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ON SUBSTITUTION AND EXTENSION OPERATORS IN BANACH-SOBOLEV FUNCTION SPACES

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Abstract. In the paper we study the Sobolev spaces $WX^m(\Omega)$ generated by some non-standard Banach function spaces $X(\Omega)$ on n-dimensional domain $\Omega \subset R^n$ with Lebesgue measure. In general, the considered non-standard spaces are not separable, therefore using classical methods in these spaces requires the essential modification of classical methods and a lot of preparation, such as correctness of substitution operator, problems connected with extension operator in such spaces, etc. To this aim, based on the shift operator, we determine the corresponding separable subspaces X_s of the above spaces, in which the set of compactly supported infinitely differentiable functions is dense. We also define the corresponding spaces $WX_s^m(\Omega)$ and $\overset{0}{WX}{}_s^m(\Omega_1)$. In case where $X(K)$ is a non-atomic rearrangement-invariant resonant Banach function space and $D, \Omega \subset K$ are bounded domains, the boundedness and other properties of the substitution operator from $X(\Omega)$ to $X(D)$ are established. The extension operators from $WX_s^m(\Omega)$ to $\overset{0}{WX}{}_s^m(\Omega_1)$ (for $\Omega_1 \supset \Omega$) are also considered. In particular, it is proved that if Ω has a sufficiently smooth boundary, then such a bounded extension exists.

1. Introduction

Partial differential equations, their solution methods and solvability problems have a long and deep history of development. Many monographs of well-known mathematicians are dedicated to these problems [25, 19, 3, 14, 28, 24, 21, 20, 23]. At present time, the theory of partial differential equations is one of the centerpieces of mathematical research. One of the principal questions of this theory is: what is understood by "the solution" of partial differential equations? In early stage, the solutions of considered equations were understood in the classical sense. But many applied problems required to introduce "solution" in a more generalized sense (for example, generalized solution, weak solution, strong solution, etc). These notions depend on considered spaces, and the emergence of new spaces dictates appropriate research in corresponding directions. It should be noted that the solvability problems of elliptic operators, various estimates, in

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particular, interior Schauder-type estimates play exceptional role in this field (see e.g., [19, 3]).

In connection with concrete problems of pure mathematics, mechanics and mathematical physics, interest in so-called nonstandard spaces of functions has greatly increased. Naturally, the emergence of new functional spaces, such as the Lebesgue space, the Morrey space, the grand-Lebesgue space, etc. requires the development of appropriate theory. That's why various problems in such spaces began to be intensively studied. (see [12, 1, 16, 17, 26, 11, 22, 13, 4, 15, 27, 10, 6, 7, 8]).

The methods of harmonic analysis in such spaces are well developed. At the same time, the various problems of differential equations in non-standard Sobolev spaces, generated by the norms of these spaces have begun to be studied. It should be noted that all these spaces are Banach function spaces (b.f.s) according to Luxemburg's classification (see [2]). First research of this kind for Morrey-type spaces has been carried out in 2000 to continue up to the present. Similar researches have been also carried out in grand-Lebesgue spaces (see [12, 1, 16, 17, 26, 4, 15, 27, 9, 30, 10]). The most of such spaces are rearrangement-invariant and fundamental results have been obtained by Calderon and Boyd.

These circumstances require the generalization of many well-known facts concerning classical Sobolev spaces to the Sobolev spaces $WX^m(\Omega)$, generated by rearrangement-invariant Banach function spaces. The weak-type L_p^{ω} -case is also covered by such spaces. The present paper is dedicated to this research direction. We study the Sobolev spaces $WX^m(\Omega)$ generated by some non-standard Banach function spaces $X(\Omega)$ on n-dimensional domain $\Omega \subset R^n$ with Lebesgue measure. In general, the considered non-standard spaces are not separable, therefore using classical methods in these spaces requires the essential modification of classical methods and a lot of preparation, such as correctness of substitution operator, problems connected with extension operator in such spaces, etc. To this aim, based on the shift operator, we determine the corresponding separable subspaces X_s of the above spaces, in which the set of compactly supported infinitely differentiable functions is dense. We also define the corresponding spaces $WX_s^m(\Omega)$ and $\overset{0}{W}X_s^m(\Omega_1)$. In case where $X(K)$ is a non-atomic rearrangement-invariant resonant Banach function space and $D, \Omega \subset K$ are bounded domains, the boundedness and other properties of the substitution operator from $X(\Omega)$ to $X(D)$ are established. The extension operators from $WX_s^m(\Omega)$ to $\overset{0}{W}X_s^m(\Omega_1)$ (for $\Omega_1 \supset \Omega$) are also considered. In particular, it is proved that if Ω has a sufficiently smooth boundary, then such a bounded extension exists.

2. Necessary information

2.1. Notations. We will use the following standard notations: Z_+ will be the set of non-negative integers and $|x| = \sqrt{x_1^2 + ... + x_n^2}$ will be the norm of $x = (x_1, ..., x_n)$. By $m = mes(M)$ we will denote the Lebesgue measure of the set M. $B_r(x_0) = \{x \in R^n : |x - x_0| < r\}$ will denote the open ball in R^n, $\Omega_r(x_0) = \Omega \bigcap B_r(x_0)$, $B_r = B_r(0)$. $\partial\Omega$ will be the boundary of the domain Ω. $\overline{\Omega} = \Omega \bigcup \partial\Omega$ will stand for the closure of Ω. For $\Omega \subset R^n$, $\delta \in R^n$ by Ω_δ we will denote the following set: $\Omega_\delta = \Omega - \delta = \{x | x + \delta \in \Omega\}$. $M_1 \Delta M_2$ will be the symmetric difference of sets M_1 and M_2. The diameter of the set Ω will be denoted by $d(\Omega) = diam\Omega$. $\rho(x, M)$ will be the distance between x and the set M and $\|T\|_{X \to Y}$ is the norm of operator T, which acts from X to Y. Unit ball in Banach function space X and its associate space will be denoted by S and S', respectively. $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ will be multiindices with the coordinates $\alpha_k \in Z_+$, $\forall k = \overline{1, n}$; $\partial_i = \frac{\partial}{\partial x_i}$ will denote the differentiation operator and $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} ... \partial_n^{\alpha_n}$. For every $\xi = (\xi_1, \xi_2, ..., \xi_n)$ we assume $\xi^\alpha = (\xi_1^{\alpha_1}, \xi_2^{\alpha_2}, ..., \xi_n^{\alpha_n})$. By m-th order diffeomorphism between two domains of R^n with sufficiently smooth boundaries we will mean the homeomorphism of these domains, i.e. an invertible function that maps one domain into another, such that both the function and its inverse are m-times differentiable.

