(\frac{\partial^2}{\partial t^2} - \nu \frac{\partial^4}{\partial t^2 \partial x^2} - \omega^2 \frac{\partial^2}{\partial x^2}\right) U(t, x), & (t, x) \in \Omega_2, \end{cases} \tag{1.1}$$

where $\Omega_1 = \{(t, x): 0 < t < b, 0 < x < l\}$, $\Omega_2 = \{(t, x): -a < t < 0, 0 < x < l\}$, ν is positive parameter, ω is positive spectral parameter, a, b are positive real numbers,

$$D^{\alpha,\gamma} = J_{0+}^{\gamma-\alpha} \frac{d}{dt} J_{0+}^{1-\gamma}, \, 0 < \alpha \leq \gamma \leq 1$$

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is Hilfer operator and

$$J_{0+}^{\alpha}\varphi(t)=\frac{1}{\Gamma(\alpha)}\int_{0}^{t}\frac{\varphi(\tau)\,d\tau}{(t-\tau)^{1-\alpha}},\quad \alpha>0$$

is Riemann–Liouville integral operator.

Problem $T_{\nu,\omega}$. It is required to find a function $U(t,x)$, which belongs to the class

$$t^{1-\gamma}\frac{\partial^k U}{\partial x^k}\in C(\overline{\Omega}_1),\ \frac{\partial^k U}{\partial x^k}\in C(\overline{\Omega}_2),\ D^{\alpha,\gamma}U\in C(\Omega_1),\ U_{tt},U_{xx}\in C(\Omega_1\cup\Omega_2),\quad (1.2)$$

$k=\overline{0,\,2}$; satisfies equation (1.1) in the domain $\Omega_1\cup\Omega_2$, boundary value conditions

$$U(t,x)\,|_{x=0}=\frac{\partial^2}{\partial x^2}U(t,x)\,|_{x=l}=0,\quad t\neq 0,\quad (1.3)$$

$$U(-a,x)=U(b,x)+\varphi(x),\quad 0\le x\le l\quad (1.4)$$

and gluing conditions

$$\begin{aligned}&\lim_{t\to+0}J_{0+}^{1-\gamma}U(t,x)=\lim_{t\to-0}U(t,x),\\&\lim_{t\to+0}J_{0+}^{1-\alpha}\tfrac{d}{dt}J_{0+}^{1-\gamma}U(t,x)=\lim_{t\to-0}\tfrac{d}{dt}U(t,x),\end{aligned}\quad (1.5)$$

where $\varphi(x)$ is given sufficiently smooth function.

For $\gamma=\alpha$ and $\gamma=1$ the Hilfer operator has the forms $D^{\alpha,0}={}_{RL}D_{0+}^{\alpha}$ and $D^{\alpha,1}={}_{C}D_{0+}^{\alpha}$. Thus, the generalized integro-differentiation operator $D^{\alpha,\gamma}$ is a continuous interpolation of the well-known fractional order differentiation operators of Riemann–Liouville and Gerasimov–Caputo, which describe diffusion processes [7, vol. 1, 47–85]. The construction of various models of theoretical physics problems using fractional calculus is described in [7, vol. 4, 5], [6, 14]. A specific physical and engineering interpretation of the generalized fractional operator $D^{\alpha,\gamma}$ is given in [7, vol. 6–8], [10, 11, 13]. In [10], in particular, were provide results on the existence and representation of solution of initial value problem for the ordinary linear fractional differential equation with generalized Riemann–Liouville fractional derivatives and constant coefficients by the method of operational calculus of Mikusinski type. In [8], the problem of source identification was studied for the generalized diffusion equation with operator $D^{\alpha,\gamma}$. In the work [2] the inverse problems are investigated for a generalized fourth-order parabolic equation with this operator $D^{\alpha,\gamma}$.

Note that boundary value conditions of the type (1.3) take place in modeling problems of the flow around a profile by a subsonic velocity stream with a supersonic zone. Some nonlocal problems for ordinary differential equations were studied in [1, 9]. Nonlocal boundary value problems for different type of partial differential equations were studied in the works of many authors, in particular, in [12, 15, 16, 17].

