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ON WELL-DEFINED SOLVABILITY OF THE DIRICHLET PROBLEM FOR A SECOND ORDER ELLIPTIC PARTIAL OPERATOR-DIFFERENTIAL EQUATION IN HILBERT SPACE

HAMIDULLA I. ASLANOV AND ROYA F. HATAMOVA

Abstract. In the paper we investigate well-defined solvability of the Dirichlet problem for a second order partial operator-differential equation in a Hilbert space. First we prove a theorem on the isomorphism of the principle part of the given equation. Then it is proved that operator coefficients of the disturbed part of the equation can be chosen from a wider class so that theorems on solvability of a boundary value problem for complete equations hold and are easily verifiable in practical problems. This feature strongly distinguishes our research from the works, where the solvability conditions entail arbitrary smallness of the disturbed part of equations and they are expressed by means of restrictions on the resolvent growth of the appropriate operator beam.

1. Introduction

This paper is devoted to the study of well-defined solvability of the Dirichlet problem for a second order partial operator-differential equation in a Hilbert space. Theory of operator-differential equations arose as generalization of classic theory of matrix differential equations in finite-dimensional spaces to the infinite-dimensional case. Theory of solvability of the Cauchy problem and boundary value problems for the first and second order linear differential equations began to develop in the works of E.Hille, K.Iosida, T.Kato, S.Agmon, R.Lax, Z.I.Khalilov and others and the main part of these problems found its reflection in the books of S.G.Krein [11], A.A.Dezin [6], V.I.Gorbachuk and M.L.Gorbachuk [10], S.Ya.Yakubov [23]. Among these studies we can note the works of M.G.Gasymov [8], B.A.Plamenevsky [18], V.G.Mazya, B.A.Plamenevsky [13], Yu.A.Dubinsky [7], S.S.Mirzoyev [14], A.A.Shkalikov [19], N.I.Yurchuk [24], A.R.Aliyev [1] and others.

Among these researches we note the elegant method suggested by M.G.Gasymov in [8] where it is shown relation of solvability of a boundary value problem for operator-differential equations with completeness of eigen and associated vectors responding to eigen-values from the left half-plane. M.G.Gasymov's results were

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developed in the papers of S.S.Mirzoyev [14], where a new method to the solvability of boundary value problems for operator-differential equations was worked out.

Compared to ordinary operator-differential equations there are a few works devoted to the investigation of solvability of partial operator-differential equations in Hilbert space.

Solvability of boundary value problems for some classes of degenerate partial operator-differential equations was considered in the works of V.B.Shakhmurov [20-21], V.B.Shakhmurov and Azad A.Babayev [22]. In these works, theorems on continuity on compactness and quasi nuclearity of an imbedding operator in abstract anisotropic spaces are proved. The obtained imbedding theorems allow to consider coercive solvability of a new class of strongly degenerate partial differential operator equations.

In the papers [2-5] G.I.Aslanov studied a unique, normal and Fredholm solvability and asymptotic behavior of solutions of higher order partial operator-differential equations. Theorems on multiple completeness of the system of eigen and associated elements of such operators were proven.

In the paper of S.S.Mirzoyev and M.F.Ismayilova [15] a well-defined solvability of a fourth order partial operator-differential equation in Hilbert space was studied.

Solvability of a boundary value problem for second order partial operator-differential equations was considered in the paper of S.S.Mirzoyev, I.J.Jafarov [16]. Similar problems were also considered in S.S.Mirzoyev, N.M.Suleymanov's paper [17].

In R.F.Gatamova's paper [9] conditions on well-defined solvability expressed by means of the coefficients of the given equation were found for a class of second order partial operator-differential equations.

Let H be a Hilbert space with a scalar product $(x,y)_H$, $x,y\in H$, while C be a self-adjoint positive-definite operator in H. The domain of definition of the operator $C^p\,(p\geq 0)$ is a Hilbert space H_p with respect to the scalar product $(x,y)_p=(C^px,C^py)_H$, $x,y\in D\left(C^p\right)$.

For $p=0$ we assume $H_0=H$, $(x,y)_{H_0}=(x,y)_H$, $x,y\in H$. Denote

$$R_+^n=\{x:x=(x_1,x_2,\ldots,x_{n-1},x_n)\,,x_i\in R=(-\infty,\infty)\,,$$

$$i=\overline{1,n-1},\ x_n\in R_+=(0,\infty)\big\}\,.$$

By $D\left(R_+^n,H_2\right)$ denote a linear set of vector-functions infinitely differentiable in R_+^n and having compact supports in R_+^n with the values in H_2 [12]. Let $W_2^2\left(R_+^n;H\right)$ be a completion of the set $D\left(R_+^n,H_2\right)$ with the norm

$$\|u\|_{W_2^2\left(R_+^n;H\right)}=\left(\sum_{k=1}^n\left\|\frac{\partial^2u(x)}{\partial x_k^2}\right\|^2_{L_2\left(R_+^n;H\right)}+\left\|C^2u(x)\right\|^2_{L_2\left(R_+^n;H\right)}\right)^{1/2}$$

Here $L_2\left(R_+^n;H\right)$ is a Hilbert space of all vector-functions $f(x)=f\left(x_1,x_2,\ldots,x_n\right)$ determined in R_+^n almost everywhere, with the values in H and with the norm

$$\|f\|_{L_2\left(R_+^n;H\right)}=\left(\int_{R_+^n}\left|f\left(x_1,x_2,\ldots x_n\right)\right|^2dx_1,dx_2,\ldots dx_n\right)^{1/2}$$

In Hilbert space H we consider the following boundary value problem for elliptic type partial operator-differential equation

$$Lu = -\sum_{k=1}^{n} a_k \frac{\partial^2 u(x)}{\partial x_k^2} + C^2 u(x) + \sum_{k=1}^{n} R_k \frac{\partial u(x)}{\partial x_k} + Tu(x) = f(x) \tag{1.1}$$

$$u\left(x_1, x_2, \ldots, x_{n-1}, 0\right) = 0 \tag{1.2}$$

where $(x_1, x_2, \ldots, x_{n-1}) \in R^{n-1}$, $(x_1, x_2, \ldots, x_{n-1}, x_n) \in R_+^n$, $f(x)$ and $u(x)$ are the vector-functions determined almost everywhere in R_+^n with the values in H and the operator coefficients of equation (1.1) satisfy the following conditions:
1) The scalar numbers $a_k > 0,\ k = 1, 2, \ldots n$,
2) C is a self-adjoint positive-definite operator in H,
3) The operators $Q_k = R_k C^{-1}, k = 1, 2, \ldots n$ and $F = TC^{-1}$ are bounded in H.

