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A-BEREZIN NUMBER OF OPERATORS

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Abstract. We introduce the notions (A,r)-adjoint of operators and A-Berezin number of operators on the reproducing kernel Hilbert space and prove some inequalities for A-Berezin number of operators. Some other related questions are also discussed.

1. Introduction

Let $\mathcal{H}=\mathcal{H}(\Omega)$ be a reproducing kernel Hilbert space on some set Ω with the reproducing kernel $k_\lambda\in\mathcal{H}$, i.e., $f(\lambda)=\langle f,k_\lambda\rangle$ for all $f\in\mathcal{H}$ and all $\lambda\in\Omega$. It is supposed that for every $\lambda\in\Omega$ there exists a function $f_\lambda\in\mathcal{H}$ such that $f_\lambda(\lambda)\neq 0$, or equivalently there is no $\lambda_0\in\Omega$ such that $f(\lambda_0)=0$ for all $f\in\mathcal{H}$.

Let $\mathcal{B}(\mathcal{H})$ denote the Banach algebra of all bounded linear operators on $\mathcal{H}$ with an inner product $\langle .,.\rangle$ and the corresponding norm $\|.\|$. For an operator $A\in\mathcal{B}(\mathcal{H})$, its Berezin symbol $\widetilde{A}$ is defined by

$$\widetilde{A}(\lambda)=\left\langle A\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle,\ \lambda\in\Omega,$$

where $\widehat{k}_\lambda:=\frac{k_\lambda}{\|k_\lambda\|_{\mathcal{H}}}$ is the normalized reproducing kernel of $\mathcal{H}$. The Berezin set, Berezin number and Berezin norm of operators are defined, respectively, by (see [7] and [15])

$$\operatorname{Ber}(A)=\operatorname{Range}\left(\widetilde{A}\right)=\left\{\widetilde{A}(\lambda):\lambda\in\Omega\right\},$$
$$\operatorname{ber}(A)=\sup_{\lambda\in\Omega}\left|\widetilde{A}(\lambda)\right|,$$

and

$$\|A\|_{\operatorname{Ber}}=\sup_{\lambda\in\Omega}\left\|A\widehat{k}_\lambda\right\|_{\mathcal{H}}.$$

Clearly $\operatorname{Ber}(A)\subseteq W(A)$ (numerical range), $\operatorname{ber}(A)\leq w(A)$ (numerical radius) and $\operatorname{ber}(A)\leq\|A\|_{\operatorname{Ber}}\leq\|A\|$ for all operators $A\in\mathcal{B}(\mathcal{H})$ (for more facts about reproducing kernel Hilbert spaces and Berezin symbol, see, Aronzajn [3] and Berezin [7]).

The null space of every operator T is denoted by $\mathcal{N}(T)$, its range by $\mathcal{R}(T)$, and adjoint of T by T^*. If S is a linear subspace of $\mathcal{H}$, then $\overline{S}$ stands for its closure in the norm topology of $\mathcal{H}$. An operator $A\in\mathcal{B}(\mathcal{H})$ is called positive,

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denoted by $A \geq 0$, if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, the absolute value of T, denoted by $|T|$, is defined as $|T| = (T^*T)^{1/2}$. Throughout the article, A denotes a non-zero positive operator on $\mathcal{H}$. Notice that any positive operator A induces a semi-inner product on $\mathcal{H}$ defined by

$$\langle x, y\rangle_A := \langle Ax, y\rangle_{\mathcal{H}}, \ \forall x, y \in \mathcal{H}.$$

The seminorm induced by $\langle ., .\rangle_A$ is given by $\|x\|_A = \sqrt{\langle x, x\rangle_A} = \left\|A^{1/2}x\right\|$ for all $x \in \mathcal{H}$.

It is easy to check that $\|.\|_A$ is norm if and only if A is injective and that the seminormed space $(\mathcal{H}, \|.\|_A)$ which is complete if and only if $\overline{\mathcal{R}(A)} = \mathcal{R}(A)$.

Definition 1.1. For $T \in \mathcal{B}(\mathcal{H})$, the A-Berezin set of $\left\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A$ is defined by

$$\mathrm{Ber}_A(T) := \left\{\left\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A : \lambda \in \Omega\right\}.$$

It should be mentioned that $\mathrm{Ber}_A(T)$ is a nonempty subset of $\mathbb{C}$ and it is in general not closed even if $\mathcal{H}$ is finite dimensional.

Definition 1.2. (i) The supremum modulus of $\mathrm{Ber}_A(T)$, denoted by $\mathrm{ber}_A(T)$, is called the A-Berezin number of T, i.e.,

$$\mathrm{ber}_A(T) := \sup_{\lambda\in\Omega} \left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A\right|.$$

(ii) A-Berezin norm of operators $T \in \mathcal{B}(\mathcal{H}(\Omega))$ is defined by

$$\|A\|_{A-\mathrm{Ber}} := \sup_{\lambda\in\Omega} \left\|AT\widehat{k}_{\mathcal{H},\lambda}\right\|_{\mathcal{H}}.$$

If $A = I$, we get the Berezin number. So, this new concept generalizes the Berezin number of reproducing kernel Hilbert space operators and the Berezin norm of operators which have recently attracted the attention of many authors (see, for instance [4, 5, 6, 16, 17, 18, 21, 22, 23, 24]).