In our work, unlike mixed parabolic-hyperbolic equations, the problem of small denominators does not arise. In addition, in our solvability problem we impose conditions that are two times weaker than in the case of parabolic-hyperbolic equations. In this paper, we consider a self-adjoint boundary value problem for a mixed type partial differential equation with Hilfer operator of fractional integro-differentiation. The spectral method of separation of variables is used taking into account the features of the fractional integro-differentiation operator. We study the solvability of the nonlocal boundary value problem (1.1)–(1.5) for various values of the spectral parameter. We prove the stability of the solution with respect to boundary function and with respect to parameter given in mixed derivative. This work is a further development and generalization of the results of [3, 4, 5].

2. Uniqueness of the solution of the problem $T_{\nu,\omega}$

We seek solutions of the nonlocal boundary value problem $T_{\nu,\omega}$ in the form of the product of two functions of different variables $U(t,x) = u(t)\cdot\vartheta(x)$. Then from equation (1.1) we arrive at the following presentations

$$\frac{D^{\alpha,\gamma}u(t)}{u(t)} - \nu\,\frac{\vartheta''(x)}{\vartheta(x)}\cdot\frac{D^{\alpha,\gamma}u(t)}{u(t)} = \frac{\vartheta''(x)}{\vartheta(x)},\ t>0,$$

$$\frac{u''(t)}{T(t)} - \nu\,\frac{\vartheta''(x)}{\vartheta(x)}\cdot\frac{u''(t)}{u(t)} = \omega^2\,\frac{\vartheta''(x)}{\vartheta(x)},\ t<0$$

or

$$\frac{\vartheta''(x)}{\vartheta(x)} = -\mu^2,\ \ 0<x<l,$$

$$\frac{D^{\alpha,\gamma}\,u(t)}{u(t)} + \nu\,\mu^2\,\frac{D^{\alpha,\gamma}\,u(t)}{u(t)} = -\mu^2,\ t>0,$$

$$\frac{u''(t)}{u(t)} + \nu\,\mu^2\,\frac{u''(t)}{u(t)} = -\mu^2\omega^2,\ \ t<0,$$

where μ^2 is constant of separation, $0<\mu$. Hence, taking into account the boundary conditions (1.3), obtain

$$\vartheta''(x) + \mu^2\,\vartheta''(x) = 0,\ \ \vartheta(0) = \vartheta(l), \tag{2.1}$$

$$D^{\alpha,\gamma}u(t) + \lambda^2(\nu)\,u(t) = 0,\ \ 0<t<b, \tag{2.2}$$

$$u''(t) + \lambda^2(\nu)\,\omega^2\,u(t) = 0,\ \ -a<t<0, \tag{2.3}$$

where $\lambda^2(\nu) = \frac{\mu^2}{1+\nu\,\mu^2}$.

The spectral problem (2.1) is self-adjoint and in the space $L_2(0;\,l)$ has a complete system of orthonormal eigenfunctions

$$\vartheta_n(x) = \sqrt{\frac{2}{l}}\,\sin\mu_n x,\ \ \mu_n = \frac{n\,\pi}{l},\ \ n\in\mathbb{N}.$$

Let $U(t,x)$ be solution of the problem $T_{\nu,\omega}$. We consider the following functions

$$u_n(t) = \sqrt{\frac{2}{l}}\int_0^l U(t,x)\,\sin\mu_n x\,dx,\ \ -a<t<b. \tag{2.4}$$

We show that functions (2.4) satisfy equations (2.2) and (2.3) in the corresponding intervals. Applying the operator $D^{\alpha,\gamma}$ with respect to t to both sides of equality

(2.4) on $0 < t < b$, differentiating (2.4) twice with respect to t on $-a < t < 0$, taking into account equation (1.1), we obtain

$$D^{\alpha,\gamma} u_n(t) = \sqrt{\frac{2}{l}} \int_0^l D^{\alpha,\gamma} U(t, x) \sin \mu_n x \, dx =$$

$$= \sqrt{\frac{2}{l}} \int_0^l (\nu D^{\alpha,\gamma} U_{xx}(t, x) + U_{xx}(t, x)) \sin \mu_n x \, dx, \tag{2.5}$$

$$u_n''(t) = \sqrt{\frac{2}{l}} \int_0^l U_{tt}(t, x) \sin \mu_n x \, dx =$$

$$= \sqrt{\frac{2}{l}} \int_0^l \left(\nu U_{ttxx}(t, x) + \omega^2 U_{xx}(t, x)\right) \sin \mu_n x \, dx. \tag{2.6}$$