Denote by

$$\overset{\circ}{W}{}_2^2\left(R_+^n; H\right) = \left\{u(x) : u(x) \in W_2^2\left(R_+^n; H\right), u\left(x_1, x_2, \ldots x_{n-1}, 0\right) = 0\right\}$$

and determine the operators

$$L_0 u = -\sum_{k=1}^{n} a_k \frac{\partial^2 u(x)}{\partial x_k^2} + C^2 u(x),\ u(x) \in \overset{\circ}{W}{}_2^2\left(R_n^+; H\right),$$

$$L_1 u = \sum_{k=1}^{n} R_k \frac{\partial u(x)}{\partial x_k} + Tu,\ u(x) \in \overset{\circ}{W}{}_2^2\left(R_n^+; H\right),$$

$$Lu = L_0 u + L_1 u,\ u(x) \in \overset{\circ}{W}{}_2^2\left(R_n^+; H\right).$$

Definition 1.1. If for $f(x) \in L_2\left(R_+^n; H\right)$ there exists a vector-function $u(x) \in W_2^2\left(R_+^n; H\right)$ satisfying equation (1.1) almost everywhere in R_+^n, then $u(x)$ is said to be a regular solution of equation (1.1).

Definition 1.2. If for any $f(x) \in L_2\left(R_+^n; H\right)$ there exists a regular solution $u(x) \in W_2^2\left(R_+^n; H\right)$ of equation (1.1) satisfying condition (1.2) in the sense of convergence

$$\lim_{x_n \to +0} \left\| u\left(x_1, x_2, \ldots x_{n-1}, x_n\right) \right\| = 0$$

almost for all $\tilde{x} \in (x_1, x_2, \ldots x_{n-1}) \in R^{n-1}$ and the estimation

$$\|u\|_{W_2^2\left(R_+^n; H\right)} \leq const \cdot \|f\|_{L_2\left(R_+^n; H\right)}$$

holds, then problem (1.1), (1.2) is said to be correctly solvable.

The goal of this paper is to find conditions on the coefficients of equation (1.1) that provide well-defined solvability of problem (1.1), (1.2). Note that for $a_k = 1$ $(k = 1, 2, \ldots n)$ a similar problem was considered in [15]. When C is a normal invertible operator, whose spectrum is contained in some vector in the right half-plane for $a_k = 1$ $(k = 1, 2)$, were considered in the papers [17,18].

2. Proof of a theorem on isomorphism of operator L_0.

We represent problem (1.1), (1.2) in the following form

$$Lu=\left(-a_n\frac{\partial^2 u(x)}{\partial x_n^2}+R_n\frac{\partial u(x)}{\partial x_n}+C^2u(x)\right)+$$
$$+\left(-\sum_{k=1}^{n-1}a_k\frac{\partial^2 u(x)}{\partial x_k^2}+\sum_{k=1}^{n-1}R_k\frac{\partial u(x)}{\partial x_k}+Tu(x)\right)=f(x) \qquad (2.1)$$
$$u\left(x_1,x_2,\ldots,x_{n-1},0\right)=0. \qquad (2.2)$$

Denote by $\hat{f}\left(\xi_1,\xi_2,\ldots\xi_{n-1},x_n\right)$ Fourier transform of the vector-function $f(x_1,x_2,\ldots x_{n-1},x_n)$ with respect to the variables $x_1,\ldots x_{n-1}$,i.e.

$$\hat{f}\left(\xi_1,\xi_2,\ldots\xi_{n-1},x_n\right)=$$
$$=\frac{1}{(2\pi)^{\frac{n-1}{2}}}\int_{R^{n-1}}f\left(x_1,x_2,\ldots x_{n-1},x_n\right)e^{-i(x_1\xi_1+x_2\xi_2+\ldots+x_{n-1},\xi_{n-1})}dx_1dx_2\ldots dx_{n-1}.$$

Then, from problem (2.1),(2.2) after Fourier transform we get:

$$\hat{L}\hat{u}\left(\xi,x_n\right)=-a_n\frac{\partial^2\hat{u}\left(\xi,x_n\right)}{\partial x_n^2}+R_n\frac{\partial\hat{u}\left(\xi,x_n\right)}{\partial x_n}+C^2\hat{u}\left(\xi,x_n\right)+$$
$$+\sum_{k=1}^{n-1}a_k\xi_k^2\hat{u}(\xi,x_n)+\sum_{k=1}^{n-1}iR_k\hat{u}(\xi,x_n)+T\hat{u}(\xi,x_n)=\hat{f}(\xi,x_n) \qquad (2.3)$$
$$\hat{u}(\xi,x_n)=0 \qquad (2.4)$$

where $\xi=(\xi_1,\xi_2\ldots,\xi_{n-1})\in R^{n-1},x_n\in R_+=(0,\infty)$.

Let

$$\overset{\circ}{W}{}_{2,\xi}^2\left(R_+^n;H\right)=\left\{v;v\in W_{2,\xi}^2\left(R_+^n;H\right),v\left(\xi_1,\xi_2,\ldots\xi_{n-1},0\right)=0\right\}.$$

Denote $B(\xi)=\sum\limits_{k=1}^{n-1}a_k\xi_k^2E+C^2$. For any $\xi=(\xi_1,\xi_2,\ldots,\xi_{n-1})\in R^{n-1}$ the operator-function $B\left(\xi\right)$ is self-adjoint, positive-definite, $B\left(\xi\right)\geq\mu_0^2E$, where μ_0 is a lower bound of the spectrum of the operator C.

Thus, we get the following boundary value problem:

$$\hat{L}\left(\hat{u}\left(\xi,x_n\right)\right)=\left(-a_n\frac{\partial^2\hat{u}\left(\xi,x_n\right)}{\partial x_n^2}+B\left(\xi\right)\hat{u}\left(\xi,x_n\right)\right)+R_n\frac{\partial\hat{u}\left(\xi,x_n\right)}{\partial x_n}+$$
$$+\sum_{k=1}^{n-1}i\xi_kR_k\hat{u}(\xi,x_n)+T\hat{u}(\xi,x_n)=\hat{f}(\xi,x_n) \qquad (2.5)$$
$$\hat{u}(\xi,0)=0 \qquad (2.6)$$