2.2. Banach function spaces. In this section we define Banach function spaces, related notions and principal characteristics of these spaces. For more details we refer to [2].

Let (M, M, μ) be a measure space, F be the set of all measurable functions whose values belong to $[-\infty; +\infty]$, F^+ be the cone of μ-measurable functions whose values lie in $[0; +\infty]$, χ_E be a characteristic function of μ-measurable subset $E \in M$. Let F_0 denote the class of functions which are finite $\mu - a.e.$, and F_s be the collection of all simple functions.

Definition 2.1. A mapping $\rho : M \to [0; +\infty]$ is called a Banach function norm if, for $\forall f, g, f_n \in F^+$, $\forall a \geq 0$, $\forall E \in M$ the following properties hold:

$(P1)$ $\rho(f) = 0 \Leftrightarrow f = 0 \ \mu - a.e,\ \rho(af) = a\rho(f),\ \rho(f + g) \leq \rho(f) + \rho(g);$

$(P2)$ $g \leq f \ \mu - a.e \Rightarrow \rho(g) \leq \rho(f);$

$(P3)$ $0 \leq f_n \uparrow f \Rightarrow \rho(f_n) \uparrow \rho(f);$

$(P4)$ $\mu(E) < +\infty \Rightarrow \rho(\chi_E) < +\infty;$

$(P5)$ $\mu(E) < +\infty \Rightarrow \int_E f d\mu \leq C_E \rho(f)$, where C_E is some positive number which may depend on E and ρ, but not on f.

Norm on these spaces can be defined as follows.

Definition 2.2. Let ρ be a function norm on F^+. The collection $X = X(\rho)$ of all functions in F for which $\rho(|f|) < +\infty$ is called a Banach function space. For each $f \in X$ the norm is defined as $\|f\|_X = \rho(|f|)$.

Let's state some theorems which describe characteristic properties of such spaces.

Theorem 2.1. *i) The following inclusions hold:* $F_s \subset X \subset F_0$.

ii) if $f_n \to f$ *in* X, *then* $f_n \to f$ *in measure. Therefore there is some subsequence of* $\{f_n\}$ *which converges pointwise to* f.

Definition 2.3. Let ρ be a function norm, its associate norm ρ' be defined on F^+ by

$$\rho'(g) = \sup \left\{ \int_M fg d\mu : \rho(f) \leq 1 \right\}.$$

Definition 2.4. Let ρ be a function norm, the space $X = X(\rho)$ be a corresponding Banach function space, ρ' be a corresponding associate norm. Then the

Banach function space $X(\rho')$ defined by associate norm ρ' is called the associate of X, denoted as $X' = X(\rho')$.

The following analogy of Hölder's inequality holds.

Theorem 2.2. *(Hölder's inequality) Let X be a Banach function space and X' be a corresponding associate space. Then for $\forall f \in X$, $\forall g \in X'$ the product fg is integrable and*

$$\int |fg|\, d\mu \leq \|f\|_X \|g\|_{X'} .$$

Theorem 2.3 and Lemma 2.1 below show that the norm of function $f \in X$ can be given by

$$\|f\|_X = \sup\left\{ \left| \int_M fg d\mu \right| : g \in X', \|g\|_{X'} \leq 1 \right\}. \tag{2.1}$$

Theorem 2.3. *(G.G. Lorentz, W.A.J. Luxemburg) Every Banach function space X coincides with its second associate space X'' and $\|f\|_X = \|f\|_{X''}$ for every $f \in X$.*

Lemma 2.1. *The norm of a function g in the associate space X' is given by*

$$\|g\|_{X'} = \sup\left\{ \left| \int_M f\, g\, d\mu \right| : f \in X, \|f\|_X \leq 1 \right\}.$$

On the example of L_∞ it is clear that "smallness in measure" doesn't mean "smallness" in the sense of norm. The following definition plays a connecting role between these notions and defines the corresponding class of functions.

Definition 2.5. Let $f \in X$. If for each sequence of measurable sets $\{E_n\}_1^\infty$ which satisfies $E_n \to \emptyset$ $\mu - a.e.$ the relation $\|f\chi_{E_n}\|_X \to 0$ holds, then it is said that f has an absolutely continuous norm. Denote the set of all functions from X with an absolutely continuous norm by X_a. If $X = X_a$, then the space X itself is said to have an absolutely continuous norm.

To obtain many structure properties of such spaces it is important to study simple functions F_s, i.e. the liner combination of characteristic functions:

$$u = \sum_{i=1}^{n} p_i E_i, \ \mu(E_i) < \infty.$$

Definition 2.6. Let X be a Banach function space. The closure in X of the set of simple functions is denoted by X_b.

The following proposition is true.

Proposition 2.1. *The subspace X_b is the closure of supported bounded functions over sets of finite measure.*

Theorem below is true for these subspaces.

Theorem 2.4. ***a)*** *The following inclusions hold:*

$$X_a \subset X_b \subset X.$$

b) *Subspaces X_a and X_b coincide if and only if for every set of finite measure χ_E has an absolutely continuous norm.*

In general, some properties of functions are lost, while translating measure spaces. The following notions are introduced for characterization of these properties and for studying many questions.