Integrating twice in parts the integrals (2.5), (2.6), taking into account conditions (1.3), we obtain the equations

$$D^{\alpha,\gamma} u_n(t) + \lambda_n^2(\nu) u_n(t) = 0, \ \ t > 0, \tag{2.7}$$

$$\frac{d^2}{dt^2} u_n(t) + \lambda_n^2(\nu) \omega^2 u_n(t) = 0, \ \ t < 0, \tag{2.8}$$

where

$$\lambda_n^2(\nu) = \frac{\mu_n^2}{1 + \nu \mu_n^2}, \ \ \mu_n = \frac{n\pi}{l}, \ \ n \in \mathbb{N}.$$

Differential equations (2.7) and (2.8) for $\lambda = \lambda_n$ coincide with equations (2.2) and (2.3), respectively. Further, taking into account conditions (1.5), from (2.4) we obtain

$$\lim_{t \to +0} J_{0+}^{1-\gamma} u_n(t) = \sqrt{\frac{2}{l}} \int_0^l \lim_{t \to +0} J_{0+}^{1-\gamma} U(t, x) \sin \mu_n x \, dx =$$

$$= \sqrt{\frac{2}{l}} \int_0^l \lim_{t \to -0} U(t, x) \sin \mu_n x \, dx = \lim_{t \to -0} u_n(t), \tag{2.9}$$

$$\lim_{t \to +0} J_{0+}^{1-\alpha} \frac{d}{dt} J_{0+}^{1-\gamma} u_n(t) = \sqrt{\frac{2}{l}} \int_0^l \lim_{t \to +0} J_{0+}^{1-\alpha} \frac{d}{dt} J_{0+}^{1-\gamma} U(t, x) \sin \mu_n x \, dx =$$

$$= \sqrt{\frac{2}{l}} \int_0^l \lim_{t \to -0} \frac{d}{dt} U(t, x) \sin \mu_n x \, dx = \lim_{t \to -0} \frac{d}{dt} u_n(t). \tag{2.10}$$

Analogously we find from (1.4) that

$$u_n(-a) = u_n(b) + \varphi_n, \tag{2.11}$$

where

$$\varphi_n = \sqrt{\frac{2}{l}} \int\limits_0^l \varphi(x) \sin \mu_n x \, dx, \quad n = 1, 2, ...$$

General form of solutions of the problem $T_{\nu,\omega}$ is

$$u_n(t) = \begin{cases} A_{1n} t^{\gamma-1} E_{\alpha,\gamma}\left(-\lambda_n^2(\nu) t^\alpha\right), & t > 0, \\ A_{2n} \sin \lambda_n(\nu) \omega t + A_{3n} \cos \lambda_n(\nu) \omega t, & t < 0, \end{cases} \tag{2.12}$$

where A_{in} are arbitrary constants, $i = \overline{1,3}$, $n = 1, 2, ...$

Further, satisfying functions (2.12) to conditions (2.9)–(2.11), we obtain the following systems of algebraic equations

$$\begin{cases} A_{1n} = A_{3n}, \quad \omega A_{2n} = -\lambda_n(\nu) A_{1n}, \\ -A_{2n} \sin \lambda_n(\nu) \omega a + A_{3n} \cos \lambda_n(\nu) \omega a - \\ -A_{1n} b^{\gamma-1} E_{\alpha,\gamma}\left(-\lambda_n^2(\nu) b^\alpha\right) = \varphi_{ni}. \end{cases} \tag{2.13}$$

This system (2.13) has a unique solution

$$A_{3n} = A_{1n} = \frac{\varphi_n}{\Delta_n(\omega)}, \quad A_{2n} = -\frac{\lambda_n(\nu)}{\omega} \frac{\varphi_n}{\Delta_n(\omega)}, \tag{2.14}$$

if for all $n \in \mathbb{N}$ there holds the condition

$$\Delta_n(\omega) = \lambda_n(\nu) \omega \sin \lambda_n(\nu) \omega a + \cos \lambda_n(\nu) \omega a - b^{\gamma-1} E_{\alpha,\gamma}\left(-\lambda_n^2(\nu) b^\alpha\right) \neq 0. \tag{2.15}$$