Denote

$$\hat{L}_0\hat{u}\left(\xi,x_n\right)=-a_n\frac{\partial^2\hat{u}(\xi,x_n)}{\partial x_n^2}+B\left(\xi\right)\hat{u}(\xi,x_n),\hat{u}(\xi,x_n)\in\overset{\circ}{W}{}_2^2\left(R_n^+;H\right), \qquad (2.7)$$

$$\hat{L}_1\hat{u}\left(\xi,x_n\right)=R_n\frac{\partial\hat{u}(\xi,x_n)}{\partial x_n}+\sum_{k=1}^{n-1}i\xi_kR_k\hat{u}(\xi,x_n)+T\hat{u}(\xi,x_n),$$
$$\hat{u}(\xi,x_n)\in\overset{\circ}{W}{}_2^2\left(R_n^+;H\right), \qquad (2.8)$$

$$\hat{L}\hat{u} = \hat{L}_0\hat{u} + \hat{L}_1\hat{u}.$$

Lemma 2.1. *For any $\varphi(\xi) \in H_{3/2}$ we have the inequality*

$$\left\| B(\xi)\, e^{-a_n^{-\frac{1}{2}} B^{1/2}(\xi) x_n} \varphi(\xi) \right\|_{L_2\left(R_+^n;H\right)} \le \frac{a_n^{\frac{1}{4}}}{\sqrt{2}} \cdot \left\| B^{\frac{3}{4}} \varphi(\xi) \right\|_H .$$

Proof. Let $\varphi(\xi) \in H_{3/2}$. Assume $B^{\frac{3}{4}}\varphi(\xi) = y(\xi)$. We have

$$\left\| B(\xi) e^{-a_n^{-\frac{1}{2}} B^{1/2}(\xi) x_n} \varphi(\xi) \right\|^2_{L_2(R_+;H)} = \left\| B^{\frac{1}{4}}(\xi) e^{-a_n^{-\frac{1}{2}} B^{1/2}(\xi) x_n} y(\xi) \right\|^2_{L_2(R_+;H)} =$$

$$= \left(B^{\frac{1}{2}}(\xi) e^{-2a_n^{-\frac{1}{2}} B^{1/2}(\xi) x_n} y(\xi), y(\xi) \right)_{L_2(R_+;H)} = \int_{\mu_0}^{\infty} \left(\sum_{k=1}^{n-1} a_k \xi_k^2 + \mu^2 \right)^{1/2} \times$$

$$\times \int_{\mu_0}^{\infty} e^{-2a_n^{-\frac{1}{2}} \left(\sum\limits_{k=1}^{n-1} a_k \xi^2 + \mu^2 \right)^{1/2} x_n} dx_n \Bigg)\, d\left(E_\mu y(\xi), y(\xi)\right) =$$

$$= \int_{\mu_0}^{\infty} \left(\sum_{k=1}^{n-1} a_k \xi_k^2 + \mu^2 \right)^{1/2} \cdot \frac{1}{2a_n^{-\frac{1}{2}}} \cdot \left(\sum_{k=1}^{n-1} a_k \xi_k^2 + \mu^2 \right)^{-\frac{1}{2}} d\left(E_\mu y(\xi), y(\xi)\right) =$$

$$= \frac{a_n^{\frac{1}{2}}}{2} \int_{\mu_0}^{\infty} d\left(E_\mu y(\xi), y(\xi)\right) = \frac{a_n^{\frac{1}{2}}}{2} \|y(\xi)\|^2 = \frac{a_n^{\frac{1}{2}}}{2} \left\| B^{\frac{3}{4}}(\xi)\, \varphi(\xi) \right\|_H^2 .$$

Hence the required inequality follows.

Now we prove the following theorem.

Theorem 2.1. *The operator $\hat{L}_0$ isomorphically maps $\overset{\circ}{W}{}^2_{2,\xi}\left(R_+^n;H\right)$ onto the space $L_2\left(R_+^n;H\right)$.*

Proof. At first we show $Ker\hat{L}_0 = \{0\}$.

The general solution of the homogeneous equation

$$-a_n \frac{\partial^2 \hat{u}(\xi, x_n)}{\partial x_n^2} + B(\xi)\hat{u}(\xi, x_n) = 0$$

from the space $W^2_{2,\xi}\left(R_+^n;H\right)$ is of the form:

$$\hat{u}(\xi, x_n) = e^{-a_n^{-\frac{1}{2}} B^{\frac{1}{2}}(\xi) x_n} \varphi(\xi), \ \ \xi \in R^{n-1}.$$

Then from the condition $\hat{u}(\xi, 0) = 0$ it follows that $\varphi(\xi) = 0$, i.e. $\hat{u}(\xi, x_n) = 0$.

We now show that $Jm\hat{L}_0 = L_2\left(R_+^n;H\right)$.

For the fixed $\xi \in R^{n-1}$ in R^n we consider the equation

$$-a_n \frac{\partial^2 \hat{u}(\xi, x_n)}{\partial x_n^2} + B(\xi)\hat{u}(\xi, x_n) = f_1(\xi, x_n) \tag{2.9}$$

where

$$f_1(\xi, x_n) = \begin{cases} \hat{f}(\xi, x_n), & x_n > 0 \\ 0, & x_n < 0 \end{cases}, \ \ \xi \in R^{n-1}.$$

After applying the Fourier transform with respect to the variable x_n we get that equation (2.9) has a solution in the form

$$V(\xi,x_n)=\frac{1}{\sqrt{2\pi}}\int_R \left(a_n\xi_n^2+B(\xi)\right)^{-1}\hat{f}_1(\xi,x_n)\,e^{ix_n\xi_n}d\xi_n,\ \xi\in R^{n-1}$$

where $\hat{f}_1(\xi,x_n)$ is Fourier transform of the vector-function $f_1(\xi,x_n)$ with respect to the variable x_n; i.e.

$$\hat{f}_1(\xi,x_n)=\frac{1}{\sqrt{2\pi}}\int_R f_1(\xi,x_n)\,e^{-i\xi_n x_n}d\xi_n,\ \xi\in R^{n-1}$$

It is easy to see that for the fixed $\xi\in R^{n-1}$ $V(\xi,x_n)$ satisfies equation (2.9) in R almost everywhere.

