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Γ-RINGS OF WEAK COSETS IN NEARNESS APPROXIMATION SPACES

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Abstract. This article introduces Γ-ring of all near weak cosets in nearness approximation spaces via new operations on the set of all near weak cosets. Additionally, some properties of nearness Γ-homomorphisms are given.

1. Introduction

As a generalization of rough sets, near sets and near approximation spaces were introduced in 2007 [12, 17]. The selection of probe functions that provide a basis for defining and distinguishing affinities between objects is the first step in near set theory. A probe function is a real-valued function representing a feature of objects such as images.

Instead of abstract points, the sets in the nearness approximation space are mainly composed of perceptual objects (non-abstract points). Perceptual objects are featured points. Feature vectors can be used to describe these points [12]. Upper approximation of a set is determined by matching descriptions of objects in the set of perceptual objects. The consideration of upper approximations of perceptual object subsets is a fundamental method in algebraic structures built on nearness approximation space. In a nearness groupoid, the binary operation has the closeness property in upper approximation of set instead of set.

Nobusawa defined the idea of a Γ-ring that is more general than a ring [8]. Barnes weakened the axioms in Nobusawa's description of the Γ-ring [1]. Barnes, Kyuno [5], and Luh [6] investigated the structure of Γ-rings and discovered a number of generalizations that are analogous to ring theory.

In 2012, İnan and Öztürk [2, 3] investigated the concept of group in nearness approximation space. Furthermore, Öztürk et al. [10] defined nearness group of weak cosets in 2013. In 2015, İnan and Öztürk [4] investigated the nearness semigroups. Also, Öztürk and İnan introduced nearness ring in 2019 [9].

The aim of this paper is to define Γ-ring of all near weak cosets in nearness approximation space via new operations on the set of all near weak cosets. Also, some properties of nearness Γ-homomorphisms are given.

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2. Preliminaries

Perceptual objects are points with feature vectors that can be defined. Let $\mathcal{O}$ be a set of perceptual objects, $X \subseteq \mathcal{O}$, $\mathcal{F}$ is a set of probe functions and $\Phi : \mathcal{O} \longrightarrow \mathbb{R}^L$ where the description length is $|\Phi| = L$.

$\Phi(x) = (\varphi_1(x), \varphi_2(x), \varphi_3(x), \cdots, \varphi_i(x), \cdots, \varphi_L(x))$ is an object description of $x \in X$ such that each $\varphi_i \in B \subseteq \mathcal{F}$ ($\varphi_i : \mathcal{O} \longrightarrow \mathbb{R}$) is a probe function representing features of sample objects $X \subseteq \mathcal{O}$ [12].

Sample objects $X \subseteq \mathcal{O}$ are near to each other iff the object descriptions are similar. Notice that each φ_i defines a description of an object and let $\Delta_{\varphi_i} = |\varphi_i(x') - \varphi_i(x)|$ where $x, x' \in \mathcal{O}$.

Let $x, x' \in \mathcal{O}$, $B \subseteq \mathcal{F}$.

$$\sim_B = \{(x, x') \in \mathcal{O} \times \mathcal{O} \mid \Delta_{\varphi_i} = 0 \text{ for all } \varphi_i \in B\}$$

is called the indiscernibility relation on $\mathcal{O}$ where description length $i \leq |\Phi|$ [12].

Definition 2.1. [7] Let $\mathcal{O}$ be a set of perceptual objects, Φ be an object description and $A \subseteq \mathcal{O}$. Then the set description of A is defined as

$$Q(A) = \{\Phi(a) \mid a \in A\}.$$

Definition 2.2. [7, 14] Let $\mathcal{O}$ be a set of perceptual objects and A, $B \subseteq \mathcal{O}$. Then the descriptive (set) intersection of A and B is defined as

$$A \underset{\Phi}{\cap} B = \{x \in A \cup B \mid \Phi(x) \in \mathcal{Q}(A) \text{ and } \Phi(x) \in \mathcal{Q}(B)\}.$$

If $Q(A) \cap Q(B) \neq \emptyset$, then A is called descriptively near B and denoted by $A\delta_\Phi B$. Also, $\xi_\Phi(A) = \{B \in \mathcal{P}(\mathcal{O}) \mid A\delta_\Phi B\}$ is a descriptive nearness collection [13].

Definition 2.3. [12] Let $X \subseteq \mathcal{O}$ and $x \in X$.

$$[x]_{B_r} = \left\{x' \in \mathcal{O} \mid x \sim_{B_r} x'\right\}$$

is called nearness class of $x \in X$.

Definition 2.4. [12] Let $X \subseteq \mathcal{O}$.

$$N_r(B)^* X = \bigcup_{[x]_{B_r} \cap X \neq \varnothing} [x]_{B_r}$$

is called upper approximation of X.

A nearness approximation space is $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B), \nu_{N_r})$ where $\mathcal{O}$ is a set of perceptual objects, $\mathcal{F}$ is a set of probe functions, "$\sim_{B_r}$" is an indiscernibility relation relative to $B_r \subseteq B \subseteq \mathcal{F}$, $N_r(B)$ is a collection of partitions and ν_{N_r} : $\wp(\mathcal{O}) \times \wp(\mathcal{O}) \longrightarrow [0, 1]$ is an overlap function that maps a pair of sets to $[0, 1]$ representing the degree of nearness between sets. The subscript r denotes the cardinality of the restricted subset B_r.

Definition 2.5. [2] Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B), \nu_{N_r})$ be a nearness approximation space and "$\cdot$" be a binary operation defined on $\mathcal{O}$. $G \subseteq \mathcal{O}$ is called a nearness group if the following properties are satisfied:

(NG_1) For all $x, y \in G$, $x \cdot y \in N_r(B)^* G$,

(NG_2) For all $x, y, z \in G$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ is satisfied in $N_r(B)^* G$,

(NG_3) There exists $e_G \in N_r(B)^* G$ such that $x \cdot e_G = e_G \cdot x = x$ for all $x \in G$ (e_G is called the near identity element of G),

(NG_4) There exists $y \in G$ such that $x \cdot y = y \cdot x = e_G$ for all $x \in G$ (y is called the near inverse of x in G and denoted as x^{-1}).