Substituting (2.14) into (2.12), we obtain the representation

$$u_n(t) = \begin{cases} \frac{\varphi_n}{\Delta_n(\omega)} t^{\gamma-1} E_{\alpha,\gamma}\left(-\lambda_n^2(\nu) t^\alpha\right), & t > 0, \\ \frac{\varphi_n}{\Delta_n(\omega)} \left(\cos \lambda_n(\nu) \omega t - \frac{\lambda_n(\nu)}{\omega} \sin \lambda_n(\nu) \omega t\right), & t < 0. \end{cases} \tag{2.16}$$

We show the uniqueness of the solution of the problem $T_{\nu,\omega}$. Suppose that the condition (2.15) and $\varphi(x) \equiv 0$ are fulfilled. Then $\varphi_n = 0$ and from the presentations (2.4) and (2.16) implies, that

$$\int\limits_0^l t^{1-\gamma} U(t, x) \vartheta_n(x) \, dx = 0, \quad t \in [0; b], \quad n = 1, 2, ...,$$

$$\int\limits_0^l U(t, x) \vartheta_n(x) \, dx = 0, \quad t \in [-a; 0], \quad n = 1, 2, ...$$

Taking into account the completeness of systems $\{\vartheta_n(x)\}_{n=1}^\infty$ in space $L_2(0; l)$, we conclude that $U(t, x) = 0$ almost everywhere on the segment $[0; l]$ for all $t \in [-a; b]$. Since $t^{1-\gamma} U(t, x) \in C\left(\overline{\Omega}_1\right)$, $U(t, x) \in C\left(\overline{\Omega}_2\right)$, then $t^{1-\gamma} U(t, x) \equiv 0$ in the domain $\overline{\Omega}$. Therefore, the solution of problem $T_{\nu,\omega}$ is unique in the domain $\overline{\Omega}$.

Thus, we have proved that the following theorem holds:

Theorem 2.1. *Suppose that there exists a solution of the problem $T_{\nu,\omega}$. Then this solution is unique, if condition* (2.15) *is fulfilled for all $n \in \mathbb{N}$.*

3. Existence of a solution of the problem $T_{\nu,\omega}$

Now we consider the case, when condition (2.15) is violated. Let $\Delta_m(\omega) = 0$ be for some ω, $\gamma \in (0;\,1)$ and $n = m$. Then the homogeneous problem $T_{\nu,\omega}$ ($\varphi(x) \equiv 0$) has nontrivial solution

$$V_m\,(t,\,x) = \upsilon_m(t)\,\vartheta_m(x), \quad (t,\,x) \in \Omega, \tag{3.1}$$

where

$$\upsilon_m(t) = \begin{cases} t^{\gamma-1} E_{\alpha,\gamma}\left(-\lambda_m^2(\nu)\,t^{\alpha}\right), & t > 0, \\ \sin\,\lambda_m(\nu)\,\omega\,t + \cos\,\lambda_m(\nu)\,\omega\,t, & t < 0. \end{cases}$$

From $\Delta_n(\omega) = 0$ we come to the trigonometric equation

$$\sqrt{1 + \omega^2\lambda_n^2(\nu)}\,\sin\,(\lambda_n(\nu)\,\omega\,a + \rho_n) - b^{\gamma-1}E_{\alpha,\gamma}\left(-\lambda_n^2(\nu)\,b^{\alpha}\right) = 0, \tag{3.2}$$

where $\rho_n = \arcsin\left(\frac{1}{\sqrt{1+\omega^2\lambda_n^2(\nu)}}\right)$. From this we obtain, that the quantity $\Delta_n(\omega)$ vanishes at

$$\omega = \frac{1}{\lambda_n(\nu)\,a}\left[(-1)^k \arcsin\,\frac{b^{\gamma-1}E_{\alpha,\gamma}\left(-\lambda_n^2(\nu)\,b^{\alpha}\right)}{\sqrt{1+\omega^2\lambda_n^2(\nu)}} + \pi k - \rho_n\right], \quad k = 1, 2, ...$$

The set of positive solutions $\Im$ of trigonometric equation (3.2) is called the set of irregular values of the spectral parameter ω. The set of remaining values of the spectral parameter $\aleph = (0;\,\infty) \setminus \Im$ is called the set of regular values of the spectral parameter ω. For all regular values of the spectral parameter ω the quantity $\Delta_n(\omega)$ is nonzero. So, for large n the values of $\Delta_n(\omega)$ can not become quite small and there the problem of small denominators does not arise. Therefore, for regular values of the spectral parameter ω the quantity $\Delta_n(\omega)$ is separated from zero.