Show that $\frac{\partial^2 V(\xi,x_n)}{\partial x_n^2}\in L_2(R_+;H)$, $B(\xi)V(\xi,x_n)\in L_2(R;H)$. Indeed, by the Plancherel theorem we have for fixed $\xi=(\xi_1\xi_2\ldots\xi_{n-1})$ the following relations

$$\begin{aligned}
&\left\|\frac{\partial^2 V(\xi,x_n)}{\partial x_n^2}\right\|^2_{L_2(R_+;H)}+\left\|B(\xi)V(\xi,x_n)\right\|^2_{L_2(R_+;H)}=\\
&=\left\|\xi_n^2\left(a_n\xi_n^2+B(\xi)\right)^{-1}\hat{f}_1(\xi,x_n)\right\|^2_{L_2(R_+;H)}+\\
&+\left\|B(\xi)\left(a_n\xi_n^2+B(\xi)\right)^{-1}\hat{f}_1(\xi,x_n)\right\|^2_{L_2(R_+;H)}\le\\
&\le\sup_{\xi_n}\left\|\xi_n^2\left(a_n\xi_n^2+B(\xi)\right)^{-1}\right\|^2\cdot\left\|\hat{f}(\xi,x_n)\right\|^2_{L_2(R_+;H)}+\\
&+\sup_{\xi_n}\left\|B(\xi)\left(a_n\xi_n^2+B(\xi)\right)^{-1}\right\|^2\cdot\left\|\hat{f}(\xi,x_n)\right\|^2_{L_2(R_+;H)}\\
&=\left(\sup_{\xi_n}\left\|\xi_n^2\left(a_n\xi_n^2+B(\xi)\right)^{-1}\right\|^2+\right.\\
&\left.+\sup_{\xi_n}\left\|B(\xi)\left(a_n\xi_n^2+B(\xi)\right)^{-1}\right\|^2\right)\cdot\left\|\hat{f}_1(\xi,x_n)\right\|_{L_2(R_+;H)}
\end{aligned}\tag{2.10}$$

Using spectral expansion of the operator C for all $\xi_n\in R$,$\xi\in R^{n-1}$ we have:

$$\begin{aligned}
&\left\|\xi_n^2\left(a_n\xi_n^2+B(\xi)\right)^{-1}\right\|=\left\|\xi_n^2\left(a_n\xi_n^2+\sum_{k=1}^{n-1}a_k\xi_k^2E+C^2\right)^{-1}\right\|=\\
&=\sup_{\mu\in\tau(c)}\left|\xi_n^2\left(a_n\xi_n^2+\sum_{k=1}^{n-1}a_k\xi_k^2+\mu^2\right)^{-1}\right|\le\sup_{\mu\in\tau(c)}\left|\xi_n^2\left(a_n\xi_n^2+\mu^2\right)^{-1}\right|\le\\
&\le\sup_{\mu\in\tau(c)}\frac{1}{a_n}\left|a_n\xi_n^2\left(a_n\xi_n^2+\mu^2\right)^{-1}\right|\le\frac{1}{a_n},
\end{aligned}\tag{2.11}$$

$$\begin{aligned}
&\left\|B(\xi)\left(a_n\xi_n^2+B(\xi)\right)^{-1}\right\|=\\
&=\sup_{\mu\in\tau(c)}\left|\left(\sum_{k=1}^{n-1}a_k\xi_k^2+\mu^2\right)\left(a_n\xi_n^2+\sum_{k=1}^{n-1}a_k\xi_k^2E+C^2\right)^{-1}\right|<1
\end{aligned}\tag{2.12}$$

Taking into account estimations (2.11), (2.12) from inequality (2.10) we get:

$$\left\| \frac{\partial V(\xi, x_n)}{\partial x_n} \right\|^2_{L_2(R_+;H)} + \|B(\xi) V(\xi, x_n)\|_{L_2(R_+;H)} \leq \left(\frac{1}{a_n} + 1 \right) \left\| \hat{f}_1(\xi, x_n) \right\|_{L_2(R_+;H)}.$$

Thus, we obtain that $V(\xi, x_n) \in W_2^2(R_+; H)$ for any $\xi \in R^{n-1}$ moreover $\frac{\partial^2 V(\xi, x_n)}{\partial x_n^2} \in L_2(R_+; H), B(\xi) V(\xi, x_n) \in L_2(R_+; H)$ for any $\xi \in R^{n-1}$.

Here $B(\xi) = \sum\limits_{k=1}^{n-1} \xi_k^2 a_k + C^2$.

Denote

$$V_1(\xi, x_n) = \begin{cases} V(\xi, x_n), & \xi \in R^{n-1}, x_n > 0, \\ 0, & \xi \in R^{n-1}, x_n < 0. \end{cases}$$

Then $\frac{\partial^2 V_1(\xi, x_n)}{\partial x_n^2} \in L_2(R_+; H), B(\xi) V_1(\xi, x_n) \in L_2(R_+; H)$.

By the theorem on traces [12] it follows that $V_1(\xi, 0) \in H_{3/2}$, moreover

$$\left\| B^{3/4}(\xi) V_1(\xi, 0) \right\|_H^2 \leq$$

$$\leq const \left(\left\| \frac{\partial^2 V_1(\xi, x_n)}{\partial x_n^2} \right\|_{L_2(R_+;H)} + \|B(\xi) V_1(\xi, x_n)\|_{L_2(R_+;H)} \right) \leq$$

$$\leq const \left(\left\| \frac{\partial^2 V(\xi, x_n)}{\partial x_n^2} \right\|_{L_2(R_+;H)} + \|B(\xi) V(\xi, x_n)\|^2_{L_2(R_+;H)} \right) \leq$$

$$\leq const \|f_1(\xi, x_n)\|^2_{L_2(R_+;H)} = const \|f(\xi, x_n)\|^2_{L_2(R_+;H)}.$$

Thus, the general solution of the equation

$$-a_n \frac{\partial^2 \hat{u}(\xi, x_n)}{\partial x_n^2} + B(\xi) \hat{u}(\xi, x_n) = \hat{f}(\xi, x_n), \; x_n > 0, \; \xi \in R^{n-1}$$

from the space $W_2^2(R_+; H)$ has the form

$$u(\xi, x_n) = V_1(\xi, x_n) + e^{-a_n^{-\frac{1}{2}} B^{1/2}(\xi) x_n} \varphi(\xi), \;\; \varphi(\xi) \in H_{3/2}.$$

From the condition $u(\xi, 0) = 0$ we get $\varphi(\xi) = -V_1(\xi, 0)$.

Consequently, $u(\xi, x_n) = V_1(\xi, x_n) - e^{-a_n^{-\frac{1}{2}} B^{\frac{1}{2}}(\xi) x_n} u(\xi, 0)$.