Additionally, if the property $x \cdot y = y \cdot x$ is satisfied in $N_r(B)^* G$ for all $x, y \in G$, then G is said to be a commutative nearness group.

Also, $S \subseteq \mathcal{O}$ is called a nearness semigroup if $x \cdot y \in N_r(B)^* S$ for all $x, y \in S$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ is satisfied in $N_r(B)^* (S)$ for all $x, y, z \in S$.

Theorem 2.1. [3] *Let G be a nearness group, H be a nonempty subset of G and $N_r(B)^* H$ be a groupoid. Then $H \subseteq G$ is a subnearness group of G if and only if $x^{-1} \in H$ for all $x \in H$.*

Theorem 2.2. [10] *Let G be a nearness group, H be a subnearness group of G and $G/_{\sim_\ell}$ be a set of all near left weak cosets of G determined by H. If $(N_r(B)^* G)/_{\sim_\ell} \subseteq N_r(B)^* (G/_{\sim_\ell})$, then $G/_{\sim_\ell}$ is a nearness group with the operation given by $xH \odot yH = (x \cdot y) H$ for all $x, y \in G$.*

Definition 2.6. [9] Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B), \nu_{N_r})$ be a nearness approximation space and "+" and "·" be binary operations defined on $\mathcal{O}$. $R \subseteq \mathcal{O}$ is called a nearness ring if the following properties are satisfied:

(NR_1) R is an abelian nearness group with "+",

(NR_2) R is a nearness semigroup with "·",

(NR_3) For all $x, y, z \in R$,

$$x \cdot (y + z) = (x \cdot y) + (x \cdot z) \text{ and } (x + y) \cdot z = (x \cdot z) + (y \cdot z)$$

is satisfied in $N_r(B)^* R$.

Additionally,

(NR_4) R is said to be a commutative nearness ring if $x \cdot y = y \cdot x$ for all $x, y \in R$,

(NR_5) R is said to be a nearness ring with identity if 1_R belongs to $N_r(B)^* R$ such that $1_R \cdot x = x \cdot 1_R = x$ for all $x \in R$.

Definition 2.7. [1] A Γ-ring (in the sense of Barnes) is a pair (M, Γ) where M and Γ are (additive) abelian groups for which exists a $M \times \Gamma \times M \to M$, the image of (a, α, b) being denoted by $a\alpha b$ for $a, b \in M$ and $\alpha \in \Gamma$, satisfying for all $a, b, c \in M$ and all $\alpha, \beta \in \Gamma$:

- $(a + b)\alpha c = a\alpha c + b\alpha c$,
- $a(\alpha + \beta)b = a\alpha b + a\beta b$,
- $a\alpha(b + c) = a\alpha b + a\alpha c$,
- $(a\alpha b)\beta c = a\alpha(b\beta c)$.

Definition 2.8. [1] Let M be a Γ-ring. A left (right) ideal of M is an additive subgroup I of M such that $M\Gamma I \subseteq I$ ($I\Gamma M \subseteq I$). If I is both a left and a right ideal, then I is called ideal of M.

Definition 2.9. [1] A mapping $\theta : M \to N$ of Γ-rings is called a Γ-homomorphism if $\theta(a + b) = \theta(a) + \theta(b)$ and $\theta(a\alpha b) = \theta(a)\alpha\theta(b)$ for all $a, b \in M$ and all $\alpha \in \Gamma$.

Definition 2.10. [16] Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B), \nu_{N_r})$ be a nearness approximation space and $M, \Gamma \subseteq \mathcal{O}$ be additive abelian nearness groups in $\mathcal{O}$. $M \subseteq \mathcal{O}$ is called an Γ-ring in nearness approximation space or shortly, nearness Γ-ring if the following properties are satisfied:

$(N\Gamma_1)$ $a\alpha b \in N_r(B)^* M$,

$(N\Gamma_2)$ $(a\alpha b)\,\beta c = a\alpha\,(b\beta c)$ property holds on $N_r\,(B)^*\,M$,

$(N\Gamma_3)$ $(a+b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha+\beta)b = a\alpha b + a\beta b$, $a\alpha(b+c) = a\alpha b + a\alpha c$ properties hold on $N_r\,(B)^*\,M$

for all $a, b, c \in M$ and all $\alpha, \beta \in \Gamma$.

In addition, M is called a commutative nearness Γ-ring if $a\alpha b = b\alpha a$ for all $a, b \in M$ and all $\alpha \in \Gamma$.

M is called a nearness Γ-ring with identity if $N_r\,(B)^*\,M$ contains 1_M such that $1_M\alpha a = a\alpha 1_M = a$ for all $a \in M$ and all $\alpha \in \Gamma$.

Lemma 2.1. [16] *Let $M \subseteq \mathcal{O}$ be a nearness Γ-ring and $0_M \in M$. If $0_M\alpha a$, $a\alpha 0_M \in M$ for all $a \in M$ and all $\alpha \in \Gamma$, then $a\alpha 0_M = a0_\Gamma b = 0_M\alpha a = 0_M$ for all $a, b \in M$ and all $\alpha \in \Gamma$.*

Definition 2.11. [16] Let $M, \Gamma \subseteq \mathcal{O}$, M be a nearness Γ-ring and $K \subseteq M$. If K additive abelian nearness group and satisfy the conditions $(N\Gamma_1) - (N\Gamma_3)$, K is called a subnearness Γ-ring of M.