Definition 1.3 ([13])**.** Let $T \in \mathcal{B}(\mathcal{H})$. An operator $S \in \mathcal{B}(\mathcal{H})$ is called an A-adjoint of T if for every $x, y \in \mathcal{H}$, identity $\langle Tx, y\rangle_A = \langle x, Sy\rangle_A$ holds.

Definition 1.4. Let $T \in \mathcal{B}(\mathcal{H}(\Omega))$. An operator $S \in \mathcal{B}(\mathcal{H}(\Omega))$ is called (A, r)-adjoint of T if for every $\lambda, \mu \in \Omega$, the identity $\left\langle T\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\mu}\right\rangle_A = \left\langle \widehat{k}_{\mathcal{H},\lambda}, S\widehat{k}_{\mathcal{H},\mu}\right\rangle_A$ holds.

Following [13], observe that the existence of an A-adjoint of T is equivalent to the existence of a solution of the equation $AX = T^*A$. This kind of equations can be studied by using a well-known theorem due to Douglas [10] (for a survey of the recent results on this theorem, the readers can consult in Moslehian, Kian and Xu [20]). Briefly, Douglas theorem says that the operator equation $TX = S$ has a bounded linear solution X if and only if $\mathcal{R}(S) \subseteq \mathcal{R}(T)$; moreover, among its many solutions it has only one, denoted by Q, which satisfies $\mathcal{R}(Q) \subseteq \overline{\mathcal{R}(T^*)}$. Such Q is called the reduced solution or Douglas solution of $TX = S$. The set of all operators in $\mathcal{B}(\mathcal{H})$ admitting A-adjoint is denoted by $\mathcal{B}_A(\mathcal{H})$. By Douglas theorem, it holds that

$$\mathcal{B}_A(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : \mathcal{R}(T^*A) \subseteq \mathcal{R}(A)\}.$$

Further, the set all operators admitting $A^{1/2}$-adjoints is denoted by $\mathcal{B}_{A^{1/2}}(\mathcal{H})$. Again, by applying Douglas theorem, we obtain

$$\mathcal{B}_{A^{1/2}}(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : \exists \lambda > 0,\ \|Tx\|_A \leq \lambda \|x\|_A, \forall x \in \mathcal{H}\}.$$

Operators in $\mathcal{B}_{A^{1/2}}(\mathcal{H})$ are called A-bounded.

If $T \in \mathcal{B}_A(\mathcal{H})$, the reduced solution (or Douglas solution) of the equation $AX = T^*A$ is a distinguished A-adjoint operator of T, which is denoted by T^{*_A}. We observe that

$$T^{*_A} = A^\dagger T^* A,$$

where $A^\dagger$ is the Moore-Penrose inverse of A (see [1, 2]). It is well-known that the operator T^{*_A} satisfies

$$AT^{*_A} = T^*A,\ \mathcal{R}(T^{*_A}) \subseteq \overline{\mathcal{R}(A)} \text{ and } \mathcal{N}(T^{*_A}) = \mathcal{N}(T^*A).$$

Also, note that if $T \in \mathcal{B}_A(\mathcal{H})$, then $T^{*_A} \in \mathcal{B}_A(\mathcal{H})$ and $(T^{*_A})^{*_A} = P_A T P_A$, where P_A denotes the orthogonal projection onto $\overline{\mathcal{R}(A)}$. Moreover, if $T \in \mathcal{B}_A(\mathcal{H})$, then $\|T^{*_A}\| = \|T\|_A$. For more results and proofs related to this class of operators, the reader can consult in [1, 2] and their references.

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be A-selfadjoint if AT is selfadjoint, that is, $AT = T^*A$. In addition, an operator T is called A-positive if $AT \geq 0$ and we write $T \geq_A 0$.

In the sequel, the Hilbert space $\left(\mathcal{R}\left(A^{1/2}\right), \langle .,. \rangle_{\mathbb{R}\left(A^{1/2}\right)}\right)$ will be simply denoted by $\mathbb{R}\left(A^{1/2}\right)$.

Using these notions, in [13], the author investigated A-numerical radius of operators.

In the present article, we introduce analogs of these new concepts by using (A, r)-adjoint of operators and prove some inequalities for A-Berezin number and A-Berezin norm of operators on the reproducing kernel Hilbert spaces.

2. A-Berezin number and A-Berezin norm of operators on reproducing kernel Hilbert spaces

Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a reproducing kernel Hilbert space with reproducing kernel $k_{\mathcal{H},\lambda}$. For $T \in \mathcal{B}(\mathcal{H})$, we define

$$\eta_A(T) := \inf_{\lambda \in \Omega} \left\| T\widehat{k}_{\mathcal{H},\lambda} \right\|_A.$$

In this section, we study the relationship between A-Berezin number and A-Berezin norm of operators and prove new inequalities.