Indeed, from the relations $\lambda_n^2(\nu) = \frac{\mu_n^2}{1+\nu\,\mu_n^2}$, $\mu_n = \frac{n\,\pi}{l}$ we see that $\lambda_n^2(\nu) \to \frac{1}{\nu}$ as $n \to \infty$. So, for regular values of the spectral parameter ω we have

$$\lim_{n\to\infty} \Delta_n(\omega) = \frac{\omega}{\nu}\,\sin\,\frac{\omega}{\nu}\,a + \cos\,\frac{\omega}{\nu}\,a - b^{\gamma-1}E_{\alpha,\gamma}\left(-\left(\frac{\omega}{\nu}\right)^2 b^{\alpha}\right) \neq 0.$$

Lemma 3.1. *Suppose that $\gamma \in (0;\,1]$, a, b are arbitrary positive real numbers, $\omega \in \aleph$ is regular. Then for arbitrary n there exists a positive constant M_0 such that there holds the following estimate*

$$|\,\Delta_n(\omega)\,| \geq M_0 > 0. \tag{3.3}$$

Proof. From (3.2) for all n and a, $b > 0$ we derive

$$|\,\Delta_n(\omega)\,| \geq \left|\,\pm\sqrt{1+\omega^2\lambda_n^2(\nu)} - b^{\gamma-1}E_{\alpha,\gamma}\left(-\lambda_n^2(\nu)\,b^{\alpha}\right)\right| \geq$$
$$\geq \left|\,1 - b^{\gamma-1}E_{\alpha,\gamma}\left(-\lambda_n^2(\nu)\,b^{\alpha}\right)\right| \geq 1 - b^{\gamma-1}E_{\alpha,\gamma}\left(-\lambda_n^2(\nu)\,b^{\alpha}\right).$$

We use the following properties of the Mittag–Leffler function [7, vol. 1, 269–295]:

1) For all $k > 0$, $\alpha,\,\gamma \in (0;\,1]$, $\alpha \leq \gamma$, $t \geq 0$ the function $t^{\alpha-1}E_{\alpha,\gamma}\,(-k\,t^{\alpha})$ is complete monotonous and there holds

$$(-1)^n\left[t^{\gamma-1}E_{\alpha,\gamma}\,(-k\,t^{\alpha})\right]^{(n)} \geq 0, \quad n = 0, 1, 2, ... \tag{3.4}$$

2) For all $\alpha \in (0;\, 2)$, $\gamma \in \mathbb{R}$ and $\arg z = \pi$ there takes place the following estimate

$$|\, E_{\alpha,\gamma}\,(z)\,| \le \frac{M}{1+|z|}, \tag{3.5}$$

where $0 < M = \text{const}$ does not depend from z.

Then, from (3.4) and (3.5) implies that there exists a number M_0 such that

$$1 - b^{\gamma-1} E_{\alpha,\gamma}\left(-\lambda_n^2\,(\nu)\, b^{\alpha}\right) = M_0 > 0.$$

Consequently, for regular values of the spectral parameter $\omega \in \aleph$ there takes place $|\,\Delta_n\,(\omega)\,| \ge M_0 > 0$. Lemma 3.1 is proved. □

By virtue of estimates (3.3) and (3.5), from (2.16) implies that the following lemma holds.

Lemma 3.2. *For regular values of the spectral parameter $\omega \in \aleph$ there holds*

$$t^{1-\gamma}\,|\,u_n\,(t)\,| \le C_1\,|\,\varphi_n\,|,\;\; t^{1-\gamma}\,|\,D^{\alpha,\gamma}\,u_n\,(t)\,| \le C_2\,|\,\varphi_n\,|,\;\; t \in [0;\, b];$$

$$|\,u_n\,(t)\,| \le C_3\,|\,\varphi_n\,|,\;\; \left|\frac{d\,u_n\,(t)}{d\,t}\right| \le C_4\,|\,\varphi_n\,|,\;\; \left|\frac{d^2 u_n\,(t)}{d\,t^2}\right| \le C_5\,|\varphi_n\,|,\;\; t \in [-a;\, 0],$$

where $C_k,\;\; k = \overline{1,5}$ are positive constants.