Theorem 2.3. [16] *Let $M, \Gamma \subseteq \mathcal{O}$, M be a nearness Γ-ring, $K \subseteq M$ and $(N_r\,(B)^*\,K, +)$ be a groupoid and $N_r\,(B)^*\,K$ be a Γ-groupoid. Then K is a subnearness Γ-ring of M iff $-k \in K$ for all $k \in K$.*

Definition 2.12. [16] Let M be a nearness Γ-ring and $I \subseteq M$. I is called a left (right) nearness Γ-ideal of M if the following properties are satisfied:

(1) $x + y \in N_r\,(B)^*\,I$,

(2) $-x \in I$,

(3) $m\alpha x \in N_r\,(B)^*\,I$ $(x\alpha m \in N_r\,(B)^*\,I)$

for all $x, y \in I$, all $\alpha \in \Gamma$ and all $m \in M$. If I is both a left and a right nearness Γ-ideal, then I is called a nearness Γ-ideal of M.

3. An Example of Nearness Γ-Ring

In [16], Example 3.3 is not a nearness Γ-ring due to some typos. Therefore another example of nearness Γ-ring is given in Example 3.1.

Example 3.1. $\mathcal{O} = \{a_{ij} \mid 0 \leq i, j \leq 4\}$ be a set of perceptual objects and $B = \{\varphi\} \subseteq \mathcal{F}$ be a subset of probe functions. Probe function

$$\varphi : \mathcal{O} \longrightarrow V = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$$

is given in Table 1.

Table 1 Probe function

	a_{00}	a_{01}	a_{02}	a_{03}	a_{04}	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}
φ	x_1	x_2	x_3	x_5	x_5	x_4	x_4	x_5	x_6	x_7

	a_{20}	a_{21}	a_{22}	a_{23}	a_{24}	a_{30}	a_{31}	a_{32}	a_{33}	a_{34}
φ	x_7	x_6	x_6	x_8	x_7	x_8	x_6	x_7	x_8	x_8

	a_{40}	a_{41}	a_{42}	a_{43}	a_{44}
φ	x_8	x_5	x_7	x_8	x_3

Thus

$$\begin{aligned} [a_{00}]_\varphi &= \{a \in \mathcal{O} \mid \varphi(a) = \varphi(a_{00}) = x_1\} \\ &= \{a_{00}\}, \end{aligned}$$

$$\begin{aligned}[a_{01}]_{\varphi} &= \{a \in \mathcal{O} \mid \varphi(a) = \varphi(a_{01}) = x_2\} \\ &= \{a_{01}\},\end{aligned}$$

$$\begin{aligned}[a_{02}]_{\varphi} &= \{a \in \mathcal{O} \mid \varphi(a) = \varphi(a_{02}) = x_3\} \\ &= \{a_{02}, a_{44}\} = [a_{44}]_{\varphi},\end{aligned}$$

$$\begin{aligned}[a_{03}]_{\varphi} &= \{a \in \mathcal{O} \mid \varphi(a) = \varphi(a_{03}) = x_5\} \\ &= \{a_{03}, a_{04}, a_{12}, a_{41}\} \\ &= [a_{04}]_{\varphi} = [a_{12}]_{\varphi} = [a_{41}]_{\varphi},\end{aligned}$$

$$\begin{aligned}[a_{10}]_{\varphi} &= \{a \in \mathcal{O} \mid \varphi(a) = \varphi(a_{10}) = x_4\} \\ &= \{a_{10}, a_{11}\} = [a_{11}]_{\varphi},\end{aligned}$$

$$\begin{aligned}[a_{13}]_{\varphi} &= \{a \in \mathcal{O} \mid \varphi(a) = \varphi(a_{13}) = x_6\} \\ &= \{a_{13}, a_{21}, a_{22}, a_{31}\} \\ &= [a_{21}]_{\varphi} = [a_{22}]_{\varphi} = [a_{31}]_{\varphi},\end{aligned}$$

$$\begin{aligned}[a_{14}]_{\varphi} &= \{a \in \mathcal{O} \mid \varphi(a) = \varphi(a_{14}) = x_7\} \\ &= \{a_{14}, a_{20}, a_{24}, a_{32}, a_{42}\} \\ &= [a_{20}]_{\varphi} = [a_{24}]_{\varphi} = [a_{32}]_{\varphi} = [a_{42}]_{\varphi},\end{aligned}$$

$$\begin{aligned}[a_{23}]_{\varphi} &= \{a \in \mathcal{O} \mid \varphi(a) = \varphi(a_{23}) = x_8\} \\ &= \{a_{23}, a_{30}, a_{33}, a_{34}, a_{40}, a_{43}\} \\ &= [a_{30}]_{\varphi} = [a_{33}]_{\varphi} = [a_{34}]_{\varphi} = [a_{40}]_{\varphi} = [a_{43}]_{\varphi}.\end{aligned}$$

Therefore

$$\xi_{\varphi} = \left\{[a_{00}]_{\varphi}, [a_{01}]_{\varphi}, [a_{02}]_{\varphi}, [a_{03}]_{\varphi}, [a_{10}]_{\varphi}, [a_{13}]_{\varphi}, [a_{14}]_{\varphi}, [a_{23}]_{\varphi}\right\}.$$

Hence a set of partitions of $\mathcal{O}$ is $N_1(B) = \{\xi_{\varphi}\}$ for $r = 1$. Thus

$$\begin{aligned}N_1(B)^* M &= \bigcup_{[a]_{\varphi} \cap M \neq \varnothing} [a]_{\varphi} \\ &= \{a_{00}\} \cup \{a_{01}\} \cup \{a_{10}, a_{11}\} \\ &= \{a_{00}, a_{01}, a_{10}, a_{11}\}\end{aligned}$$

and

$$\begin{aligned}N_1(B)^* \Gamma &= \bigcup_{[a]_{\varphi} \cap \Gamma \neq \varnothing} [a]_{\varphi} \\ &= \{a_{00}, a_{02}, a_{44}\}\end{aligned}$$

where $M = \{a_{00}, a_{01}, a_{10}\}$, $\Gamma = \{a_{00}, a_{02}\} \subseteq \mathcal{O}$.