Definition 2.7. Let $f \in F_0$. By distribution function of f we mean the following function defined on $[0; \infty)$: $\mu_f(\lambda) = \mu\{x \in M : |f(x)| > \lambda\}$. Two functions $f, g \in F_0$, with the same distribution functions, are called equimeasurable. That is $\mu_f(\lambda) = \mu_g(\lambda)$ for all $\lambda \geq 0$.

The class of all equimeasurable functions with g will be denoted by E_g.

Definition 2.8. For every function $f \in F_0$ the decreasing rearrangement function is defined by the relation

$$f^*(t) = \inf\{\lambda : \mu_f(\lambda) \leq t\} = \sup\{\lambda : \mu_f(\lambda) > t\} = m_{\mu_f}(t), \quad (t \geq 0).$$

Definition 2.9. Let (M,M, μ) be a totally σ- finite measure space. If for each pair $f, g \in F_0$ the equality

$$\int_0^\infty f^*(t)\, g^*(t)\, dt = \sup_{\tilde{g} \in E_g} \left\{ \int_M |f\tilde{g}|\, dt \right\}$$

holds, then the space (M,M, μ) is called resonant. If for each pair $f, g \in F_0$ there is some function $\tilde{g} \in E_g$ for which the supremum is achieved, then the space (M,M, μ) is called strongly resonant.

Theorem 2.5. *A totally σ- finite measure space is strongly resonant if and only if it is a finite measure space of one of the following two types:*
i) nonatomic,
ii) completely atomic and all atoms have equal measure.

In particular, if $\Omega \subset R^n$, and μ is a Lebesgue measure, then $X(\Omega)$ is the resonant space and if $mes(\Omega) < \infty$, then it is strongly resonant.

Definition 2.10. Let (M,M, μ) be totally σ- finite measure space. If for every pair of equimeasurable functions $f, g \in F_0^+$ the equality $\rho(f) = \rho(g)$ holds, then the norm ρ is called rearrangement-invariant norm. The Banach function space generated by rearrangement-invariant norm is called rearrangement-invariant Banach function space or symmetric space.

Let $X = X(\rho)$ be rearrangement-invariant resonant Banach space. The following lemma holds [18, p.133, Corollary 2].

Lemma 2.2. *The rearrangement-inavriant Banach function space X has the property: if $f \in X$ and the inequality*

$$\mu_g(\lambda) \leq C\mu_f(\lambda) \ \ (\forall \lambda > 0)$$

holds, then $g \in X$ and

$$\|g\|_X \leq \max\{1; C\}\, \|f\|_X\, .$$

3. Lebesgue measure case and substitution operator

Hereinafter, we will assume the following: let $K = \left\{(x_1, ..., x_n) : |x_i| < \frac{d}{2}\right\} \subset R^n$ be any cube, $X(K)$ be a rearrangement- invariant Banach function space defined on K, with Lebesgue measure and the function norm ρ. If $\Omega \subset K : \overline{\Omega} \subset K$ is a connected domain, by $X(\Omega)$ we mean the space of restrictions of all functions from $X(K)$ to Ω, with corresponding norm, i.e.

$$X(\Omega) = \left\{ f \in X(K) : \|f\|_{X(\Omega)} = \|f\chi_\Omega\|_{X(K)} < \infty \right\}.$$

We assume that sufficiently small neighborhoods of any boundary point are simply connected set.

It is suggested that $\Omega + \Omega \subset K$. $\Omega + \delta = \{t + \delta : t \in \Omega\}$ means that $\delta \in R^n$ is such that $\Omega + \delta \subset K$. For $\forall f \in X(\Omega)$ we assume $f\Big|_{K\backslash\overline{\Omega}} \equiv 0$.

By T_δ for arbitrary $\delta \in R^n : |\delta| < dist(\partial\Omega, \partial K)$ we denote the additive shift operator, defined in the following way: $(T_\delta f)(x) = f(x + \delta)$, for every $f \in X(\Omega)$. By X_s $(X_s^{'})$ we denote the subspace of functions of $X(\Omega)$, which have the following property:

$$\begin{array}{ll} \alpha) & \|T_\delta(f) - f\|_X \to 0,\ \delta \to 0,\ f \in X_s, \\ & (\|T_\delta(f) - f\|_{X'} \to 0,\ \delta \to 0,\ f \in X'), \end{array} \tag{3.1}$$

where $\delta \in R^n$ is a shift vector and $T_\delta f(x) = f(x + \delta)$ is a corresponding shift operator.

The convolution of functions f, g defined on $\Omega \subset K : \Omega + \Omega \subset K$ will be denoted as $f * g$, i.e.

$$(f * g)(x) = \int_\Omega f(x - y)\, g(y)\, dy. \tag{3.2}$$

Let $\omega : [0; \infty) \to R_+$ be an infinitely differentiable function which is equal to zero for $t \geq 1$ and has positive value for arbitrary $t < 1$. Then the cap function is defined in the following way:

$$\omega_r(x) = cr^{-n}\omega\left(\frac{|x|^2}{r^2}\right), \tag{3.3}$$

where c is chosen in such a way that $\int_{R^n} \omega_r(x)\, dx = 1$.

For example,

$$\omega(t) = \begin{cases} \exp\left(\frac{1}{t-1}\right), & t < 1, \\ 0, & t \geq 1, \end{cases}$$

may be considered as a cap function.

Let f be any integrable function defined on $\Omega : \overline{\Omega} \subset K$. We introduce

$$f_r(x) = (\omega_r * f)(x) = \int_\Omega \omega_r(x - y)\, f(y)\, dy. \tag{3.4}$$

Recall that f is equal to zero on $K\backslash\overline{\Omega}$ and we are considering such $r > 0 \Rightarrow \operatorname{supp} f_r \subset K$.