On the other hand, by lemma 2.1 we have:

$$\left\| \frac{\partial^2 u(\xi, x_n)}{\partial x_n^2} \right\|_{L_2(R_+;H)} \leq$$

$$\leq \|V_1(\xi, x_n)\|_{L_2(R_+;H)} + \left\| a_n B(\xi) e^{-a_n^{-\frac{1}{2}} B(\xi) x_n} \hat{u}(\xi, 0) \right\|_{L_2(R_+;H)} \leq$$

$$\leq const \cdot \left\| \hat{f}(\xi, x_n) \right\|_{L_2(R_+;H)} + const \cdot \left\| B^{3/4} \hat{u}(\xi, 0) \right\|_{H_0} \leq$$

$$\leq const \left\| \hat{f}(\xi, x_n) \right\|_{L_2(R_+;H)} + const \cdot \left\| \hat{f}(\xi, x_n) \right\|_{L_2(R_+;H)}.$$

$$B(\xi) \cdot \hat{u}(\xi, x_n) = \hat{f}(\xi, x_n) + a_n \frac{\partial^2 \hat{u}(\xi, x_n)}{\partial x_n^2} \in L_2(R_+; H)$$

and

$$\left\| B(\xi)\,\hat{u}(\xi,x_n) \right\|_{L_2(R_+;H)} \le const \left\| \hat{f}(\xi,x_n) \right\|_{L_2(R_+;H)}.$$

Consequently,

$$\left\| \frac{\partial^2 \hat{u}(\xi,x_n)}{\partial x_n^2} \right\|^2_{L_2(R_+;H)} + \left\| \left(\sum_{k=1}^{n-1} |\xi_k|^2 + C^2 \right)^{-1} \right\|^2 \left\| \hat{u}(\xi,x_n) \right\|^2_{L_2(R_+;H)} \le$$

$$\le const \left\| \hat{f}(\xi,x_n) \right\|^2_{L_2(R_+;H)}$$

In what follows, integrating with respect to ξ in R^{n-1}, we get

$$\left\| \hat{u}(\xi,x_n) \right\|^2_{W_2^2\left(R_+^n;H\right)} \le const \left\| \hat{f}(\xi,x_n) \right\|^2_{L_2\left(R_+^n;H\right)} \tag{2.13}$$

The obtained inequality shows that all the conditions of the Banach theorem on an inverse operator are fulfilled for the operator $\hat{L}_0$. Therefore, the operator $\hat{L}_0 : W_2^2\left(R_+^n;H\right) \to L_2\left(R_+^n;H\right)$ is an isomorphism. Theorem 2.1 has been proved.

We now prove the following auxiliary results

Lemma 2.2. *For any* $\hat{u}(\xi,x_n) \in \overset{0}{W}{}_2^2\left(R_+^n;H\right)$ *we have the inequality*

$$\left\| -a_n \frac{\partial^2 \hat{u}(\xi,x_n)}{\partial x_n^2} + B(\xi)\,\hat{u}(\xi,x_n) \right\|_{L_2\left(R_+^n;H\right)} = a_n^2 \left\| \frac{\partial^2 \hat{u}(\xi,x)}{\partial x_n^2} \right\|^2_{L_2\left(R_+^n;H\right)} +$$

$$+ \left\| B(\xi)\,\hat{u}(\xi,x_n) \right\|^2_{L_2\left(R_+^n;H\right)} + 2 \left\| B^{\frac{1}{2}}(\xi)\,\frac{\partial \hat{u}(\xi,x_n)}{\partial x_n} \right\|^2_{L_2\left(R_+^n;H\right)} \tag{2.14}$$

Proof. Let $\xi \in R^{n-1}$, $x_n \in R_+ = (0,\infty)$. Having in mind $\hat{u}(\xi,0) = 0$, we can write that

$$\left\| -a_n \frac{\partial^2 \hat{u}(\xi,x_n)}{\partial x_n^2} + B(\xi)\,\hat{u}(\xi,x_n) \right\|_{L_2\left(R_+;H\right)} = a_n^2 \left\| \frac{\partial^2 \hat{u}(\xi,x_n)}{\partial x_n^2} \right\|_{L_2\left(R_+;H\right)} +$$

$$+ \left\| B(\xi)\,\hat{u}(\xi,x_n) \right\|^2_{L_2(R_+;H)} - 2a_n \left\| \frac{\partial^2 \hat{u}(\xi,x_n)}{\partial x_n^2}, B(\xi)\,\hat{u}(\xi,x_n) \right\|_{L_2(R_+;H)}. \tag{2.15}$$

On the other hand,

$$\left(\frac{\partial^2 \hat{u}(\xi,x_n)}{\partial x_n^2}, B(\xi)\hat{u}(\xi,x_n) \right)_{L_2(R_+;H)} = \int_0^\infty \left(\frac{\partial^2 \hat{u}(\xi,x_n)}{\partial x_n^2}, B(\xi)\hat{u}(\xi,x_n) \right)_H dx_n =$$

$$= - \left(B^{\frac{3}{4}}(\xi)\hat{u}(\xi,x_n), B^{\frac{1}{4}}(\xi)\frac{\partial \hat{u}(\xi,x_n)}{\partial x_n} \right)\Bigg|_0^\infty -$$

$$- \int_0^\infty \left(B^{\frac{1}{2}}(\xi)\frac{\partial \hat{u}(\xi,x_n)}{\partial x_n}, B^{\frac{1}{2}}\frac{\partial \hat{u}(\xi,x_n)}{\partial x_n} \right)_H dx_n =$$

$$= - \int_0^\infty \left\| B^{\frac{1}{2}}(\xi)\frac{\partial \hat{u}(\xi,x_n)}{\partial x_n} \right\|^2 dx_n. \tag{2.16}$$

Integrating in R^{n-1} with respect to the variable $\xi = (\xi_1, \xi_2, \ldots \xi_{n-1})$ from (2.16) we get the statement of the lemma.

Theorem 2.2. *For any* $\hat{u}(\xi, x_n) \in \overset{\circ}{W}{}^2_{2,\xi}\left(R^n_+; H\right)$ *we have the following estimations:*

$$\|B(\xi)\hat{u}(\xi, x_n)\|_{L_2\left(R^n_+;H\right)} \leq \left\|\hat{L}_0\hat{u}(\xi, x_n)\right\|_{L_2\left(R^n_+;H\right)} \tag{2.17}$$

$$\left\|B^{\frac{1}{2}}(\xi)\hat{u}(\xi, x_n)\right\|_{L_2\left(R^n_+;H\right)} \leq \frac{1}{2\sqrt{a_n}}\left\|\hat{L}_0\hat{u}(\xi, x_n)\right\|_{L_2\left(R^n_+;H\right)} \tag{2.18}$$

$$\|\xi C\hat{u}(\xi, x_n)\|_{L_2\left(R^n_+;H\right)} \leq \frac{1}{2\sqrt{a_n}}\left\|\hat{L}_0\hat{u}(\xi, x_n)\right\|_{L_2\left(R^n_+;H\right)} \tag{2.19}$$

$$\left\|C^2\hat{u}(\xi, x_n)\right\|_{L_2\left(R^n_+;H\right)} \leq \left\|\hat{L}_0\hat{u}(\xi, x_n)\right\|_{L_2\left(R^n_+;H\right)} \tag{2.20}$$

Proof. Inequality (2.17) follows from the statement of theorem 2.1, i.e. from inequality (2.13). We prove inequality (2.18). Let $\hat{u}(\xi, x_n) \in \overset{\circ}{W}{}^2_{2,\xi}\left(R^n_+; H\right)$.