Let

$$+_1 \; : \; \begin{array}{ll} \mathcal{O} \times \mathcal{O} & \longrightarrow \mathcal{O} \\ (a_{ij}, a_{mn}) & \longmapsto a_{ij} +_1 a_{mn} \end{array}$$

be a binary operation on M such that

$$a_{ij} +_1 a_{mn} = a_{pr} \quad, i + m \equiv p \ (\mathrm{mod}\ 2) \text{ and } j + n \equiv r \ (\mathrm{mod}\ 2).$$

Then $(M, +_1)$ is an abelian nearness group.

Furthermore, let

$$+_2 \; : \; \begin{array}{ll} \mathcal{O} \times \mathcal{O} & \longrightarrow \mathcal{O} \\ (a_{ij}, a_{mn}) & \longmapsto a_{ij} +_2 a_{mn} \end{array}$$

be a binary operation on Γ such that

$$a_{ij} +_2 a_{mn} = a_{st} \quad, i + m \equiv s \ (\mathrm{mod}\ 4) \text{ and } j + n \equiv t \ (\mathrm{mod}\ 4)\ .$$

Then $(\Gamma, +_2)$ is an abelian nearness group.

Since $a_{01} + a_{10} = a_{11} \notin M$, $M \subseteq \mathcal{O}$ is not a group with binary operation "$+_1$" and so M is not a Γ-ring.

Let

$$\begin{array}{ll} \mathcal{O} \times \Gamma \times \mathcal{O} & \longrightarrow \mathcal{O} \\ (a_{ij}, a_{kl}, a_{mn}) & \longmapsto a_{ij}a_{kl}a_{mn} \end{array}$$

be an operation such that

$$a_{ij}a_{kl}a_{mn} = a_{uv} \quad , u = \min\{i, k, m\} \text{ and } v = \min\{j, l, n\} \; .$$

From Definition 2.10, it is easily shown that

$(N\Gamma_1)$ $a\alpha b \in N_r(B)^* M$,

$(N\Gamma_2)$ $(a\alpha b)\beta c = a\alpha(b\beta c)$ property holds on $N_r(B)^* M$,

$(N\Gamma_3)$ $(a+b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha+\beta)b = a\alpha b + a\beta b$, $a\alpha(b+c) = a\alpha b + a\alpha c$ properties hold on $N_r(B)^* M$

for all $a, b, c \in M$ and all $\alpha, \beta \in \Gamma$.

Consequently, M is a nearness Γ-ring.

4. Nearness Γ-Rings of Weak Cosets

Let M be a nearness Γ-ring and K be a subnearness Γ-ring of M. The relation "$\sim_r$" defined as

$$x \sim_r y :\Leftrightarrow x + (-y) \in K \cup \{0_M\}$$

where $x, y \in M$.

Theorem 4.1. *Let M be a nearness Γ-ring. "$\sim_r$" is a right weak equivalence relation on M.*

Proof. Since M is a nearness Γ-ring, $-x \in M$ for all $x \in M$. Due to $x + (-x) = 0_M \in K \cup \{0_M\}$, $x \sim_r x$. Let $x \sim_r y$ for all $x, y \in M$. Then $x + (-y) \in K \cup \{0_M\}$, that is, $x + (-y) \in K$ or $x + (-y) \in \{0_M\}$. If $x + (-y) \in K$, since K is a subnearness Γ-ring, then $-(x + (-y)) = y + (-x) \in K$. Hence $y \sim_r x$. Also if $x + (-y) \in \{0_M\}$, then $x + (-y) = 0_M$. Therefore $y + (-x) = -(x + (-y)) = -0_M = 0_M$ and so $y \sim_r x$. Consequently, "$\sim_r$" is a right weak equivalence relation on M. □

A class that contains the element $x \in M$, determined by relation "$\sim_r$" is

$$\tilde{x}_r = \{k + x | k \in K, x \in M, k + x \in M\} \cup \{x\} \, .$$

Definition 4.1. Let M be a nearness Γ-ring. A class determined by right weak equivalence relation "$\sim_r$" is called near right weak coset.

Similarly, the relation "$\sim_\ell$" defined as

$$x \sim_\ell y :\Leftrightarrow (-x) + y \in K \cup \{0_M\}$$

where $x, y \in M$.

Theorem 4.2. *Let M be a nearness Γ-ring. "$\sim_\ell$" is a left weak equivalence relation on M.*

Proof. Since M is a nearness Γ-ring, $-x \in M$ for all $x \in M$. Due to $(-x) + x = 0_M \in K \cup \{0_M\}$, $x \sim_\ell x$. Let $x \sim_\ell y$ for all $x, y \in M$. Then $(-x) + y \in K \cup \{0_M\}$, that is, $(-x) + y \in K$ or $(-x) + y \in \{0_M\}$. If $(-x) + y \in K$, since K is a subnearness Γ-ring, then $-((-x) + y) = (-y) + x \in K$. Hence $y \sim_\ell x$. Also if $(-x) + y \in \{0_M\}$, then $(-x) + y = 0_M$. Therefore $(-y) + x = -((-x) + y) = -0_M = 0_M$ and so $y \sim_\ell x$. Consequently, "$\sim_\ell$" is a left weak equivalence relation on M. □

A class that contains the element $x \in M$, determined by relation "$\sim_\ell$" is

$$\tilde{x}_\ell = \{x + k | k \in K, x \in M, x + k \in M\} \cup \{x\}.$$

Definition 4.2. Let M be a nearness Γ-ring. A class determined by left weak equivalence relation "$\sim_\ell$" is called near left weak coset.

We can easily show that $\tilde{x}_r = K + x$ and $\tilde{x}_\ell = x + K$. Since $(M, +)$ is an abelian nearness group, $\tilde{x}_r = \tilde{x}_\ell$.