Let's introduce the following characteristics:

$$\beta)\ \ \forall E_n \to \emptyset \Rightarrow \|\chi_{E_n}\|_X \to 0. \tag{3.5}$$

Property 3.1. Let $\beta)$ holds, and let $E \subset K$be any measurable set. Then

$$\|\chi_{E\Delta E_\delta}\|_X \underset{\delta\to 0}{\to} 0\,.$$

Proof. This is evident. Indeed,

$$mes\,(E\Delta E_\delta) = \int_K |\chi_E(x) - \chi_{E_\delta}(x)|\,dx = \|T_\delta \chi_E(x) - \chi_E(x)\|_{L_1} \underset{\delta\to 0}{\to} 0 \Rightarrow$$
$$\overset{\text{by } \beta)}{\Rightarrow} \|\chi_{E\Delta E_\delta}\|_X \to 0,\ \delta \to 0\,.$$

The property is proved. □

Lemma 3.1. *If $\beta)$ holds, then $X_s = X_a = X_b = \overline{C_0^\infty(\Omega)}$ (the closure is taken in topology of X).*

Proof. $X_a = X_b$ follows from Theorem 2.4 and definition of X_b. (Indeed, it is true for arbitrary rearrangement-invariant Banach function space with property $\beta)$).

The case $X_s = \overline{C_0^\infty(\Omega)}$ is proved in [5, Lemma 2.3].

On the other hand, by Property 3.1 any characteristic, and consequently any simple function belongs to X_s. Therefore, $X_b \overset{Th.2.1}{=} \overline{F_s} \subset X_s$.

□

Proposition 3.1. *Let $X = X(K,\rho)$ be a non-atomic rearrangement-invariant Banach function space and $\Omega : \overline{\Omega} \subset K$. Then for each $\delta > 0$: $\delta < dist\,(\partial\Omega, \partial K)$:*

a) $T_\delta : X_s(\Omega) \to X_s(\Omega_\delta)$, $T_\delta : X_a(\Omega) \to X_a(\Omega_\delta)$;

b) If $\beta)$ holds, then $T_\delta : X_b(\Omega) \to X_b(\Omega_\delta)$.

Proof. The statement a) is evident. Indeed, $f \in X_s(\Omega)$ means that

$$\delta' \to 0 \Rightarrow \left\|f\left(\cdot + \delta'\right) - f(\cdot)\right\|_X \to 0.$$

Since for arbitrary δ the relation

$$\left\|f\left((\cdot + \delta) + \delta'\right) - f(\cdot + \delta)\right\|_X = \left\|f\left(\cdot + \delta'\right) - f(\cdot)\right\|_X \to 0,\ \delta' \to 0$$

holds, we have $T_\delta(f) \in X_s(\Omega_\delta)$.

Let's consider $f \in X_a$. Since X is a rearrangement-invariant space, the following holds:

$$\{E_n\}_{n=1}^\infty, \mu(E_n) \to 0 \Rightarrow \|\chi_{E_n+\delta}\|_X = \|\chi_{E_n}\|_X \to 0.$$

Therefore,

$$\|(T_\delta f)\cdot \chi_{E_n}\|_X = \|f \cdot \chi_{E_n-\delta}\|_X \to 0,\ n \to \infty.$$

b) From Definition 2.6 it follows that it is sufficient to prove this fact for $f \in F_s$. Indeed, let $f \in X_b$. Then $\exists g \in F_s$ for which the value $\|f - g\|_X$ is sufficiently small. On the other hand, the functions $(f-g)(.)$ and $(f-g)(.+\delta)$ are equimeasurable. Therefore, $\|T_\delta f - T_\delta g\|_X = \|f - g\|_X$ is also small. Besides, for $f = \sum_1^k a_i \chi_{E_i}$ (here $E_i, i = \overline{1,k}$ are measurable subsets of Ω with finite measure) we have

$$\|T_\delta f - f\|_X = \left\|\sum_1^k a_i\left(\chi_{E_i+\delta} - \chi_{E_i}\right)\right\|_X \le \sum_1^k |a_i|\,\|(\chi_{E_i+\delta} - \chi_{E_i})\|_X\,.$$

Hence it immediately follows that it is sufficient to prove this statement for $f = \chi_E$, $\mu(E) < \infty$, where E is measurable. But this is also evident:

$$\|\chi_E(x+\delta) - \chi_E(x)\|_X = \left\|\chi_{(E+\delta)\Delta E}\right\|_X \to 0,\ \delta \to 0.$$

The proposition is proved. □

Proposition 3.2. *If* $f \in X_s(\Omega)$ *and* $\|T_\delta f - f\|_{X(K)} \to 0,\ \delta \to 0$, *then*

$$\|f_r - f\|_X \to 0\ \ r \to 0.$$

Proof. Since

$$f_r(x) - f(x) = \int_\Omega \omega_r(x-y)\left(f(y) - f(x)\right) dy,$$

we have

$$\|f_r - f\|_{X(K)} = \sup\left\{\int_\Omega \left(\int_{|x-y|\le r} \omega_r(x-y)\,|f(y) - f(x)|\,dy\right)|v(x)|\,dx : v \in S'\right\}.$$

On the other hand, if we substitute $x - y = z$, we obtain

$$\begin{aligned}&\int_\Omega \left(\int_{|x-y|\le r} \omega_r(x-y)\,|f(y)-f(x)|\,dy\right)|v(x)|\,dx = \\ &= \int_{|z|\le r} \omega_r(z)\,dz \int_\Omega |f(x-z) - f(x)|\,|v(x)|\,dx \le \\ &\le \sup_{\substack{v \in S' \\ |z| \le r}} \int_\Omega |f(x-z) - f(x)|\,|v(x)|\,dx.\end{aligned}$$

As a result,

$$\|f_r - f\|_X \le \sup_{|z|\le r} \|f(x-z) - f(x)\|_X \to 0,\ r \to 0.$$

The proposition is proved. □

Lemma 3.2. *If* $\beta)$ *holds, then* $\forall \varphi \in L_\infty(\Omega)$ *the relation*

$$\varphi f \in X_s \;\&\; \|\varphi f\|_X \le \|\varphi\|_{L_\infty} \|f\|_{X(\Omega)}$$

is true.