By virtue of presentation (2.16), for regular values of the spectral parameter ω we write the formal solution of the problem $T_{\nu,\omega}$ in the form of Fourier series

$$U\,(t,\,x) = \sum_{n=1}^{\infty} \frac{\varphi_n}{\Delta_n\,(\omega)} t^{\gamma-1}\, E_{\alpha,\gamma}\left(-\lambda_n^2\,(\nu)\,t^{\alpha}\right)\,\vartheta_n\,(x),\;\; (t,\,x) \in \Omega_1, \tag{3.6}$$

$$U(t,x) = \sum_{n=1}^{\infty} \frac{\varphi_n}{\Delta_n(\omega)}\Big[\cos\lambda_n(\nu)\omega t - \frac{\lambda_n(\nu)}{\omega}\sin\lambda_n(\nu)\omega t\Big]\vartheta_n(x), (t,x) \in \Omega_2. \tag{3.7}$$

Now formally differentiating term-by-term the series (3.6) the required number of times, we obtain the series

$$D^{\alpha,\gamma}\,U\,(t,x) = \sum_{n=1}^{\infty} D^{\alpha,\gamma}\,u_n\,(t)\,\vartheta_n\,(x),\;\; t > 0, \tag{3.8}$$

$$\frac{\partial^k U\,(t,\,x)}{\partial\, x^k} = \sum_{n=1}^{\infty} u_n(t)\frac{d^k\vartheta_n(x)}{d\,x^k} = (-1)^{k+1}\sum_{n=1}^{\infty} u_n(t)\,\mu_n^k\vartheta_n(x),\;\; t > 0, \tag{3.9}$$

$$\frac{\partial^2\,U\,(t,\,x)}{\partial\,t^2} = \sum_{n=1}^{\infty} \frac{d^2 u_n\,(t)}{d\,t^2}\,\vartheta_n\,(x),\;\; t < 0, \tag{3.10}$$

$$\frac{\partial^k U\,(t,x)}{\partial\,x^k} = \sum_{n=1}^{\infty} u_n(t)\,\frac{d^k\vartheta_n(x)}{d\,x^k} = (-1)^{k+1}\sum_{n=1}^{\infty} u_n(t)\,\mu_n^k\vartheta_n(x),\;\; t < 0, \tag{3.11}$$

$k = 1,\, 2$.

By virtue of the validity of lemma 3.1 and lemma 3.2, we obtain that the series (3.7), (3.10) and (3.11) are majorized by the following series

$$C_6\sum_{n=1}^{\infty} n^2\,|\,\varphi_n\,|,\;\; C_6 = \text{const}. \tag{3.12}$$

From series (3.8) and (3.9) by the aid of term-by-term multiplication by $t^{1-\gamma}$ we obtain the series

$$\sum_{n=1}^{\infty} t^{1-\gamma} D^{\alpha,\gamma} u_n(t)\,\vartheta_n(x), \quad \sum_{n=1}^{\infty} t^{1-\gamma} u_n(t)\,\frac{d^k \vartheta_n(x)}{d x^k}, \quad k=1,\,2, \quad t>0. \quad (3.13)$$

The series in (3.13) are also majorized by the series (3.12). Taking into account the fact that the function $\varphi(x)$ is sufficiently smooth and integrating by parts three times the integral

$$\varphi_n = \int_0^l \varphi(x)\,\vartheta_n(x)\,dx,$$

we obtain

$$\varphi_n = -\frac{1}{\mu_n^3}\varphi_n''' = -\frac{1}{\mu_n^3}\int_0^l \varphi'''(x)\,\vartheta_n(x)\,dx, \quad \mu_n = \frac{n\pi}{l}.$$

By virtue of these presentations, we apply the Cauchy-Schwartz inequality and Bessel inequality to (3.12). Then we have

$$\sum_{n=1}^{\infty} n^2 |\varphi_n| \le C_7 \sum_{n=1}^{\infty} \frac{1}{n}\,|\varphi_n'''| \le$$

$$\le C_7 \left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{1/2} \left(\sum_{n=1}^{\infty} |\varphi_n'''|^2\right)^{1/2} \le C_8\,\|\varphi'''(x)\|_{L_2(0;\,l)} < \infty,$$

where $C_7,\,C_8 = \text{const}$.

From this estimate implies that the series (3.6)–(3.11) converge absolutely and uniformly in the domains $\overline{\Omega}_1$ and $\overline{\Omega}_2$, respectively. Therefore, the function $U(t,\,x)$, represented by series (3.6) and (3.7), possesses properties (1.2) and satisfies conditions (1.3)–(1.5).