It is natural to define the A-Berezin symbol of operator $T \in \mathcal{B}(\mathcal{H}(\Omega))$ by the formula

$$\widetilde{T}_A(\lambda) := \left\langle T\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda} \right\rangle_A = \left\langle AT\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda} \right\rangle,\ \lambda \in \Omega.$$

We denote the set of all operators in $\mathcal{B}(\mathcal{H}(\Omega))$ admitting (A, r)-adjoints by $\mathcal{B}_{A,r}(\mathcal{H}) := \mathcal{B}_{A,r}(\mathcal{H}(\Omega))$. Our first result proves inequalities between $\|T\|_{A-\mathrm{Ber}}$ and $w_A(T)$ with A in $\mathcal{B}_{A,r}(\mathcal{H})$.

Theorem 2.1. *Let $T \in \mathcal{B}_{A,r}(\mathcal{H})$. Then*

$$\frac{1}{2}\|T\|_{A-\mathrm{Ber}} \leq \sqrt{\frac{1}{4}\|T\|_{A-\mathrm{Ber}}^2 + \frac{1}{4}\left[\max\left(\eta_A(T), \eta_A(T^{*_A})\right)\right]^2} \leq \mathrm{ber}_A(T).$$

Proof. Let $\lambda \in \Omega$ be arbitrary. An elementary calculus shows that

$$\left\|T\widehat{k}_{\mathcal{H},\lambda}\right\|_A^2 + \left\|T^{*_A}\widehat{k}_{\mathcal{H},\lambda}\right\|_A^2 = \frac{1}{2}\left[\left\|T\widehat{k}_{\mathcal{H},\lambda} - T^{*_A}\widehat{k}_{\mathcal{H},\lambda}\right\|_A^2 + \left\|T\widehat{k}_{\mathcal{H},\lambda} + T^{*_A}\widehat{k}_{\mathcal{H},\lambda}\right\|_A^2\right].$$

This implies that

$$\eta_A^2(T) + \left\|T^{*_A}\widehat{k}_{\mathcal{H},\lambda}\right\|_A^2 \leq \frac{1}{2}\left[\left\|T\widehat{k}_{\mathcal{H},\lambda} - T^{*_A}\widehat{k}_{\mathcal{H},\lambda}\right\|_A^2 + \left\|T\widehat{k}_{\mathcal{H},\lambda} + T^{*_A}\widehat{k}_{\mathcal{H},\lambda}\right\|_A^2\right].$$

By taking supremum over all $\lambda \in \Omega$ and using the fact that $\|T\|_{A-\mathrm{Ber}} = \|T^{*_A}\|_{A-\mathrm{Ber}}$, we get

$$\eta_A^2(T) + \|T^{*_A}\|_{A-\mathrm{Ber}}^2 \leq \frac{1}{2}\left[\|T - T^{*_A}\|_{A-\mathrm{Ber}}^2 + \|T + T^{*_A}\|_{A-\mathrm{Ber}}^2\right]. \tag{2.1}$$

Since $AT^{*_A} = T^*A$, it is easy to see that

$$\begin{aligned}
\left\langle (T + T^{*_A})\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\mu}\right\rangle_A &= \left\langle A(T + T^{*_A})\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\mu}\right\rangle \\
&= \left\langle (AT + AT^{*_A})\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\mu}\right\rangle \\
&= \left\langle AT\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\mu}\right\rangle + \left\langle AT^{*_A}\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\mu}\right\rangle \\
&= \left\langle (T^{*_A})^* A\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\mu}\right\rangle + \left\langle T^* A\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\mu}\right\rangle \\
&= \left\langle (T + T^{*_A})^* A\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\mu}\right\rangle \\
&= \left\langle A\widehat{k}_{\mathcal{H},\lambda}, (T + T^{*_A})\widehat{k}_{\mathcal{H},\mu}\right\rangle
\end{aligned}$$

for all $\lambda, \mu \in \Omega$. This means that $T + T^{*_A}$ is an (A, r)-selfadjoint operator. Then, it can be proved by using some arguments of the paper [14] (which is omitted) that

$$\|T + T^{*_A}\|_{A-\mathrm{Ber}} = \mathrm{ber}_A(T + T^{*_A}).$$

This implies that

$$\|T + T^{*_A}\|_{A-\mathrm{Ber}} \leq 2\mathrm{ber}_A(T),$$

which, in turn, implies, by using, by using (2.1), that

$$\eta_A^2(T) + \|T^{*_A}\|_{A-\mathrm{Ber}}^2 \leq 2\left(\mathrm{ber}_A^2(T) + \|\mathrm{Im}_A(T)\|_{A-\mathrm{Ber}}^2\right),$$

where $\mathrm{Im}_A(T) = \frac{T - T^{*_A}}{2i}$. By similar arguments as above, it can be proven that

$$\eta_A^2(T^{*_A}) + \|T\|_{A-\mathrm{Ber}}^2 \leq 2\left(\mathrm{ber}_A^2(T) + \|\mathrm{Im}_A(T)\|_{A-\mathrm{Ber}}^2\right).$$

So, we obtain

$$\|T^{*_A}\|_{A-\mathrm{Ber}}^2 + [\max(\eta_A(T), \eta_A(T^{*_A}))]^2 \leq 2\left(\mathrm{ber}_A^2(T) + \|\mathrm{Im}_A(T)\|_{A-\mathrm{Ber}}^2\right). \tag{2.2}$$