Proof. As it is shown in Lemma 3.1, in this case $X_s = X_a = X_b$. Let $ess\sup|\varphi| = A$. Then

$$\forall E \in M \Rightarrow \|\varphi f \chi_E\| \le A\,\|f\chi_E\|.$$

Thus, φf also has absolutely continuous norm, therefore $\varphi f \in X_a$ and

$$|\varphi f| \le A\,|f| \Rightarrow \|\varphi f\|_X \le A\,\|f\|_X = \|\varphi\|_{L_\infty} \|f\|_X.$$

The lemma is proved. □

Substitution operator. Let (M_1, μ), (M_2, ν) be measure spaces, $X(M_1)$ and $X(M_2)$ be Banach function spaces, and $\varphi : M_1 \to M_2$ be an injective mapping of M_1 onto M_2, itself and its inverse translate measure sets onto measure sets. Let ϕ be the operator defined in the following way:

$$\phi : F_0(M_2, \nu) \to F_0(M_1, \mu),$$

$$(\phi f)(\cdot) = f(\varphi(\cdot)),\ \forall f \in F_0(M_2, \nu).$$

What is the relationship between the spaces $X(M_1)$ and $X(M_2)$ with regard to the above mentioned mapping?

We will try to answer this question under the assumptions that $D = M_1 \subset K$; $\Omega = M_2 \subset K$ are bounded domains, $X(K)$ is a non-atomic rearrangement-invariant resonant Banach function space with norm ρ, measure μ, and D, Ω are μ-measurable subsets of K.

Theorem 3.1. *a) Let $\varphi : D \to \Omega$ be an injective mapping from D onto itself, its inverse transform measurable sets to measurable sets and*

$$0 < c_1 \leq 1,\ c_2 \geq 1 : \forall E \in (D, \mu) \Rightarrow c_1 \mu(E) \leq \mu(\varphi(E)) \leq c_2 \mu(E). \tag{3.6}$$

Then the substitution operator ϕ is an isomorphism between $X(\Omega)$ and $X(D)$. Furthermore,

$$c_2^{-1} \leq \|\phi\| \leq {c_1}^{-1}.$$

b) If $X(D)$ and $X(\Omega)$ have the property β), then the operator ϕ is an isomorphism between $X(D)$ and $X(\Omega)$ if and only if the relation (3.6) *holds.*

Proof. a) Let $f \in X(\Omega)$, and

$$E_f(\lambda) = \{\omega \in \Omega : |f(\omega)| > \lambda\},\ E_{\phi f}(\lambda) = \{t \in D : (\phi f)(t) > \lambda\}.$$

Then

$$|(\phi f)(t)| = |f(\varphi(t))| > \lambda \Leftrightarrow \varphi(t) \in E_f(\lambda).$$

Thus

$$E_{\phi f}(\lambda) = \varphi^{-1}(E_f(\lambda)).$$

Using (3.6) we obtain

$$\mu_{\phi f}(\lambda) \leq {c_1}^{-1} \mu_f(\lambda).$$

Thus, using Lemma 2.2 we get

$$\|\phi f\|_{X(D)} \leq {c_1}^{-1} \|f\|_{X(\Omega)}.$$

b) Assume the relation (3.6) doesn't hold. Then, there exists a consequence of measurable subsets of D such that the following relation holds;

$$\exists m > 0\ \exists E_n \subset D : \ \mu(E_n) \to 0,\ \ \mu(\varphi(E_n)) \geq m,\ \ n \to \infty.$$

But, by property β) the following holds:

$$\|\chi_{E_n}\| \to 0 \Rightarrow \chi_{\varphi(E_n)} \to 0.$$

Since $X(\Omega)$ is a rearrangement-invariant space, we have

$$\begin{aligned} &\forall U \subset \Omega : \ \mu(U) \leq m \Rightarrow \|\chi_U\|_{X(\Omega)} \leq \\ &\leq \left\|\chi_{\varphi(E_n)}\right\|_{X(\Omega)} \to 0,\ (\forall n) \Rightarrow \|\chi_U\|_{X(\Omega)} = 0. \end{aligned}$$

Thus

$$\forall U \subset \Omega : \ \mu(U) \leq m \Rightarrow \|\chi_U\|_{X(\Omega)} = 0.$$

But, this is impossible.

The theorem is proved. □

In particular, the following corollary holds.

Corollary 3.1. *Let $D, \Omega : \overline{D}, \overline{\Omega} \subset K \subset R^n$ be bounded domains in R^n, $\varphi : D \to \Omega$ be a C^1-class diffeomorphism, and $X(K)$ be a rearrangement-invariant space with Lebesgue measure. Then the substitution operator ϕ is an isomorphism between $X(D)$ and $X(\Omega)$.*

Let's consider the following case (see [2, pp. 79-80]).

Definition 3.1. Let (M_1, μ_1) and (M_2, μ_2) be totally σ-finite measure spaces. A mapping φ from M_1 into M_2 is said to be a measure-preserving transformation if, whenever E is a μ_2 measurable subset of M_2, the set $\varphi^{-1}E = \{x \in M_1 : \varphi(x) \in M_2\}$ is a μ_1-measurable subset of M_1 and $\mu_1(\varphi^{-1}E) = \mu_2(E)$.

It is clear that if $M_1 = \Omega \subset R^n$, $M_2 = \Omega_{-\delta} = \Omega + \delta$, $\mu_1 = \mu_2$ is a Lebesgue measure, then the mappings $\varphi_1 : x \mapsto x + \delta$, $\varphi_2 : (x_1, ..., x_{n-1}, x_n) \mapsto (x_1, ..., x_{n-1}, -x_n)$ are measure-preserving transformations.

We will use the following proposition.