Let M be a nearness Γ-ring and K be a subnearness Γ-ring of M. Then

$$M/_\sim = \{x + K | x \in M\}$$

is a set of all near weak cosets of M determined by K.

If we consider $N_r(B)^* M$ instead of nearness Γ-ring M

$$(N_r(B)^* M)/_\sim = \{x + K | x \in N_r(B)^* M\}.$$

Hence

$$x + K = \{x + k | k \in K, x \in N_r(B)^* M, x + k \in M\} \cup \{x\}.$$

Definition 4.3. Let M be a nearness Γ-ring and K be a subnearness Γ-ring of M. For $x, y \in M$, let $x + K$ and $y + K$ be two near weak cosets that determined the elements x and y, respectively. Then sum of two near weak cosets that determined by $x + y \in N_r(B)^* M$ can be defined as

$$\{(x + y) + k | k \in K, x + y \in N_r(B)^* M, (x + y) + k \in M\} \cup \{x + y\}$$

and denoted by

$$(x + K) \oplus (y + K) = (x + y) + K.$$

Definition 4.4. Let M be a nearness Γ-ring and K be a subnearness Γ-ring of M. For $x, y \in M$, let $x + K$ and $y + K$ be two near weak cosets that determined the elements x and y, respectively. Then product of two near weak cosets that determined by $x\alpha y \in N_r(B)^* M$, $\alpha \in \Gamma$ can be defined as

$$\{(x\alpha y) + k | k \in K, x\alpha y \in N_r(B)^* M, (x\alpha y) + k \in M\} \cup \{x\alpha y\}$$

and denoted by

$$(x + K)\,\alpha\,(y + K) = (x\alpha y) + K.$$

Definition 4.5. Let $M/_\sim$ be a set of all near weak cosets of M determined by K and $\xi_\Phi(A)$ a descriptive nearness collection of $A \in P(\mathcal{O})$. Then

$$N_r(B)^* (M/_\sim) = \bigcup_{\xi_\Phi(A) \underset{\Phi}{\cap} M/_\sim \neq \emptyset} \xi_\Phi(A)$$

is called upper approximation of $M/_\sim$.

Theorem 4.3. *Let M be a nearness Γ-ring, K be a subnearness Γ-ring of M and $M/_{\sim}$ be a set of all near weak cosets of M determined by K. If*

$$(N_r(B)^* M)/_{\sim} \subseteq N_r(B)^* (M/_{\sim}),$$

then $M/_{\sim}$ is a nearness Γ-ring with the operations given by

$$(x+K) \oplus (y+K) = (x+y)+K$$

and

$$(x+K)\,\alpha\,(y+K) = (x\alpha y)+K$$

for all $x, y \in M$ and all $\alpha \in \Gamma$.

Proof. Let $(N_r(B)^* M)/_{\sim} \subseteq N_r(B)^* (M/_{\sim})$. Since M is a nearness Γ-ring, $(M/_{\sim}, \oplus)$ is an abelian nearness group of all near weak cosets of M determined by K from Theorem 2.2.

$(N\Gamma_1)$ Since M is a nearness Γ-ring, $x\alpha y \in N_r(B)^* M$ for all $x, y \in M$, all $\alpha \in \Gamma$ and $(x+K)\,\alpha\,(y+K) = (x\alpha y)+K \in (N_r(B)^* M)/_{\sim}$ for all $x+K, y+K \in M/_{\sim}$ and all $\alpha \in \Gamma$. From the hypothesis, $(x+K)\,\alpha\,(y+K) = (x\alpha y)+K \in N_r(B)^* (M/_{\sim})$ for all $x+K, y+K \in M/_{\sim}$ and all $\alpha \in \Gamma$.

$(N\Gamma_2)$ Since M is a nearness Γ-ring, associative property is satisfied in $N_r(B)^* M$. Hence

$$\begin{aligned} & ((x+K)\,\alpha\,(y+K))\,\beta\,(z+K) \\ = & ((x\alpha y)+K)\,\beta\,(z+K) \\ = & ((x\alpha y)\,\beta z)+K \\ = & (x\alpha\,(y\beta z))+K \\ = & (x+K)\,\alpha\,((y\beta z)+K) \\ = & (x+K)\,\alpha\,((y+K)\,\beta\,(z+K)) \end{aligned}$$

is satisfied in $(N_r(B)^* M)/_{\sim}$ for all $x+K, y+K, z+K \in M/_{\sim}$ and all $\alpha, \beta \in \Gamma$. From the hypothesis, associative property is satisfied in $N_r(B)^* (M/_{\sim})$.

$(N\Gamma_3)$ Since M is a nearness Γ-ring, left distributive property $x\alpha\,(y+z) = (x\alpha y)+(x\alpha z)$ is satisfied in $N_r(B)^* M$ for all $x, y, z \in M$ and all $\alpha \in \Gamma$. Then

$$\begin{aligned} & (x+K)\,\alpha\,((y+K) \oplus (z+K)) \\ & = (x+K)\,\alpha\,((y+z)+K) \\ & = (x\alpha\,(y+z))+K = ((x\alpha y)+(x\alpha z))+K \\ & = ((x\alpha y)+K) \oplus ((x\alpha z)+K) \\ & = ((x+K)\,\alpha\,(y+K)) \oplus ((x+K)\,\alpha\,(z+K)) \end{aligned}$$

for all $x+K, y+K, z+K \in M/_{\sim}$ and all $\alpha \in \Gamma$. Hence left distributive property is satisfied in $(N_r(B)^* M)/_{\sim}$. Also, right distributive property

$$((x+K) \oplus (y+K))\,\alpha\,(z+K) = ((x+K)\,\alpha\,(z+K)) \oplus ((y+K)\,\alpha\,(z+K))$$

is satisfied in $(N_r(B)^* M)/_{\sim}$, that is, from the hypothesis, it is satisfied in $N_r(B)^* (M/_{\sim})$ for all $x+K, y+K, z+K \in M/_{\sim}$ and all $\alpha \in \Gamma$. Similarly,

$$(x+K)\,(\alpha+\beta)\,(y+K) = ((x+K)\,\alpha\,(y+K)) \oplus ((x+K)\,\beta\,(y+K))$$

is satisfied for $x+K, y+K, z+K \in M/_{\sim}$ and all $\alpha, \beta \in \Gamma$. From the hypothesis, distributive properties are satisfied in $N_r(B)^* (M/_{\sim})$. As a result, $M/_{\sim}$ is a nearness Γ-ring. □