We note that $\Delta_n(\omega)=0$ for irregular values of the spectral parameter ω and $n=k_1,\ldots,\,k_s$, $1\le k_1<k_1<\ldots<k_s$, $s\in\mathbb{N}$, $(\gamma\neq 1)$. Then, for the solvability of systems (2.13), it is necessary and sufficient that the orthogonality conditions are satisfied

$$\varphi_n = \int_0^l \varphi(x)\,\vartheta_n(x)\,dx = 0, \quad n=k_1,\ldots k_s. \quad (3.14)$$

In this case, the solutions of problem $T_{\nu,\omega}$ are represented as sum of the series

$$U(t,\,x) = \left[\sum_{n=1}^{k_1-1} + \sum_{n=k_1+1}^{k_2-1} + \ldots + \sum_{n=k_s+1}^{\infty}\right] u_n(t)\,\vartheta_n(x) + \sum_m C_m V_m(t,\,x), \quad (3.15)$$

where $m=k_1,\ldots,\,k_s$, C_m are arbitrary constants, functions $V_m(t,\,x)$ are determined from (3.1).

Thus, it is proved that there following theorem holds.

Theorem 3.1. *Suppose that the following conditions are fulfilled:*

$$\varphi(x) \in C^2[0; l], \ \ \varphi'''(x) \in L_2(0; l), \varphi(0) = \varphi(l) = 0.$$

Then the boundary value problem $T_{\nu,\omega}$ is uniquely solvable for regular values of spectral parameter ω and this solution is represented in the form of the Fourier series (3.6) *and* (3.7) *in the domains Ω_1 and Ω_2, respectively.*

For irregular values of the spectral parameter ω and for some $n = k_1, ..., k_s$ problem $T_{\nu,\omega}$ has an infinite number of solutions in the form of series (3.15). *The solvability condition is formula* (3.14).

4. Stability of solution of the problem $T_{\nu,\omega}$

For regular values of the spectral parameter ω we consider the questions of stability of the solution of boundary value problem $T_{\nu,\omega}$ with respect to the function $\varphi(x)$ of condition (1.4) and with respect to parameter ν. To this end, we introduce the norm in the space of continuous functions as follows

$$\| U(t, x) \|_{C(\overline{\Omega})} = \max_{(t,x)\in\overline{\Omega}_1} \left| t^{1-\gamma} U(t, x) \right| + \max_{(t,x)\in\overline{\Omega}_2} | U(t, x) | .$$

4.1. Stability with respect to function $\varphi(x)$.

Theorem 4.1. *Suppose that all conditions of theorem 3.1 are fulfilled. Then, the solution of the problem $T_{\nu,\omega}$ for regular values of the spectral parameter $\omega \in \aleph$ is stable with respect to given function $\varphi(x)$.*

Proof. We show that the solution $U(t, x)$ of the mixed differential equation (1.1) is stable with respect to the given function $\varphi(x)$. Let $U_1(t, x)$ and $U_2(t, x)$ be two different solutions of the boundary value problem $T_{\nu,\omega}$, corresponding to functions $\varphi_1(x)$ and $\varphi_2(x)$, respectively.

We put that $|\varphi_{1n} - \varphi_{2n}| < \delta_n$, where $0 < \delta_n$ is sufficiently small quantity and the series $\sum_{n=1}^{\infty} |\delta_n|$ is convergent. Then, taking this fact into account, by virtue of the conditions of the theorem, from (3.6) and (3.7) it is easy to see that

$$\| U_1(t, x) - U_2(t, x) \|_{C(\overline{\Omega})} \le C_9 \sum_{n=1}^{\infty} |\varphi_{1n} - \varphi_{2n}| < C_9 \sum_{n=1}^{\infty} |\delta_n| ,$$

where $C_9 = \text{const}$. From this estimate we finally obtain assertions of stability of the solution of differential equation (1.1) with respect to the given function $\varphi(x)$, if we put $\varepsilon = C_9 \sum\limits_{n=1}^{\infty} |\delta_n|$. The theorem 4.1 is proved. □

4.2. Stability with respect to parameter ν. Now we show that the solution $U(t, x)$ of the mixed differential equation (1.1) is stable with respect to a given parameter ν in mixed derivatives of this equation.