Then we get:

$$a_n \int_0^\infty \left\|B^{1/2}(\xi)\frac{\partial \hat{u}(\xi, x_n)}{\partial x_n}\right\|^2 dx_n =$$

$$= a_n \int_0^\infty \left(B^{1/2}(\xi)\frac{\partial \hat{u}(\xi, x_n)}{\partial x_n}, B^{1/2}(\xi)\frac{\partial \hat{u}(\xi, x_n)}{\partial x_n}\right)_H dx_n =$$

$$= a_n \left(B^{\frac{3}{4}}(\xi)\hat{u}(\xi, x_n), B^{\frac{1}{4}}(\xi)\frac{\partial \hat{u}(\xi, x_n)}{\partial x_n}\right)\Big|_0^\infty -$$

$$-a_4 \int_0^\infty \left(B(\xi)\hat{u}(\xi, x_n), \frac{\partial \hat{u}(\xi, x_n)}{\partial x_n}\right)_H dx_n =$$

$$= -a_n \int_0^\infty \left(B(\xi)\hat{u}(\xi, x_n), \frac{\partial \hat{u}(\xi, x_n)}{\partial x_n}\right) dx_n \leq$$

$$\leq a_n \|B(\xi)\hat{u}(\xi, x_n)\|_{L_2(R_+;H)} \cdot \left\|\frac{\partial \hat{u}(\xi, x_n)}{\partial x_n}\right\|_{L_2(R_+;H)} \leq$$

$$\leq a_n \left(\frac{\varepsilon}{2} \cdot \|B(\xi)\hat{u}(\xi, x_n)\|^2_{L_2(R_+;H)}\right) + \frac{1}{2\varepsilon}\left\|\frac{\partial \hat{u}(\xi, x_n)}{\partial x_n}\right\|^2_{L_2(R_+;H)}. \tag{2.21}$$

Assuming $\varepsilon = \frac{1}{a_n}$ we get:

$$a_n \left\|B^{1/2}(\xi)\hat{u}(\xi, x_n)\right\|^2_{L_2(R_+;H)} \leq$$

$$\leq a_n \left(\frac{1}{2a_n}\|B(\xi)\hat{u}(\xi, x_n)\|^2_{L_2(R_+;H)} + \frac{a_n}{2}\left\|\frac{\partial \hat{u}(\xi, x_n)}{\partial x_n}\right\|^2_{L_2(R_+;H)}\right) =$$

$$= \frac{1}{2}\left(\|B(\xi)\hat{u}(\xi, x_n)\|^2_{L_2(R_+;H)} + a_n \left\|\frac{\partial \hat{u}(\xi, x_n)}{\partial x_n}\right\|^2_{L_2(R_+;H)}\right) \tag{2.22}$$

Integrating with respect to the variable $\xi = (\xi_1, \xi_2, \dots \xi_{n-1})$ in R^{n-1} we get:

$$a_n \|B(\xi)\hat{u}(\xi, x_n)\|^2_{L_2\left(R^n_+;H\right)} \leq$$

$$\leq \frac{1}{2}\left(\|B(\xi)\hat{u}(\xi, x_n)\|^2_{L_2\left(R^n_+;H\right)} + a_n^2 \left\|\frac{\partial \hat{u}(\xi, x_n)}{\partial x_n}\right\|^2_{L_2\left(R^n_+;H\right)}\right.$$

From theorem 2.1 it follows that

$$\|B\hat{u}(\xi,x_n)\|^2_{L_2(R^n_+;H)} + a_n^2\left\|\frac{\partial\hat{u}(\xi,x_n)}{\partial x_n}\right\|^2_{L_2(R^n_+;H)} = \left\|\hat{L}_0\hat{u}(\xi,x_n)\right\|^2_{L_2(R^n_+;H)} -$$

$$-2a_n\left\|B^{1/2}(\xi)\frac{\partial\hat{u}(\xi,x_n)}{\partial x_n}\right\|^2_{L_2(R^n_+;H)}$$

Taking this equality into account in inequality (2.22), we get:

$$a_n\left\|B^{1/2}(\xi)\frac{\partial\hat{u}(\xi,x_n)}{\partial x_n}\right\|^2_{L_2(R^n_+;H)} \le$$

$$\le \frac{1}{2}\left(\left\|\hat{L}_0\hat{u}(\xi,x_n)\right\|^2_{L_2(R^n_+;H)} - 2a_n\left\|B^{1/2}(\xi)\frac{\partial\hat{u}(\xi,x_n)}{\partial x_n}\right\|^2_{L_2(R^n_+;H)}\right)$$

Hence

$$2a_n\left\|B^{1/2}(\xi)\frac{\partial\hat{u}(\xi,x_n)}{\partial x_n}\right\|^2_{L_2(R^n_+;H)} \le \frac{1}{2}\left\|\hat{L}_0\hat{u}(\xi,x_n)\right\|^2_{L_2(R^n_+;H)}$$

or

$$\left\|B^{1/2}(\xi)\frac{\partial\hat{u}(\xi,x_n)}{\partial x_n}\right\|_{L_2(R^n_+;H)} \le \frac{1}{2\sqrt{a_n}}\left\|\hat{L}_0\hat{u}(\xi,x_n)\right\|_{L_2(R^n_+;H)}$$

Inequality (2.18) has also been proved. We now prove inequalities (2.19) and (2.20).