Definition 4.6. Let M be a nearness Γ-ring and K be a subnearness Γ-ring of M. The nearness Γ-ring $M/_{\sim}$ is called a nearness Γ-ring of all near weak cosets of M determined by K and denoted by $M/_{\omega}K$.

Definition 4.7. Let $M_1, M_2 \subseteq \mathcal{O}$ be two nearness Γ-rings and

$$\psi : N_r(B)^* M_1 \to N_r(B)^* M_2$$

be a mapping. If

$$\psi(x+y) = \psi(x) + \psi(y)$$

and

$$\psi(x\alpha y) = \psi(x)\,\alpha\psi(y)$$

for all $x, y \in M_1$ and all $\alpha \in \Gamma$, then ψ is called a nearness Γ-homomorphism.

A nearness Γ-homomorphism $\psi : N_r(B)^* M_1 \to N_r(B)^* M_2$ is called

(1) a nearness Γ-monomorphism if ψ is one-one,

(2) a nearness Γ-epimorphism if ψ is onto,

(3) a nearness Γ-isomorphism if ψ is one-one and onto.

Set of all nearness Γ-homomorphisms from $N_r(B)^* M_1$ into $N_r(B)^* M_2$ is denoted by $Hom(N_r(B)^* M_1, N_r(B)^* M_2)$.

Also, if ψ is onto, M_1 is called near homomorphic to M_2 and denoted by $M_1 \simeq_{\Gamma} M_2$.

Theorem 4.4. *Let M_1, M_2 be two nearness Γ-rings and ψ be a nearness Γ-homomorphism from $N_r(B)^* M_1$ into $N_r(B)^* M_2$. Then*

(i) $\psi(0_{M_1}) = 0_{M_2}$ where $0_{M_2} \in N_r(B)^* M_2$ is the near zero of M_2.

(ii) $\psi(-x) = -\psi(x)$ for all $x \in M_1$.

Proof. (i) Since $0_{M_1} = 0_{M_1} + 0_{M_1}$ and ψ is a nearness Γ-homomorphism, $\psi(0_{M_1}) = \psi(0_{M_1} + 0_{M_1}) = \psi(0_{M_1}) + \psi(0_{M_1})$. Hence $\psi(0_{M_1}) = 0_{M_2}$ by the near identity element is unique.

(ii) $x + (-x) = 0_{M_1}$ for all $x \in M_1$. Then $0_{M_2} = \psi(0_{M_1}) = \psi(x + (-x)) = \psi(x) + \psi(-x)$ by (i). Similarly, $0_{M_2} = \psi(-x) + \psi(x)$ for all $x \in M_1$. Since $\psi(x)$ has a unique near inverse, $\psi(-x) = -\psi(x)$ for all $x \in M_1$. □

Definition 4.8. Let $M_1, M_2 \subseteq \mathcal{O}$ be two nearness Γ-rings and $\psi \in Hom(N_r(B)^* M_1, N_r(B)^* M_2)$. The set

$$Ker\psi = \{x \in M_1 \mid \psi(x) = 0_{M_2}\}$$

is called kernel of nearness Γ-homomorphism ψ.

Theorem 4.5. *Let $M_1, M_2 \subseteq \mathcal{O}$ be two nearness Γ-rings, $\psi \in Hom(N_r(B)^* M_1, N_r(B)^* M_2)$, $(N_r(B)^* Ker\,\psi, +)$ be a groupoid and $N_r(B)^* Ker\,\psi$ be a Γ-groupoid. Then $Ker\psi$ is a subnearness Γ-ring of M_1.*

Proof. Let $x \in Ker\psi$. Then $\psi(x) = 0_{M_2}$. Since $M_1, M_2 \subseteq \mathcal{O}$ are two nearness Γ-rings, $0_{M_1} \in N_r(B)^* M_1$ and $0_{M_2} \in N_r(B)^* M_2$, $\psi(0_{M_1}) = 0_{M_2}$ by Theorem 4.4 (i). Hence $0_{M_2} = \psi(0_{M_1}) = \psi(x + (-x)) = \psi(x) + \psi(-x)$ and so $\psi(-x) = 0_{M_2}$ from $\psi(x) = 0_{M_2}$. Thus from Definition 4.8, $-x \in Ker\,\psi$. Therefore $Ker\psi$ is a subnearness Γ-ring of M_1 from Theorem 2.3. □

Theorem 4.6. *Let $M_1, M_2 \subseteq \mathcal{O}$ be two nearness Γ-rings, $\psi \in Hom\left(N_r(B)^* M_1, N_r(B)^* M_2\right)$, $\left(N_r(B)^* Ker\,\psi, +\right)$ be a groupoid and $N_r(B)^* Ker\,\psi$ be a Γ-groupoid. Then $Ker\psi$ is a nearness Γ-ideal of M_1.*

Proof. Let $x \in Ker\psi$. Since $\psi(-x) = -\psi(x) = -0_{M_2} = 0_{M_2}$, $-x \in Ker\psi$ by Theorem 4.4 (ii). Hence $Ker\psi$ is an additive subnearness group of M_1.