Theorem 4.2. *Suppose that all conditions of theorem 3.1 are fulfilled. Then, the solution of the problem $T_{\nu,\omega}$ for regular values of spectral parameter ω is stable with respect to given parameter ν.*

Proof. Let $U_1(t,\, x)$ and $U_2(t,\, x)$ be two different solutions of the boundary value problem $T_{\nu,\,\omega}$, corresponding to two different values of the parameter ν_1 and ν_2, respectively.

By virtue of $\lambda_n^2(\nu)=\frac{\mu_n^2}{1+\nu\,\mu_n^2}$, we derive the following estimates

$$\left|\,E_{\alpha,\gamma}\left(-\lambda_n^2(\nu_1)\,t^{\alpha}\right)-E_{\alpha,\gamma}\left(-\lambda_n^2(\nu_2)\,t^{\alpha}\right)\right|\leq C_{10}\left|\,\lambda_n^2(\nu_1)-\lambda_n^2(\nu_2)\,\right|\leq$$

$$\leq C_{10}\left|\int\limits_{\nu_1}^{\nu_2}\frac{d}{d\,\nu}\,\frac{\mu_n^2}{1+\nu\,\mu_n^2}\right|\leq C_{11}\,|\,\nu_1-\nu_2\,|\,;$$

$$|\cos\lambda_n(\nu_1)\,\omega\,t-\cos\lambda_n(\nu_2)\,\omega\,t\,|\leq\left|\int\limits_{\nu_1}^{\nu_2}\frac{d}{d\,\nu}\,\cos\lambda_n(\nu)\,\omega\,t\right|\leq$$

$$\leq\left|\int\limits_{\nu_1}^{\nu_2}\frac{d\,\nu}{\lambda_n(\nu)}\right|\leq C_{12}\,|\,\nu_1-\nu_2\,|\,;$$

$$|\lambda_n(\nu_1)\sin\lambda_n(\nu_1)\omega\,t-\lambda_n(\nu_2)\sin\lambda_n(\nu_2)\omega\,t\,|\leq|[\lambda_n(\nu_1)-\lambda_n(\nu_2)]\sin\lambda_n(\nu_2)\omega\,t\,|+$$

$$+\,|\,\lambda_n(\nu_1)\,[\sin\lambda_n(\nu_1)\,\omega\,t-\sin\lambda_n(\nu_2)\,\omega\,t]\,|\leq C_{13}\,|\,\nu_1-\nu_2\,|\,,$$

where C_{1i}, $i=0,\,1,\,2,\,3$ are constants. We put that $|\,\nu_1-\nu_2\,|<\delta$, where $0<\delta$ is sufficiently small real number. Then, by virtue of above derived estimates, from series (3.6) and (3.7) we obtain

$$\|\,U_1(t,x)-U_2(t,x)\,\|_{C(\overline{\Omega})}\leq C_{14}\|\,\varphi'''(x)\,\|_{C[0;l]}\,|\,\nu_1-\nu_2\,|<C_{14}\|\,\varphi'''(x)\,\|_{C[0;l]}\delta,$$

where C_{14} is constant. If we put $\varepsilon=C_{14}\,\|\,\varphi'''(x)\,\|_{C\,[0;\,l]}\,\delta$, then we obtain

$$\|\,U_1(t,\,x)-U_2(t,\,x)\,\|_{C\,(\overline{\Omega})}<\varepsilon.$$

The theorem 4.2 is proved. □

5. Conclusions

We studied the boundary value problem $T_{\nu,\,\omega}$ with following assumption

$$\varphi\,(x)\in C^2\,[0;\,l],\ \ \varphi'''\,(x)\in L_2\,(0;\,l),\ \varphi\,(0)=\varphi\,(l)=0.$$

If these conditions fulfilled, then the boundary value problem $T_{\nu,\,\omega}$ is uniquely solvable for regular values of the spectral parameter $\omega\in\aleph$ and this solution is represented in the form of the Fourier series (3.6) and (3.7) in the domains Ω_1 and Ω_2, respectively.

For irregular values of the spectral parameter $\omega\in\Im$ and for some $n=k_1,...,\,k_s$ the problem $T_{\nu,\,\omega}$ has an infinite number of solutions in the form of series (3.15). The solvability condition is formula (3.14). For regular values of the spectral parameter ω we studied the questions of the stability of the solution of the boundary value problem $T_{\nu,\,\omega}$ with respect to the function $\varphi\,(x)$ and with respect to parameter ν.

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