By replacing T by iT in (2.2), we have

$$\|T\|_{A-\mathrm{Ber}}^2 + [\max(\eta_A(T), \eta_A(T^{*_A}))]^2 \leq 2\left(\mathrm{ber}_A^2(T) + \|\mathrm{Re}_A(T)\|_{A-\mathrm{Ber}}^2\right). \tag{2.3}$$

By combining (2.2) together with (2.3), we get

$$\|T\|_{A-\mathrm{Ber}}^2 + [\max(\eta_A(T), \eta_A(T^{*_A}))]^2 \leq 2\left(\mathrm{ber}_A^2(T) + m_A(T)\right), \tag{2.4}$$

where

$$m_A(T) := \min\left(\|\mathrm{Re}_A(T)\|_{A-\mathrm{Ber}}^2, \|\mathrm{Im}_A(T)\|_{A-\mathrm{Ber}}^2\right).$$

On the other hand, for $\lambda \in \Omega$, we have that

$$\begin{aligned}\left|\left\langle T\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right|^2 &= \left|\left\langle \mathrm{Re}_A(T)\,\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A + i\left\langle \mathrm{Im}_A(T)\,\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right|^2 \\ &= \left|\left\langle \mathrm{Re}_A(T)\,\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right|^2 + \left|\left\langle \mathrm{Im}_A(T)\,\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right|^2 \\ &\geq \left|\left\langle \mathrm{Re}_A(T)\,\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right|^2. \qquad (2.5)\end{aligned}$$

Now, using that

$$\mathrm{ber}_A(\mathrm{Re}_A(T)) = \|\mathrm{Re}_A(T)\|_{A-\mathrm{Ber}}^2$$

(since $\mathrm{Re}_A(T)$ is (A,r)-selfadjoint), and so $\mathrm{ber}_A^2(T) \geq m_A(T)$, by taking the supremum over all $\lambda \in \Omega$ in inequality (2.5), we have by considering (2.4) that

$$\|T\|_{A-\mathrm{Ber}}^2 + \left[\max\left(\eta_A(T), \eta_A(T^{*_A})\right)\right]^2 \leq 4\mathrm{ber}_A^2(T).$$

This yields that

$$\frac{1}{2}\|T\|_{A-\mathrm{Ber}} \leq \sqrt{\frac{1}{4}\|T\|_{A-\mathrm{Ber}}^2 + \frac{1}{4}\left[\max\left(\eta_A(T), \eta_A(T^{*_A})\right)\right]^2} \leq \mathrm{ber}_A(T),$$

as desired. The theorem is proved. □

Proposition 2.1. *Let $T \in \mathcal{B}(\mathcal{H}(\Omega))$ be an (A,r)-selfadjoint operator. Then T^{*_A} is (A,r)-selfadjoint and*

$$(T^{*_A})^{*_A} = T^{*_A}. \qquad (2.6)$$

Proof. Since T is an (A,r)-selfadjoint operator, $T \in \mathcal{B}_{A,r}(\mathcal{H}(\Omega))$. For every $\lambda, \mu \in \Omega$, we have

$$\begin{aligned}\left\langle AT^{*_A}\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\mu}\right\rangle &= \left\langle T^{*}A\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\mu}\right\rangle \\ &= \left\langle AT\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\mu}\right\rangle \ \text{(since } T \text{ is } (A,r)\text{-selfadjoint)} \\ &= \left\langle \widehat{k}_{\mathcal{H},\lambda}, T^{*_A}\widehat{k}_{\mathcal{H},\mu}\right\rangle_A = \left\langle (T^{*_A})^{*} A\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\mu}\right\rangle.\end{aligned}$$

Hence, T^{*_A} is (A,r)-selfadjoint. This implies that T^{*_A} is a solution of the equation $AX = (T^{*_A})^{*}A$, since $\mathrm{span}\left\{\widehat{k}_{\mathcal{H},\lambda} : \lambda \in \Omega\right\} = \mathcal{H}$. Moreover, we have $\mathcal{R}(T^{*_A}) \subseteq \overline{\mathcal{R}(A)}$. Therefore, by the uniqueness of the Douglas solution, (2.6) follows. The proposition is proved. □

Now we give A-Berezin number inequalities for sum of operators. We need the following lemma (see, [14] and [8]).

Lemma 2.1. *For any $x, y, z \in \mathcal{H}$, we have*

$$\left|\langle x,y\rangle_A\right|^2 + \left|\langle x,z\rangle_A\right|^2 \leq \|x\|_A^2\left(\max\left\{\|y\|_A^2, \|z\|_A^2\right\} + \left|\langle y,z\rangle_A\right|\right). \qquad (2.7)$$

Proof. Indeed, by the proof of [11, Theorem 3] we have

$$\left|\langle x,y\rangle\right|^2 + \left|\langle x,z\rangle\right|^2 \leq \|x\|^2\left(\max\left\{\|y\|^2, \|z\|^2\right\} + \left|\langle y,z\rangle\right|\right), \qquad (2.8)$$

for every $x, y, z \in \mathcal{H}$. Now

$$|\langle x, y\rangle_A|^2 + |\langle x, z\rangle_A|^2 = \left|\left\langle A^{1/2}x, A^{1/2}y\right\rangle\right|^2 + \left|\left\langle A^{1/2}x, A^{1/2}z\right\rangle\right|^2.$$