Proposition 3.3. *Let $X(K)$ be a rearrangement-invariant space with Lebesgue measure, $D, \Omega : \overline{D}, \overline{\Omega} \subset K \subset R^n$ be bounded domains in R^n, and $\varphi : D \to \Omega$ be a measure-preserving transformation from D onto Ω. Then the substitution operator ϕ is an isometric isomorphism between $X(D)$ and $X(\Omega)$.*

Proof. It is clear that the functions $\forall f \in X(D)$ and $\phi f \in X(\Omega)$ are equimeasurable. Indeed,

$$E_\lambda(f) = \{x : |f(x)| > \lambda\} \Rightarrow E_\lambda(\phi f) = \{z : |\phi f(z)| > \lambda\} =$$
$$= \{z : |f(\varphi(z))| > \lambda\} = \{z : \varphi(z) \in E_\lambda(f)\} \Rightarrow E_\lambda(f) = \varphi^{-1}E_\lambda(\phi f) \Rightarrow$$
$$\Rightarrow mesE_\lambda(f) = mesE_\lambda(\phi f) \Rightarrow \|f\|_{X(K)} = \|\phi f\|_{X(K)}.$$

The proposition is proved. □

Remark 3.1. In particular, the following relations are true:

$$a) f \in X(\Omega) \Rightarrow \|f\|_{X(\Omega)} = \|T_\delta f\|_{X(\Omega_\delta)}, \ (|\delta| < dist(\partial K, \partial\Omega)), \tag{3.7}$$

b) Let $\forall x = (x_1, ..., x_{n-1}, x_n) \in \Omega_+ \Rightarrow x_n \geq 0$. Also let $\Omega_- = \{x : (x_1, ..., x_{n-1}, -x_n)\}$ and for $\forall f_+ \in X(\Omega)$ define $f_- \in X(\Omega_-) : f_-(x_1, ..., x_{n-1}, x_n) = f_+(x_1, ..., x_{n-1}, -x_n)$. Then

$$\|f_+\|_{X(\Omega_+)} = \|f_-\|_{X(\Omega_-)}. \tag{3.8}$$

4. Banach-Sobolev function spaces and extension theorems

In this section we will assume that $K \subset R^n$ is any cube with edges d, $K = \left\{(x_1, ..., x_n) : |x_i| < \frac{d}{2}\right\}$ and $\Omega \subset K : \overline{\Omega} \subset K$ is an arbitrary domain, all assumptions made at the beginning of Section 3 hold. $X(K)$ is a rearrangement-invariant Banach function space (with Lebesgue measure), which has the property β). Without loss of generality, $B_1 \subset K$ can be assumed.

Introduce the following spaces of functions:

$$WX^m(\Omega) = \left\{f \in X : \partial^p f \in X, \forall p \in Z_+^n, \ |p| \leq m\right\},$$
$$WX_s^m(\Omega) = \left\{f \in WX^m : \|T_\delta f - f\|_{WX^m(\Omega)} \to 0, \delta \to 0\right\},$$

with corresponding norm

$$\|f\|_{WX^m} = \sum_{|p|\leq m} \|\partial^p f\|_X . \tag{4.1}$$

When $\Omega = B_r$ we will use the notations $X(r)$, $X_s(r)$, $WX^m(r)$, $WX_s^m(r)$. The shift operator is continuous on $WX_s^m(\Omega)$, then $WX_s^m(\Omega)$ is a closed subspace of $WX^m(\Omega)$.

Subspace: $\overset{0}{WX_s^m}(\Omega) = \overline{C_0^\infty}(\Omega)$ (closure is taken in the space $WX^m(\Omega)$). It is clear that every function from $\overset{0}{WX_s^m}(\Omega)$ can be extended by zero on all K.

Lemma 4.1. *(Minkowski-type inequality) Let $\Omega_1 \subset R^n$, $\Omega_2 \subset R^k$ be domains, $X(\Omega_1)$ be a Banach function space, $f : \Omega_1 \times \Omega_2 \to R$ be a measurable function. If $f(.,y) \in X(\Omega_1)$ for $m-a.e.$ $y \in \Omega_2$ and $\|f(.,y)\|_X \in L_1(\Omega_2)$, then the following inequality holds:*

$$\left\| \int_{\Omega_2} f(x,y)\,dy \right\|_X \leq \int_{\Omega_2} \|f(\cdot,y)\|_X\,dy. \tag{4.2}$$

Proof. Let $F(x) = \int_{\Omega_2} f(x,y)\,dy$. Using (2.1), we obtain

$$\begin{aligned} \|F\|_X &= \sup_{g\in S'} \left\{ \textstyle\int_{\Omega_1} |F(x)|\,|g(x)|\,dx : \right\} \leq \\ &\leq \sup_{g\in S'} \textstyle\int_{\Omega_1} \left(\int_{\Omega_2} |f(x,y)|\,dy \right) |g(x)|\,dx = \\ &= \sup_{g\in S'} \textstyle\int_{\Omega_2} \left(\int_{\Omega_1} |f(x,y)|\,|g(x)|\,dx \right) dy \leq \\ &\leq \textstyle\int_{\Omega_2} \|f(.,y)\|_X\,dy = \| \|f(.,y)\|_X \|_{L_1(\Omega_2)} . \end{aligned}$$

The lemma is proved. □

Remark 4.1. Let $f \in X(\Omega)$, $\forall h > 0$. Let us consider the function defined in the following way:

$$\begin{aligned} f_{i,h}(x) &= \textstyle\int_{x_i}^{x_i+h} f(x_1,...,x_{i-1},\tau,x_{i+1},...,x_n)\,d\tau = \\ &= \textstyle\int_0^h f(x_1,...,x_{i-1},x_i+\tau,x_{i+1},...,x_n)\,d\tau. \end{aligned}$$

Applying (4.2) to this function, we have

$$\|f_{i,h}\|_X \leq h\,\|f\|_X . \tag{4.3}$$

Theorem 4.1. *Let $u \in \overset{0}{WX_s^m}(\Omega)$. Then $u_r \overset{WX^m(K)}{\to} u$.*

Proof. It is well known that if $v = D^\alpha u \in \overset{0}{WX_s^m}(\Omega)$, then

$$\forall x : r < dist\,(x : d\Omega) \Rightarrow v_r(x) = D^\alpha u_r(x).$$

Indeed

$$D^\alpha u_r(x) = \int_\Omega D_x^\alpha \omega_r(x-y)\,u(y)\,dy = (-1)^{|\alpha|} \int_\Omega u(y)\,D_y^\alpha \omega_r(x-y)\,dy.$$