It is easily seen that the inequalities are valid for $j=1,2,\dots n-1$

$$\|\xi_j C\hat{u}(\xi,x_n)\|^2_{L_2(R^n_+;H)} = \left\|\xi_j CB^{-1}(\xi)B(\xi)\hat{u}(\xi,x_n)\right\|^2_{L_2(R^n_+;H)} \le$$

$$\le \sup_{\xi\in R^{n-1}}\left\|\xi_j CB^{-1}(\xi)\right\|^2\cdot\|B(\xi)\hat{u}(\xi,x_n)\|^2_{L_2(R^n_+;H)} \tag{2.23}$$

Using spectral expansion of the operator C, we get:

$$\left\|\xi_j CB^{-1}(\xi)\right\| = \left\|\xi_j C\left(\sum_{k=1}^{n-1}a_k\xi_k^2 + C^2\right)^{-1}\right\| =$$

$$= \sup_{\mu\in\tau(c)}\left|\xi_j\mu\left(\sum_{k=1}^{n-1}a_k\xi_k^2+\mu^2\right)^{-1}\right| \le$$

$$\le \sup_{\mu\in\tau(c)}\left|\xi_j\mu\left(a_j\xi_j^2+\mu^2\right)^{-1}\right| \le \sup_{\mu\in\tau(c)}\left|\left(a_j^{\frac{1}{2}}\xi_j\right)\cdot a_j^{\frac{1}{2}}\mu\left(a_j\xi_j^2+\mu^2\right)^{-1}\right| \le$$

$$\le \frac{1}{2}a_j^{-\frac{1}{2}}\left|\left(a_j\xi_j^2+\mu^2\right)\cdot\left(a_j\xi_j^2+\mu^2\right)^{-1}\right| \le a_j^{-\frac{1}{2}} = \frac{1}{\sqrt{a_j}}.$$

From inequality (2.17) we get:

$$\|\xi_j C\hat{u}(\xi,x_n)\| \le \frac{1}{2\sqrt{a_n}}\left\|\hat{L}_0\hat{u}(\xi,x_n)\right\|_{L_2(R^n_+;H)}$$

This shows validity of inequality (2.19). Finally, we show the validity of inequality (2.20). We have:

$$\left\|C^2\hat{u}\left(\xi,x_n\right)\right\|_{L_2\left(R_+^n;H\right)}=\left\|C^2B^{-1}\left(\xi\right)B\left(\xi\right)\hat{u}\left(\xi,x_n\right)\right\|_{L_2\left(R_+^n;H\right)}\le$$

$$\le\sup_{\xi}\left\|C^2B^{-1}\left(\xi\right)\right\|_H\cdot\left\|B\left(\xi\right)\hat{u}\left(\xi,x_n\right)\right\| \tag{2.24}$$

Then we have

$$\left\|C^2B^{-1}\left(\xi\right)\right\|=\left\|C^2\left(\sum_{k=1}^{n-1}a_k\xi_k^2+C^2\right)^{-1}\right\|=$$

$$=\sup_{\mu\in\tau(C)}\left|\mu^2\left(\sum_{k=1}^{n-1}a_k\xi_k^2+\mu^2\right)^{-1}\right|\le 1$$

Taking into account inequalities (2.17) and (2.20), from inequality (2.24) we get:

$$\left\|C^2\hat{u}\left(\xi,x_n\right)\right\|_{L_2\left(R_+^n;H\right)}=\left\|\hat{L}_0\hat{u}\left(\xi,x_n\right)\right\|_{L_2\left(R_+^n;H\right)}.$$

Thus, theorem 2.2 is completely proved.

In the next section we prove the main theorem.

3. The main result

.

Theorem 3.1. *Let scalar numbers* $a_k>0\,(k=1,2,\ldots n)$, *C be a self-adjoint, positive-definite operator in H, the operators* $Q_k=R_kC^{-1}\,(k=1,2,\ldots n)$ *and* $F=TC^{-2}$ *be bounded in H and the inequality*

$$q=\frac{1}{2}\sum_{k=1}^{n}\frac{\|Q_k\|}{\sqrt{a_k}}+\|F\|<1$$

hold. Then the operator $\hat{L}$ isomorphically maps the space $\overset{\circ}{W}{}_{2,\xi}^2\left(R_+^n;H\right)$ *onto* $L_2\left(R_+^n;H\right)$.

Proof. By theorem 2.1 the operator $\hat{L}_0$ isomorphically maps the space $\overset{\circ}{W}{}_{2,\xi}^2\left(R_+^n;H\right)$ onto $L_2\left(R_+^n;H\right)$. From the equation $\hat{L}_0\hat{u}\left(\xi,x_n\right)+\hat{L}_1\hat{u}\left(\xi,x_n\right)=\hat{f}\left(\xi,x_n\right)$ where $\hat{u}\left(\xi,x_n\right)\in\overset{\circ}{W}{}_{2,\xi}^2\left(R_+^n;H\right)$, $\hat{f}\left(\xi,x_n\right)\in L_2\left(R_+^n;H\right)$ after substitution of $\hat{L}_0\hat{u}\left(\xi,x_n\right)=\hat{\omega}\left(\xi,x_n\right)$ we get:

$$\hat{\omega}\left(\xi,x_n\right)+\hat{L}_1\hat{L}_0\omega\left(\xi,x_n\right)=f\left(\xi,x_n\right)$$

or $\left(E+\hat{L}_1\hat{L}_0^{-1}\right)\omega\left(\xi,x_n\right)=\hat{f}\left(\xi,x_n\right)$, where E is a unit operator in $L_2\left(R_+^n;H\right)$. Since the operator $\hat{L}_0$ is an isomorphism, then for any $\hat{\omega}\left(\xi,x_n\right)\in L_2\left(R_+^n;H\right)$ we have:

$$\left\|\hat{L}_1\hat{L}_0^{-1}\hat{\omega}\left(\xi,x_n\right)\right\|_{L_2\left(R_+^n;H\right)}=\left\|\hat{L}_1\hat{u}\left(\xi,x_n\right)\right\|_{L_2\left(R_+^n;H\right)}=$$

$$= \left\| R_n \frac{\partial \hat{u}(\xi, x_n)}{\partial x_n} + \sum_{k=1}^{n-1} i\xi_k R_k \hat{u}(\xi, x_n) + T\hat{u}(\xi, x_n) \right\|_{L_2\left(R_+^n;H\right)} \le$$

$$\le \left\| R_n B^{-1/2}(\xi) \cdot B^{1/2}(\xi) \frac{\partial \hat{u}(\xi, x_n)}{\partial x_n} \right\|_{L_2\left(R_+^n;H\right)} + \sum_{k=1}^{n-1} \left\| \xi_k R_k \hat{u}(\xi, x_n) \right\|_{L_2\left(R_+^n;H\right)} + + \left\| T\hat{u}(\xi, x_n) \right\|_{L_2\left(R_+^n;H\right)}.$$