Let $z \in M_1\Gamma(Ker\psi)$. Then $z = m\alpha x$ where $m \in M_1$, $\alpha \in \Gamma$, $x \in Ker\psi$. $\psi(z) = \psi(m\alpha x) = \psi(m)\alpha\psi(x) = \psi(m)\alpha 0_{M_2} = 0_{M_2}$ by Lemma 2.1. Hence $z \in Ker\psi$ and since $Ker\psi \subseteq N_r(B)^*(Ker\psi)$, $z \in N_r(B)^*(Ker\psi)$. Therefore $M_1\Gamma(Ker\psi) \subseteq N_r(B)^*(Ker\psi)$ and so $Ker\psi$ is a left nearness Γ-ideal of M_1. Similarly, $Ker\psi$ is also a right nearness Γ-ideal of M_1. □

Theorem 4.7. *Let $M_1, M_2 \subseteq \mathcal{O}$ be two nearness Γ-rings, $\psi \in Hom\left(N_r(B)^* M_1, N_r(B)^* M_2\right)$, $\left(N_r(B)^* \psi(K), +\right)$ be a groupoid and $N_r(B)^* \psi(K)$ be a Γ-groupoid. If K is a subnearness Γ-ring of M_1 and*

$$\psi\left(N_r(B)^* K\right) = N_r(B)^* \psi(K),$$

then $\psi(K) = \{\psi(x) \,|\, x \in K\}$ is a subnearness Γ-ring of M_2.

Proof. Since K is a subnearness Γ-ring of M_1, $0_K \in N_r(B)^* K$. By Theorem 4.4 (i), $\psi(0_K) = 0_{M_2}$ where $0_{M_2} \in N_r(B)^* M_2$. Thus $0_{M_2} = \psi(0_K) \in \psi\left(N_r(B)^* K\right) = N_r(B)^* \psi(K)$. Hence $N_r(B)^* \psi(K) \neq \emptyset$, that is, $\psi(K) \neq \emptyset$. Since K is a subnearness Γ-ring of M_1, $-x \in K$ for all $x \in K$ from Theorem 2.3. Therefore $-\psi(x) = \psi(-x) \in \psi(K)$ for all $\psi(x) \in \psi(K)$ by Theorem 4.4 (ii). Consequently, $\psi(K)$ is a subnearness Γ-ring of M_2 from Theorem 2.3. □

Theorem 4.8. *Let $M_1, M_2 \subseteq \mathcal{O}$ be two nearness Γ-rings, $L \subseteq M_2$, $\psi \in Hom\left(N_r(B)^* M_1, N_r(B)^* M_2\right)$, $\left(N_r(B)^* L, +\right)$ be a groupoid and $N_r(B)^* L$ be a Γ-groupoid. If L is a subnearness Γ-ring of M_2, then*

$$\psi^{-1}(L) = \{x \in M_1 | \psi(x) \in L\}$$

is a subnearness Γ-ring of M_1.

Proof. Let $x \in \psi^{-1}(L)$. Then $\psi(x) \in L$. Since L is a subnearness Γ-ring of M_2, $-\psi(x) \in L$ from Theorem 2.3. Hence $\psi(-x) \in L$ and so $-x \in \psi^{-1}(L)$ by Theorem 4.4 (ii). Consequently, $\psi^{-1}(L)$ is a subnearness Γ-ring of M_1 from Theorem 2.3. □

Theorem 4.9. *Let M be a nearness Γ-ring and K be a subnearness Γ-ring of M. Then the mapping $\Pi : N_r(B)^* M \to N_r(B)^* (M/_w K)$ defined by $\Pi(x) = x + K$ for all $x \in M$ is a nearness Γ-homomorphism.*

Proof. From the definition of Π, Definitions 4.3 and 4.4,

$$\Pi(x+y) = (x+y) + K = (x+K) \oplus (y+K) = \Pi(x) \oplus \Pi(y),$$

$$\Pi(x\alpha y) = (x\alpha y) + K = (x+K)\alpha(y+K) = \Pi(x)\alpha\Pi(y)$$

for all $x, y \in M$ and all $\alpha \in \Gamma$. Thus Π is a nearness Γ-homomorphism from Definition 4.7. □

Definition 4.9. The nearness Γ-homomorphism Π is called a natural nearness Γ-homomorphism from $N_r(B)^* M$ into $N_r(B)^* (M/_w K)$.

Definition 4.10. Let $M_1, M_2 \subseteq \mathcal{O}$ be two nearness Γ-rings and $K \subseteq M_1$. Let

$$\tau : N_r(B)^* M_1 \longrightarrow N_r(B)^* M_2$$

be a mapping and

$$\tau_K = {}^{\tau}\big|_K : K \longrightarrow N_r(B)^* M_2$$

a restricted mapping. If

$$\tau(x+y) = \tau_K(x+y) = \tau_K(x) + \tau_K(y) = \tau(x) + \tau(y)$$

and

$$\tau(x\alpha y) = \tau_K(x\alpha y) = \tau_K(x)\,\alpha\tau_K(y) = \tau(x)\,\alpha\tau(y)$$

for all $x, y \in K$ and all $\alpha \in \Gamma$, then τ is called a restricted nearness Γ-homomorphism and also, M_1 is called restricted near homomorphic to M_2, denoted by $M_1 \simeq_{rn} M_2$.