On the other hand, $r < dist\{x : d\Omega\} \Rightarrow \omega_r(x-y) \in C_0^\infty(\Omega)$. Consequently, by the definition of derivative, the relation

$$(-1)^{|\alpha|}\int_\Omega u(y)\, D_y^\alpha \omega_r(x-y)\,dy = \int_\Omega v(y)\,\omega_r(x-y)\,dy = v_r(x),$$

holds. If $u \in WX_s^m(\Omega)$, then from this equality and Proposition 3.1 it follows that

$$\forall \Omega' : \overline{\Omega'} \subset \Omega \Rightarrow D^\alpha u_r \to D^\alpha u \text{ in } X\left(\Omega'\right).$$

Taking into account that every function from $\overset{0}{WX_s^m}(\Omega)$ can be extended by zero on all K, we have

$$\forall p : |p| \le m \Rightarrow \partial^p u \in X_s(\Omega) \subset X_s(\Omega_1) \Rightarrow$$
$$\Rightarrow \partial^p u_r = (\partial^p u)_r \overset{X(\Omega)}{\to} \partial^p u \Rightarrow u_r \overset{WX^m(\Omega)}{\to} u,\ (r \to 0).$$

The theorem is proved. □

Theorem 4.2. *Let $D, \Omega : \overline{D}, \overline{\Omega} \subset K$ and $\varphi : D \to \Omega$ be a C^m-class diffeomorphism. If $u \in WX_s^m(\Omega)$, then $v = u \circ \varphi \in WX_s^m(D)$ and the following inequality holds:*

$$c_1 \|u\|_{WX_s^m(\Omega)} \le \|\nu\|_{WX_s^m(D)} \le c_2 \|u\|_{WX_s^m(\Omega)}, \tag{4.4}$$

where the constants depend only on norm φ and φ^{-1}.

Proof. Let $x = (x_1, ..., x_n) \in D$, $y = (y_1, ..., y_n) \in \Omega$ and

$$y_1 = \varphi_1(x_1, ..., x_n); ....; y_n = \varphi_n(x_1, ..., x_n).$$

If $u \in WX_s^m(\Omega)$, then for $\forall |p| = \overline{0,m}$ the derivatives $\partial^p v$ can be represented as

$$\frac{\partial^p \nu}{\partial x^p} = \sum_{|q| \le |p|} \left(\sum_{i,j,q} a_{i,j,q} \frac{\partial^j \varphi_i}{\partial x^j} \right) \frac{\partial^q u(\varphi(\cdot))}{\partial y^q}.$$

Under conditions of the theorem, the inclusion $\sum_{i,j,k} a_{i,j,k} \frac{\partial^j \varphi_i}{\partial x^j} \in L_\infty$ holds. Taking into account Lemma 3.2 and Theorem 3.1, we obtain

$$\left\| \frac{\partial^p v}{\partial x^p} \right\|_{X(D)} \le C_1 \sum_{|q| \le |p|} \left\| \frac{\partial^q u}{\partial y^q} \right\|_{X(\Omega)} \Rightarrow \|v\|_{WX_s^m(D)} \le C_2 \|u\|_{WX_s^m(\Omega)},$$

where the constants C_1 and C_2 depend only on φ.

The left-hand side of (4.4) follows from this fact, if to consider φ^{-1}.

The theorem is proved. □

We say that a bounded domain Ω belongs to a class C^l, if there exists a covering of this domain by finite number of open sets $\{\Omega_i\}_{i=1}^k$, such that if the intersection $\Omega_i \cap \partial\Omega \ne \emptyset$, then there exists a diffeomorphism of the class C^l, which maps Ω_i on B, $\Omega_i \cap \Omega$ on B_+ and $\Omega_i \cap \partial\Omega$ on $\partial B_+ \setminus \{|x| = 1\}$, where

$$B = B(1) = \{x \in R^n : |x| = 1\}$$

$$B_+ = \{x \in B : x_n > 0\}$$

$$B_- = \{x \in B : x_n < 0\}.$$

Theorem 4.3. *Let* $\Omega : \partial\Omega \in C^1$ *be a bounded domain and* $\overline{\Omega} \subset \Omega_1$. *Then there is a bounded extension (continuation) operator* θ *acting from* $WX_s^1(\Omega)$ *to* $\overset{0}{WX_s^1}(\Omega_1)$ *such that*

$$\forall u \in WX_s^1(\Omega), \forall x \in \Omega \Rightarrow (\theta u)(x) = u(x),$$

and

$$\exists c > 0 \, \forall u \in WX_s^1(\Omega) \Rightarrow \|\theta u\|_{\overset{0}{WX_s^1}(\Omega_1)} \leq c \|u\|_{WX^1(\Omega)}. \tag{4.5}$$

Proof. As in classical case, this theorem will be proved in three steps. Let us carry out the proof according to the scheme of [29, pp. 151-153]

1. Let Ω be a semi-ball $B_+ = \{x : |x| < 1, \ x_n > 0\}$, and $u \in WX^1(B_+)$ be equal to zero near $S_+ = \{\partial B_+ : \ |x| = 1\}$. We extend this function on semi-ball $B_- = \{x : |x| < 1, \ x_n < 0\}$ by putting

$$v(x) = \begin{cases} u(x), & x \in B_+, \\ u(x_{n'}, -x_n), & x \in B_-. \end{cases}$$

It is well known that the function defined this way belongs to $W_1^1(B(1))$. On the other hand, by Remark 3.1,

$$\|v\chi_{B_+}\|_{WX_s^1(K)} = \|v\chi_{B_-}\|_{WX_s^1(K)} = \|u\|_{WX_s^1(B_-)} \Rightarrow$$

$$\Rightarrow \|v\|_{WX_s^1(B(1))} = \|v\chi_{B_+} + v\chi_{B_-}\|_{W_s^1(B(1))} \leq 2\|u\|_{WX_s^1(B_+)}.$$