So, by applying (2.8), we have

$$|\langle x, y\rangle_A|^2 + |\langle x, z\rangle_A|^2 \leq \left\|A^{1/2}x\right\|^2 \left(\max\left\{\left\|A^{1/2}y\right\|^2, \left\|A^{1/2}z\right\|^2\right\} + \left|\left\langle A^{1/2}y, A^{1/2}z\right\rangle\right|\right).$$

Hence, we have (2.7), as desired. This proves the lemma. □

Theorem 2.2. *Let $T, S \in \mathcal{B}_{A,r}(\mathcal{H}(\Omega))$. Then*

$$\mathrm{ber}_A(T+S) \leq$$
$$\sqrt{\frac{1}{2}\left(\mathrm{ber}_A\left(TT^{*_A} + SS^{*_A}\right) + \mathrm{ber}_A\left(TT^{*_A} - SS^{*_A}\right)\right) + \mathrm{ber}_A\left(ST^{*_A}\right) + 2\mathrm{ber}_A(T)\,\mathrm{ber}_A(S)}.$$

Proof. It is well known that

$$\max\{t, s\} = \frac{1}{2}(t + s + |t - s|) \tag{2.9}$$

for every $t, s \in \mathbb{R} := (-\infty, +\infty)$.

Let $\lambda \in \Omega$ be arbitrary. Using Lemma 2.1 we get

$$\begin{aligned}
&\left|\left\langle (T+S)\,\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right|^2 \\
&\leq \left|\left\langle \widehat{k}_{\mathcal{H},\lambda}, T^{*_A}\widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right|^2 + \left|\left\langle \widehat{k}_{\mathcal{H},\lambda}, S^{*_A}\widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right|^2 \\
&+ 2\left|\left\langle T\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right|\left|\left\langle S\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right| \\
&\leq \max\left\{\left\|T^{*_A}\widehat{k}_{\mathcal{H},\lambda}\right\|_A^2, \left\|S^{*_A}\widehat{k}_{\mathcal{H},\lambda}\right\|_A^2\right\} + \left|\left\langle ST^{*_A}\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right| \\
&+ 2\left|\left\langle T\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right|\left|\left\langle S\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right| \\
&= \frac{1}{2}\left(\left\|T^{*_A}\widehat{k}_{\mathcal{H},\lambda}\right\|_A^2 + \left\|S^{*_A}\widehat{k}_{\mathcal{H},\lambda}\right\|_A^2 + \left|\left\|T^{*_A}\widehat{k}_{\mathcal{H},\lambda}\right\|_A^2 - \left\|S^{*_A}\widehat{k}_{\mathcal{H},\lambda}\right\|_A^2\right|\right) \\
&+ \left|\left\langle ST^{*_A}\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right| \\
&+ 2\left|\left\langle T\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right|\left|\left\langle S\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right| \text{ (by (2.9))} \\
&\leq \frac{1}{2}\left(\left\langle \left(TT^{*_A} + SS^{*_A}\right)\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A + \left|\left\langle \left(TT^{*_A} - SS^{*_A}\right)\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right|\right) \\
&+ \left|\left\langle ST^{*_A}\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right| + 2\left|\left\langle T\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right|\left|\left\langle S\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right| \\
&\leq \frac{1}{2}\left(\mathrm{ber}_A\left(TT^{*_A} + SS^{*_A}\right) + \mathrm{ber}_A\left(TT^{*_A} - SS^{*_A}\right)\right) \\
&+ \mathrm{ber}_A\left(ST^{*_A}\right) + 2\mathrm{ber}_A(T)\,\mathrm{ber}_A(S),
\end{aligned}$$

this proves the theorem. □

The following lemma in [8] is needed for our next result.

Lemma 2.2. *Let $x, y, z \in \mathcal{H}$ with $\|e\|_A = 1$. Then*

$$|\langle x, e\rangle_A \langle e, y\rangle_A| \leq \frac{1}{2}\left(|\langle x, y\rangle_A| + \|x\|_A \|y\|_A\right).$$

Proposition 2.2. *Let $T, S \in \mathcal{B}_{A,r}(\mathcal{H}(\Omega))$. Then*

$$\operatorname{ber}_A(T+S) \leq \sqrt{\operatorname{ber}_A^2(T) + \operatorname{ber}_A^2(S) + \frac{1}{2}\left(\operatorname{ber}_A\left(T^{*_A}T + SS^{*_A}\right)\right) + \operatorname{ber}_A(ST)}. \tag{2.10}$$

Proof. Let $\lambda \in \Omega$ be arbitrary. It is easy to check that

$$\begin{aligned}
\left|\left\langle (T+S)\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right|^2 &\leq \left|\left\langle T\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right|^2 + \left|\left\langle S\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right|^2 \\
&\quad + 2\left|\left\langle T\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right|\left|\left\langle S\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right| \\
&= \left|\left\langle T\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right|^2 + \left|\left\langle S\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right|^2 \\
&\quad + 2\left|\left\langle T\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right|\left|\left\langle \widehat{k}_{\mathcal{H},\lambda}, S^{*_A}\widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right|.
\end{aligned}$$