Theorem 4.10. *Let $M_1, M_2 \subseteq \mathcal{O}$ be two nearness Γ-rings and $\tau \in Hom(N_r(B)^* M_1, N_r(B)^* M_2)$. Let $(N_r(B)^* Ker\tau, +)$ be a groupoid, $N_r(B)^* Ker\tau$ be a Γ-groupoid, $(\tau(M_1), +)$ be a groupoid, $\tau(M_1)$ be a Γ-groupoid and $(N_r(B)^* M_1)/_{\sim_\ell}$ be a set of all near weak cosets of $N_r(B)^* M_1$ determined by $Ker\tau$. If*

$$(N_r(B)^* M_1)/_\sim \subseteq N_r(B)^* (M_1/_\sim)$$

and

$$N_r(B)^* \tau(M_1) = \tau(N_r(B)^* M_1),$$

then

$$M_1/_\sim \simeq_{rn} \tau(M_1).$$

Proof. Since $(N_r(B)^* Ker\tau, +)$ is a groupoid and $N_r(B)^* Ker\tau$ is a Γ-groupoid, $Ker\tau$ is a subnearness Γ-ring of M_1 from Theorem 4.5. Since $Ker\tau$ is a subnearness Γ-ring of M_1 and $(N_r(B)^* M_1)/_\sim \subseteq N_r(B)^* (M_1/_\sim)$, then $M_1/_\sim$ is a nearness Γ-ring of all near weak cosets of M_1 determined by $Ker\tau$, from Theorem 4.3. Since $N_r(B)^* \tau(M_1) = \tau(N_r(B)^* M_1)$, $\tau(M_1)$ is a subnearness Γ-ring of M_2 from Theorem 4.7. Let

$$\begin{array}{rcl} \sigma : N_r(B)^* (M_1/_\sim) & \longrightarrow & N_r(B)^* \tau(M_1) \\ A & \longmapsto & \sigma(A) = \begin{cases} \sigma_{M_1/_\sim}(A) & , A \in (N_r(B)^* M_1)/_\sim \\ 0_{\tau(M_1)} & , A \notin (N_r(B)^* M_1)/_\sim \end{cases} \end{array}$$

be a mapping where

$$\begin{array}{rcl} \sigma_{M_1/_\sim} = {}^{\sigma}\big|_{M_1/_\sim} : M_1/_\sim & \longrightarrow & N_r(B)^* \tau(M_1) \\ x + Ker\tau & \longmapsto & \sigma_{M_1/_\sim}(x + Ker\tau) = \tau(x) \end{array}$$

for all $x + Ker\tau \in M_1/_\sim$.
Since

$$x + Ker\tau = \{x + k \mid k \in Ker\tau, x + k \in M_1\} \cup \{x\},$$
$$y + Ker\tau = \left\{y + k' \mid k' \in Ker\tau, y + k' \in M_1\right\} \cup \{y\}$$

and τ is a nearness Γ-homomorphism, for $x, y \in M_1$

$$\begin{array}{ll} & x + Ker\tau = y + Ker\tau \\ \Rightarrow & x \in y + Ker\tau \\ \Rightarrow & x \in \{y + k' \mid k' \in Ker\tau, y + k' \in M_1\} \text{ or } x \in \{y\} \\ \Rightarrow & x = y + k' \ (k' \in Ker\tau, y + k' \in M_1) \text{ or } x = y \\ \Rightarrow & -y + x = (-y + y) + k' \ (k' \in Ker\tau) \text{ or } \tau(x) = \tau(y) \\ \Rightarrow & -y + x = k' \ (k' \in Ker\tau) \\ \Rightarrow & -y + x \in Ker\tau \\ \Rightarrow & \tau(-y + x) = 0_{\tau(M_1)} \\ \Rightarrow & \tau(-y) + \tau(x) = 0_{\tau(M_1)} \\ \Rightarrow & -\tau(y) + \tau(x) = 0_{\tau(M_1)} \\ \Rightarrow & \tau(x) = \tau(y) \\ \Rightarrow & \sigma_{M_1/\sim}(x + Ker\tau) = \sigma_{M_1/\sim}(y + Ker\tau). \end{array}$$

Therefore $\sigma_{M_1/\sim}$ is well defined.
For $A, B \in N_r(B)^*(M_1/\sim)$, let $A = B$. Since $\sigma_{M_1/\sim}$ is well defined,

$$\begin{aligned} \sigma(A) &= \begin{cases} \sigma_{M_1/\sim}(A) & , A \in (N_r(B)^* M_1)/\sim \\ 0_{\tau(M_1)} & , A \notin (N_r(B)^* M_1)/\sim \end{cases} \\ &= \begin{cases} \sigma_{M_1/\sim}(B) & , B \in (N_r(B)^* M_1)/\sim \\ 0_{\tau(M_1)} & , B \notin (N_r(B)^* M_1)/\sim \end{cases} \\ &= \sigma(B). \end{aligned}$$

Hence σ is well defined.
For all $x + Ker\tau$, $y + Ker\tau \in M_1/\sim \subset N_r(B)^*(M_1/\sim)$ and all $\alpha \in \Gamma$,

$$\begin{array}{ll} & \sigma((x + Ker\tau) \oplus (y + Ker\tau)) \\ = & \sigma((x + y) + Ker\tau) \\ = & \sigma_{M_1/\sim}((x + y) + Ker\tau) \\ = & \tau(x + y) \\ = & \tau(x) + \tau(y) \\ = & \sigma_{M_1/\sim}(x + Ker\tau) + \sigma_{M_1/\sim}(y + Ker\tau) \\ = & \sigma(x + Ker\tau) + \sigma(y + Ker\tau) \end{array}$$

and

$$\begin{array}{ll} & \sigma((x + Ker\tau)\,\alpha\,(y + Ker\tau)) \\ = & \sigma((x\alpha y) + Ker\tau) \\ = & \sigma_{M_1/\sim}((x\alpha y) + Ker\tau) \\ = & \tau(x\alpha y) \\ = & \tau(x)\,\alpha\tau(y) \\ = & \sigma_{M_1/\sim}(x + Ker\tau)\,\alpha\sigma_{M_1/\sim}(y + Ker\tau) \\ = & \sigma(x + Ker\tau)\,\alpha\sigma(y + Ker\tau). \end{array}$$

Therefore σ is a restricted nearness Γ-homomorphism by Definition 4.10. As a result, $M_1/\sim \simeq_{rn} \tau(M_1)$. □

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