2. Let the function $u \in WX_s^1(\Omega)$ be supported in the neighborhood U of some boundary point $s \in \partial\Omega$ and $\overline{U} \subset \Omega_1$, $\overline{\Omega} \subset \Omega_1$. Suppose that f is some C^1-diffeomorphism, which transforms U to B, and $f(\Omega \bigcap U) = B_+$ & $f(U \bigcap \partial\Omega) = \partial B_+ \backslash S_+$. It is evident that $\tilde{u} = u \circ (f^{-1}) \in WX_s^1(B_+)$ and it equals to zero in the neighborhood of S_+. Extend this function on all ball by means of the above mentioned method and denote new function by $\tilde{v}$. As stated above, it follows that $\tilde{v} \in \overset{0}{WX_s^1}(B)$, because $v = \tilde{v} \circ f \in \overset{0}{WX_s^1}(U)$. The function v may be extended by zero on $\Omega_1 \backslash U$. It is clear that $v \in \overset{0}{WX_s^1}(\Omega_1)$, coincides with u on Ω and the following inequality

$$\|v\|_{WX_s^1(\Omega_1)} \leq c\|u\|_{\overset{0}{WX_s^1}(\Omega)}$$

holds, where the constant c depends only on the norm f and f^{-1} in C^1.

3. General case. Since Ω is a bounded domain, it may be covered by a finite number of open sets $U_1, U_2, ..., U_k$, each of which is a subset of Ω with its closure, or is a neighborhood of boundary points, as in the second case, and $\bigcup_{i=\overline{1,k}} U_i = \Omega_1 \supset \overline{\Omega}$. Let $\{\varepsilon_i\}_1^m$ be the corresponding partition of unity of Ω_1 such that

$$\forall i \Rightarrow \varepsilon_i \in C_0^\infty(R^n), \, \text{supp}\varepsilon_i \subset U_i, \sum_1^k \varepsilon_i(x)|_{\overline{\Omega}} = 1,$$

$$u = \sum_1^k u\varepsilon_i = \sum_1^k u_i, \ \ (where \ u_i = u\varepsilon_i).$$

If $\overline{U_i} \subset \Omega \Rightarrow u_i \in \overset{0}{WX_s^1}(\Omega)$ and u_i may be extended by zero on all Ω_1. This extended function is denoted by v_i. If $U_i \bigcap \partial\Omega \neq \emptyset$, then we may apply secondary method, i.e.

$$v_i \in \overset{0}{WX_s^1}(\Omega_1): \;\; v_i(x)|_\Omega = u(x) \;\; \& \;\; \|v_i\|_{WX_s^1(\Omega_1)} \le \|u_i\|_{\overset{0}{WX_s^1}(\Omega)}.$$

Thus $v = \sum_1^k v_i \in WX_s^1(\Omega_1)$. As a result, we have

$$\|v\|_{WX_s^1(\Omega_1)} \le c\,\|u\|_{WX_s^1(\Omega)},$$

where c depends only on Ω and Ω_1.

The theorem is proved. □

Theorem 4.4. *Let Ω be a C^m-class bounded domain and $\overline{\Omega} \subset \Omega_1$. Then there exists a bounded extension (continuation) operator θ acting from $WX_s^m(\Omega)$ to $\overset{0}{WX_s^m}(\Omega_1)$ such that*

$$u \in WX_s^m(\Omega),\; v = \theta u \Rightarrow (\forall x \in \Omega \Rightarrow v(x) = u(x)),$$

and

$$\exists c > 0 \Rightarrow \|v\|_{WX_s^m(\Omega_1)} \le c\,\|v\|_{WX_s^m(\Omega)}, \tag{4.6}$$

where c is independent of $u(\cdot)$.

Proof. In this case, the function $u \in C^\infty\left(\overline{B_+}\right)$ similar to the classic case, is continued across $x_n = 0$ in the following way:

$$v(x) = \begin{cases} u(x), & x \in B_+, \\ \sum_0^m c_j u\left(x_n', -2^{-j}x_n\right), & x \in B_-, \end{cases}$$

$$\frac{\partial^k v\left(x_n', +0\right)}{\partial x_n^k} = \frac{\partial^k v\left(x_n', -0\right)}{\partial x_n^k}, \quad k = 0, 1, ..., m.$$

For the correctness of this function, it is sufficient to solve the following system of liner equations:

$$\sum_{j=0}^m c_j\left(-2^{-j}\right)^k = 1, \;\; k = 0, ..., m.$$

□

Corollary 4.1. *If there exists the bounded extension operator from $WX_s^m(\Omega)$ to $\overset{0}{WX_s^m}(\Omega_1)$, then $\overline{C^\infty\left(\overline{\Omega}\right)} = WX_s^m(\Omega)$ in topology of $WX_s^m(\Omega)$.*

Corollary 4.2. *If there exists the bounded extension operator from $WX_s^m(\Omega)$ to $WX_s^m(\Omega_1)$, then there exists a bounded extension operator from $WX_s^m(\Omega)$ to $\overset{0}{WX_s^m}(\Omega_1)$.*

Proof. Under these conditions, there is a domain Ω' with sufficiently smooth boundary, such that

$$\Omega \subset \Omega' \subset \Omega_1, \;\; \overline{\Omega} \subset \Omega', \;\; \overline{\Omega'} \subset \Omega_1.$$

The restriction of the extension operator θ from $WX_s^m(\Omega)$ to $WX_s^m(\Omega_1)$, defined by $\theta_1 = \chi_{\Omega'}\theta$, is an extension operator from $WX_s^m(\Omega)$ to $WX_s^m(\Omega')$. Then by

Theorem 4.4, there exists the extension operator θ' from $WX_s^m(\Omega')$ to $\overset{0}{WX_s^m}(\Omega_1)$. Finally, the composition $\theta'\theta_1$ will be a bounded extension operator from $WX_s^m(\Omega)$ to $\overset{0}{WX_s^m}(\Omega_1)$.

The corollary is proved. □

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