Using the Lemma 2.2, we get

$$\begin{aligned}
\left|\left\langle (T+S)\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right|^2 &\leq \left|\left\langle T\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right|^2 + \left|\left\langle S\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right|^2 \\
&\quad + \left\|T\widehat{k}_{\mathcal{H},\lambda}\right\|_A \left\|S^{*_A}\widehat{k}_{\mathcal{H},\lambda}\right\|_A + \left|\left\langle T\widehat{k}_{\mathcal{H},\lambda}, S^{*_A}\widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right| \\
&= \left|\left\langle T\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right|^2 + \left|\left\langle S\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right|^2 \\
&\quad + \sqrt{\left\langle T^{*_A}T\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A \left\langle SS^{*_A}\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A} \\
&\quad + \left|\left\langle ST\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right|.
\end{aligned}$$

Now, by using the arithmetic-geometric mean inequality we get

$$\begin{aligned}
&\left|\left\langle (T+S)\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right|^2 \\
&\leq \operatorname{ber}_A^2(T) + \operatorname{ber}_A^2(S) \\
&+ \frac{1}{2}\left(\left|\left\langle T^{*_A}T\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right| + \left|\left\langle SS^{*_A}\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A\right|\right) + \operatorname{ber}_A(ST) \\
&= \operatorname{ber}_A^2(T) + \operatorname{ber}_A^2(S) + \frac{1}{2}\left\langle \left(T^{*_A}T + SS^{*_A}\right)\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle_A + \operatorname{ber}_A(ST) \\
&\leq \operatorname{ber}_A^2(T) + \operatorname{ber}_A^2(S) + \frac{1}{2}\operatorname{ber}_A\left(T^{*_A}T + SS^{*_A}\right) + \operatorname{ber}_A(ST),
\end{aligned}$$

hence, we get (2.10) as required. □

Corollary 2.1. *Let $T \in \mathcal{B}_{A,r}(\mathcal{H}(\Omega))$. Then*

$$\operatorname{ber}_A(T) \leq \frac{1}{2}\sqrt{2\operatorname{ber}_A^2(T) + \operatorname{ber}_A\left(T^2\right) + \frac{1}{2}\operatorname{ber}_A\left(T^{*_A}T + TT^{*_A}\right)}.$$

Proof. To put $S = T$ in Proposition 2.2. □

Proposition 2.3. *For* $i=1,2,...,k$, *let* $S_i \in \mathcal{B}_{A,r}(\mathcal{H}(\Omega))$. *Then*

$$\operatorname{ber}_A^{4n}\left(\sum_{i=1}^{k} S_i\right) \leq \frac{k^{4n-1}}{4}\left[\left\|\left(S_i^{*_A} S_i\right)^{2n}+\left(S_i S_i^{*_A}\right)^{2n}\right\|_{A-\mathrm{Ber}}\right.$$
$$\left.+2\sum_{i=1}^{k} \operatorname{ber}_A\left(\left(S_i^{*_A} S_i\right)^{n}+\left(S_i S_i^{*_A}\right)^{n}\right)\right]$$

for all $n=1,2,....$

Proof. Let $\lambda \in \Omega$ be arbitrary. Since $S_i \in \mathcal{B}_{A,r}(\mathcal{H}(\Omega))$, so $S_i \in \mathcal{B}(\mathcal{H}(\Omega))$. Then we have :

$$\left|\left\langle\left(\sum_{i=1}^{k} S_i\right)\widehat{k}_{\mathcal{H},\lambda},\widehat{k}_{\mathcal{H},\lambda}\right\rangle\right|^{4n} \leq \left(\sum_{i=1}^{k}\left|\left\langle S_i\widehat{k}_{\mathcal{H},\lambda},\widehat{k}_{\mathcal{H},\lambda}\right\rangle\right|\right)^{4n}$$
$$\leq k^{4n-1}\sum_{i=1}^{k}\left|\left\langle S_i\widehat{k}_{\mathcal{H},\lambda},\widehat{k}_{\mathcal{H},\lambda}\right\rangle\right|^{4n}$$
$$\leq k^{4n-1}\sum_{i=1}^{k}\left\langle |S_i|\,\widehat{k}_{\mathcal{H},\lambda},\widehat{k}_{\mathcal{H},\lambda}\right\rangle^{2n}\left\langle |S_i^*|\,\widehat{k}_{\mathcal{H},\lambda},\widehat{k}_{\mathcal{H},\lambda}\right\rangle^{2n},$$

$$\left(\left\langle S_i\widehat{k}_{\mathcal{H},\lambda},\widehat{k}_{\mathcal{H},\lambda}\right\rangle^{2} \leq \left\langle |S_i|\,\widehat{k}_{\mathcal{H},\lambda},\widehat{k}_{\mathcal{H},\lambda}\right\rangle\left\langle |S_i^*|\,\widehat{k}_{\mathcal{H},\lambda},\widehat{k}_{\mathcal{H},\lambda}\right\rangle\right)$$
$$\leq k^{4n-1}\sum_{i=1}^{k}\left\langle |S_i|^{2n}\,\widehat{k}_{\mathcal{H},\lambda},\widehat{k}_{\mathcal{H},\lambda}\right\rangle\left\langle |S_i^*|^{2n}\,\widehat{k}_{\mathcal{H},\lambda},\widehat{k}_{\mathcal{H},\lambda}\right\rangle,$$