Here using inequalities (2.17)-(2.20) proved in theorem 2.1, we get:

$$\left\| \hat{L}_1 \hat{L}_0 \hat{\omega}(\xi, x_n) \right\|_{L_2\left(R_+^n;H\right)} \le \left\| R_n C^{-1} \cdot C B^{-\frac{1}{2}}(\xi) B^{\frac{1}{2}}(\xi) \frac{\partial \hat{u}(\xi, x_n)}{\partial x_n} \right\|_{L_2\left(R_+^n;H\right)} +$$

$$+ \sum_{k=1}^{n-1} \left\| R_k C^{-1} \xi_k C \hat{u}(\xi, x_n) \right\|_{L_2\left(R_+^n;H\right)} + \left\| T C^{-2} C^2 \hat{u}(\xi, x_n) \right\|_{L_2\left(R_+^n;H\right)} \le$$

$$\le \sup_{\xi} \left\| R_n \cdot C^{-1} C \cdot B^{-\frac{1}{2}}(\xi) \right\| \cdot \left\| B^{\frac{1}{2}}(\xi) \frac{\partial \hat{u}(\xi, x_n)}{\partial x_n} \right\|_{L_2\left(R_+^n;H\right)} +$$

$$+ \sum_{k=1}^{n-1} \left\| R_k C^{-1} \right\| \cdot \left\| \xi_k C \hat{u}(\xi, x_n) \right\|_{L_2\left(R_+^n;H\right)} + \left\| T C^{-2} \right\| \left\| C^2 \hat{u}(\xi, x_n) \right\|_{L_2\left(R_+^n;H\right)} \le$$

$$\le \left\| R_n C \right\| \cdot \sup_{\xi} \left\| C B^{-\frac{1}{2}}(\xi) \right\| \cdot \left\| B^{\frac{1}{2}}(\xi) \frac{\partial \hat{u}(\xi, x_n)}{\partial x_n} \right\|_{L_2\left(R_+^n;H\right)} +$$

$$+ \sum_{k=1}^{n-1} \left\| R_k C^{-1} \right\| \cdot \left\| \xi_k C \hat{u}(\xi, x_n) \right\|_{L_2\left(R_+^n;H\right)} + \left\| T C^{-2} \right\| \left\| C^2 \hat{u}(\xi, x_n) \right\|_{L_2\left(R_+^n;H\right)} \le$$

$$+ \left\| Q_n \right\| \cdot \sup \left\| C B^{-\frac{1}{2}}(\xi) \right\| \cdot \frac{1}{2\sqrt{a_n}} \left\| \hat{L}_0 \hat{u}(\xi, x_n) \right\|_{L_2\left(R_+^n;H\right)} +$$

$$+ \sum_{k=1}^{n-1} \left\| Q_k \right\| \cdot \frac{1}{2\sqrt{a_k}} \left\| \hat{L}_0 \hat{u}(\xi, x_n) \right\|_{L_2\left(R_+^n;H\right)} + \left\| F \right\| \left\| L_0 \hat{u}(\xi, x_n) \right\|_{L_2\left(R_+^n;H\right)} \qquad (3.1)$$

On the other hand, for any $\xi \in R^{n-1}$ we have

$$\left\| C B^{-\frac{1}{2}}(\xi) \right\| = \left\| C \left(\sum_{k=1}^{n-1} a_k^2 \xi_k^2 E + C^2 \right)^{-1/2} \right\| =$$

$$= \sup_{\mu \in \tau(C)} \left| \mu \left(\sum_{k=1}^{n-1} a_k^2 \xi^2 + \mu^2 \right)^{-\frac{1}{2}} \right| \le \sup_{\mu \in \tau(C)} \left| \mu^2 \left(\sum_{k=1}^{n-1} a_k \xi_k^2 + \mu^2 \right)^{-\frac{1}{2}} \right| \le 1.$$

Therefore, it follows from (3.1) that

$$\left\| \hat{L}_1 \hat{L}_0^{-1} \hat{\omega}(\xi, x_n) \right\| \le$$

$$\le \left(\left\| Q_n \right\| \cdot \frac{1}{2\sqrt{a_n}} + \sum_{k=1}^{n-1} \left\| Q_k \right\| \frac{1}{2\sqrt{a_n}} + \left\| F \right\| \right) \left\| \hat{L}_0 \hat{u}(\xi, x_n) \right\|_{L_2\left(R_+^n;H\right)} \le$$

$$\le q \left\| \hat{L}_0 \hat{u}(\xi, x_n) \right\|_{L_2\left(R_+^n;H\right)} = q \left\| \hat{\omega}(\xi, x_n) \right\|_{L_2\left(R_+^n;H\right)}$$

Since $0 < q < 1$, then the operator $\left(E + \hat{L}_1 \hat{L}_0^{-1}\right)$ is invertible in the space $\overset{\circ}{W}{}_2^2\left(R_+^n; H\right)$. Then

$$\hat{\omega}\left(\xi, x_n\right) = \left(E + \hat{L}_1 \hat{L}_0^{-1}\right)^{-1} \hat{f}\left(\xi, x_n\right)$$

or

$$\hat{u}\left(\xi, x_n\right) = \hat{L}_0^{-1} \left(E + \hat{L}_1 \hat{L}_0^{-1}\right)^{-1} \hat{f}\left(\xi, x_n\right)$$

Hence it follows that

$$\left\|\hat{u}\left(\xi, x_n\right)\right\|_{W_2^2\left(R_+^n; H\right)} \leq const \left\|\hat{f}\left(\xi, x_n\right)\right\|_{L_2\left(R_+^n; H\right)}.$$

Obviously, knowing the function $\hat{u}\left(\xi, x_n\right)$ and having made the inverse Fourier transform, we can find the solution of problem (1.1), (1.2) $u(x) \in W_2^2\left(R_+^n; H\right)$. Since the Fourier transform is a unit operator, we get

$$\left\|u\left(x\right)\right\|_{W_2^2\left(R_+^n; H\right)} \leq const \left\|f\left(x\right)\right\|_{L_2\left(R_+^n; H\right)}.$$

Using the Banach theorem on the inverse operator we get the statement of theorem 3.1. Theorem 3.1 has been proved.

Hence it follows

Theorem 3.2. *Let the numbers $a_k > 0\,(k = 1, 2, \ldots n)$, C be a self-adjoint, positive-definite operator, the operators $Q_k = R_k C^{-1}\left(k = 1, 2, \ldots n\right)$ and $F = TC^{-2}$ be bounded in the space H and the following condition be fulfilled:*

$$q = \frac{1}{2} \sum_{k=1}^{n} \frac{\|Q_k\|}{\sqrt{a_k}} + \|F\| < 1.$$

Then problem (1.1),(1.2) *is correctly solvable.*

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Hamidulla I. Aslanov

Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan

E-mail address: `aslanov.50@mail.ru`

Roya F. Hatamova

Sumgait State University, Sumgait, Azerbaijan

E-mail address: `hetemova_roya@mail.ru`

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