$$\left(\left\langle S\widehat{k}_{\mathcal{H},\lambda},\widehat{k}_{\mathcal{H},\lambda}\right\rangle^{r} \leq \left\langle S^r\widehat{k}_{\mathcal{H},\lambda},\widehat{k}_{\mathcal{H},\lambda}\right\rangle,\ S\geq 0,\ r\geq 1\right)$$
$$= k^{4n-1}\sum_{i=1}^{k}\left\langle |S_i|^{2n}\,\widehat{k}_{\mathcal{H},\lambda},\widehat{k}_{\mathcal{H},\lambda}\right\rangle\left\langle \widehat{k}_{\mathcal{H},\lambda},|S_i^*|^{2n}\,\widehat{k}_{\mathcal{H},\lambda}\right\rangle$$
$$\leq \frac{k^{4n-1}}{2}\sum_{i=1}^{k}\left(\left\||S_i|^{2n}\,\widehat{k}_{\mathcal{H},\lambda}\right\|\left\||S_i^*|^{2n}\,\widehat{k}_{\mathcal{H},\lambda}\right\|\right.$$
$$\left.+\left|\left\langle |S_i|^{2n}\,\widehat{k}_{\mathcal{H},\lambda},|S_i^*|^{2n}\,\widehat{k}_{\mathcal{H},\lambda}\right\rangle\right|\right)$$
$$\leq \frac{k^{4n-1}}{2}\sum_{i=1}^{k}\left(\frac{1}{2}\left(\left\||S_i|^{2n}\,\widehat{k}_{\mathcal{H},\lambda}\right\|^{2}+\left\||S_i^*|^{2n}\,\widehat{k}_{\mathcal{H},\lambda}\right\|^{2}\right)\right.$$
$$\left.+\left|\left\langle |S_i|^{2n}\,|S_i^*|^{2n}\,\widehat{k}_{\mathcal{H},\lambda},\widehat{k}_{\mathcal{H},\lambda}\right\rangle\right|\right)$$
$$\leq \frac{k^{4n-1}}{4}\sum_{i=1}^{k}\left\langle\left(|S_i|^{4n}+|S_i^*|^{4n}\right)\widehat{k}_{\mathcal{H},\lambda}\widehat{k}_{\mathcal{H},\lambda}\right\rangle$$

$$+\frac{k^{4n-1}}{2}\sum_{i=1}^{k}\left|\left\langle |S_i|^{2n}|S_i^*|^{2n}\widehat{k}_{\mathcal{H},\lambda},\widehat{k}_{\mathcal{H},\lambda}\right\rangle\right|$$
$$=\frac{k^{4n-1}}{4}\left\langle\left(\sum_{i=1}^{k}\left(|S_i|^{4n}+|S_i^*|^{4n}\right)\right)\widehat{k}_{\mathcal{H},\lambda}\widehat{k}_{\mathcal{H},\lambda}\right\rangle$$
$$+\frac{k^{4n-1}}{2}\sum_{i=1}^{k}\left|\left\langle |S_i|^{2n}|S_i^*|^{2n}\widehat{k}_{\mathcal{H},\lambda},\widehat{k}_{\mathcal{H},\lambda}\right\rangle\right|$$
$$\leq\frac{k^{4n-1}}{4}\left\|\sum_{i=1}^{k}|S_i|^{4n}+|S_i^*|^{4n}\right\|_{\text{Ber}}+\frac{k^{4n-1}}{2}\sum_{i=1}^{k}\text{ber}\left(|S_i|^{2n}|S_i^*|^{2n}\right),$$

where the second inequality follows from Bohr's inequality (see [12]), i.e., if for $i=1,2,...,n$, a_i be a positive real number, then

$$\left(\sum_{i=1}^{k}a_i\right)^r\leq k^{r-1}\sum_{i=1}^{k}a_i^r,\ r\geq 1$$

and the fifth inequality follows from Bozano's inequality (see [9]), i.e., if $x,y,e\in\mathcal{H}$ with $\|e\|=1$, then

$$|\langle x,e\rangle|+|\langle e,y\rangle|\leq\frac{1}{2}\left(\|x\|\,\|y\|+|\langle x,y\rangle|\right).$$

Taking the supremum over all $\lambda\in\Omega$, we get

$$\text{ber}^{4n}\left(\sum_{i=1}^{k}S_i\right)\leq\frac{k^{4n-1}}{4}\left\|\sum_{i=1}^{k}|S_i|^{4n}+|S_i^*|^{4n}\right\|_{\text{Ber}}+\frac{k^{4n-1}}{2}\sum_{i=1}^{k}\text{ber}\left(|S_i|^{2n}|S_i^*|^{2n}\right).\tag{2.11}$$

Since $\mathcal{B}_{A,r}(\mathcal{H}(\Omega))\subseteq\mathcal{B}_{A^{1/2},r}(\mathcal{H}(\Omega))$, for each $i=1,2,...,k$, we have that $S_i\in\mathcal{B}_{A^{1/2},r}(\mathcal{H}(\Omega))$. Therefore, there exists unique $\widehat{S}_i$ in $\mathcal{B}\left(\mathbb{R}\left(A^{1/2}\right)\right)$ such that $\mathbb{Z}_AS_i=\widehat{S}_i\mathbb{Z}_A$. Now, $\mathbb{R}\left(A^{1/2}\right)$ being a complex Hilbert space, so (2.11) implies that

$$\text{ber}^{4n}\left(\sum_{i=1}^{k}\widehat{S}_i\right)\leq\frac{k^{4n-1}}{4}\left\|\sum_{i=1}^{k}\left|\widehat{S}_i\right|^{4n}+\left|\widehat{S_i^*}\right|^{4n}\right\|_{\mathcal{B}\left(\mathbb{R}\left(A^{\frac{1}{2}}\right)\right)}$$
$$+\frac{k^{4n-1}}{2}\sum_{i=1}^{k}\text{ber}\left(\left|\widehat{S}_i\right|^{2n}\left|\widehat{S_i^*}\right|^{2n}\right).$$

On the other hand, for $S,T\in\mathcal{B}_{A^{1/2},r}(\mathcal{H})$ we have $\widehat{S+\lambda T}=\widehat{S}+c\widehat{T}$ and $\widehat{ST}=\widehat{S}\widehat{T}$ for all $c\in\mathbb{C}$ (see [14]). Therefore, from the above inequality we have :

$$\text{ber}^{4n}\left(\widehat{\sum_{i=1}^{k}S_i}\right)\leq\frac{k^{4n-1}}{4}\left\|\sum_{i=1}^{k}\left(\left(\widehat{S}_i\right)^*\widehat{S}_i\right)^{2n}+\left(\widehat{S}_i\left(\widehat{S}_i\right)^*\right)^{2n}\right\|_{\mathcal{B}\left(\mathbb{R}\left(A^{1/2}\right)\right)}$$
$$+\frac{k^{4n-1}}{2}\sum_{i=1}^{k}\text{ber}\left(\left(\left(\widehat{S}_i\right)^*\widehat{S}_i\right)^n\left(\widehat{S}_i\left(\widehat{S}_i\right)^*\right)^n\right).$$

Moreover, since $\left(\widehat{S}_i\right)^* = \widehat{S_i^{*_A}}$, we have

$$\operatorname{ber}^{4n}\left(\widehat{\sum_{i=1}^{k} S_i}\right) \le \frac{k^{4n-1}}{4}\left\|\sum_{i=1}^{k}\left(\widehat{S_i^{*_A}}\widehat{S}_i\right)^{2n} + \left(\widehat{S}_i\widehat{S_i^{*_A}}\right)^{2n}\right\|_{\mathcal{B}\left(\mathbb{R}\left(A^{\frac{1}{2}}\right)\right)}$$
$$+ \frac{k^{4n-1}}{2}\sum_{i=1}^{k}\operatorname{ber}\left(\left(\widehat{S_i^{*_A}}\widehat{S}_i\right)^{n}\left(\widehat{S}_i\widehat{S_i^{*_A}}\right)^{n}\right)$$
$$= \frac{k^{4n-1}}{4}\left\|\sum_{i=1}^{k}\left(\left(S_i^{*_A}S_i\right)^{2n} + \left(S_iS_i^{*_A}\right)^{2n}\right)\hat{}\,\right\|_{\mathcal{B}\left(\mathbb{R}\left(A^{\frac{1}{2}}\right)\right)}$$
$$+ \frac{k^{4n-1}}{2}\sum_{i=1}^{k}\operatorname{ber}\left(\left(\left(S_i^{*_A}S_i\right)^{n}\left(S_iS_i^{*_A}\right)^{n}\right)\hat{}\,\right).$$

Thus

$$\operatorname{ber}_A^{4n}\left(\sum_{i=1}^{k} S_i\right) \le \frac{k^{4n-1}}{4}\left\|\left(S_i^{*_A}S_i\right)^{2n} + \left(S_iS_i^{*_A}\right)^{2n}\right\|_{A-\mathrm{Ber}}$$
$$+ \frac{k^{4n-1}}{2}\sum_{i=1}^{k}\operatorname{ber}_A\left(\left(S_i^{*_A}S_i\right)^{n} + \left(S_iS_i^{*_A}\right)^{n}\right).$$

This completes the proof. □

Corollary 2.2. *Let $S \in \mathcal{B}_{A,r}\left(\mathcal{H}\left(\Omega\right)\right)$. Then*

$$\operatorname{ber}_A^{4}\left(S\right) \le \frac{1}{4}\left\|\left(S^{*_A}S\right)^{2} + \left(SS^{*_A}\right)^{2}\right\|_{A-\mathrm{Ber}} + \frac{1}{2}\operatorname{ber}_A\left(S^{*_A}S^{2}S^{*_A}\right).$$

Proof. Put $k = 1$ and $n = 1$ in Proposition 2.3. □

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