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TO THE INVERSE SPECTRAL PROBLEM FOR A PERTURBED OSCILLATOR ON THE SEMIAXIS

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Abstract. The perturbed oscillator $T=-\frac{d^2}{dx^2}+x^2+q\left(x\right)$ on the half-axis $0\leq x<\infty$ with the Dirichlet boundary condition is considered. By the method of transformation operators, we study the spectral problem; deduce the main integral equation for the inverse problem. We give rigorous derivation of the main integral equation of the inverse problem and substantiate the proof. We also propose an effective algorithm of reconstruction of the perturbation potential.

1. Introduction

Inverse spectral problems for Schrodinger operators defined on an infinite interval in various contexts were studied in the works of many authors (see [11], [13] and the references therein). The main research apparatus in these works is the transformation operator method. McKean and Trubowitz [15] and Poschel and Trubowitz [16] have used a different approach to inverse problems, based on the smoothness of the mapping that translates the potential into spectral data and on an explicit reconstruction procedure for the special case where only one spectral parameter is changed.

Of particular interest is also the Schrodinger equation with a quadratic potential, which is a problem about a quantum oscillator and arises in the description of vibrational motions of atoms in molecules and crystals (see [3]). In this direction, we note the McKean and Trubowitz's work [21], in which the inverse spectral problem for perturbed oscillators

$$T\left(q\right)=-\frac{d^2}{dx^2}+x^2+q\left(x\right),\ -\infty<x<\infty,$$

having the same spectrum, with potentials $q(x)$ from the class of sufficiently smooth and rapidly decreasing functions at $\pm\infty$, was considered. The latter problem was also investigated in Levitan's work [12]. Some uniqueness theorems were proved by Gesztesy, Simon [7] and Chelkak, Kargaev, Korotyaev [4]. Chelkak, Kargaev, Korotyaev [5] obtained the characterization and described the isospectral set for the case on the real line for $q\in H=\{q: x_q\in L_2(-\infty,\infty), q'\in L_2(-\infty,\infty)\}$.

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In the work [6], the Trubowitz approach was used in solving the inverse problem for operator $T(q)$ with Dirichlet boundary conditions. In this paper, it is proved that the mapping that transforms a potential into spectral data is one-to-one and onto. Moreover, the complete characterization of the spectral data which corresponds to the class of potentials $H_+ = \{q : xq \in L_2(0,\infty), q' \in L_2(0,\infty)\}$ is given.

Recently, papers [2], [8] have appeared, in which it is shown that the method of transformation operators is effectively applied to inverse spectral problems for operators $T(q)$ with Dirichlet and Neumann boundary conditions. Moreover, unlike in the work [6], using the formalism of the Gel'fand-Levitan-Marchenko integral equations an effective algorithm for the reconstruction of the potential $q(x)$ is indicated.

However, in papers [2],[8] the derivation of the main integral equations of Marchenko-type is based on symbolic equalities related to the Dirac delta function. In this regard, the question arises of a rigorous substantiation of the derivation of the main integral equation.

Consider a boundary-value problem generated on the semiaxis $0 \le x < \infty$ by a differential equation

$$-y'' + x^2 y + q(x)\, y = \lambda y, \quad \lambda \in C, \tag{1.1}$$

with a boundary condition

$$y(0) = 0 \tag{1.2}$$

in the case where the function $q(x)$ is real and satisfies the conditions

$$q(x) \in C^{(1)}[0,\infty), \sigma_j(0) = \int_0^\infty x^j |q(x)|\, dx < \infty, \; 0 \le j \le 5, \tag{1.3}$$

where $\sigma_j(x) = \int_x^\infty t^j |q(t)|\, dt$, $x \ge 0$,which are assumed to hold throughout the work.

In the present paper, by the method of transformation operators, we study the inverse spectral problem for the boundary-value problem (1.1), (1.2) in the class of potentials (1.3). We obtain a Marchenko -type integral equation for problem (1.1)–(1.2). A rigorous derivation of this equation is given. An efficient algorithm is proposed for the reconstruction of the potential $q(x)$.

2. Preliminary information

We consider the unperturbed equation

$$-y'' + x^2 y = \lambda y, \; 0 \le x < \infty, \quad \lambda \in C. \tag{2.1}$$

It is known (see [1], [4]) that equation (2.1) has a solution $f_0(x,\lambda)$ that can be represented as $f_0(x,\lambda) = D_{\frac{\lambda-1}{2}}\left(\sqrt{2}x\right)$, which satisfies the initial conditions $f_0(0,\lambda) = 2^{\frac{\lambda-1}{4}} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3-\lambda}{4}\right)}$, $f_0'(0,\lambda) = 2^{\frac{\lambda-1}{4}} \frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(\frac{1-\lambda}{4}\right)}$, where $D_\nu(x)$ is the Weber function and $\Gamma(\cdot)$ is the Gamma function. The spectrum of problem (2.1)–(1.2) consists of simple eigenvalues $\hat{\lambda}_n = 4n+3$, $n = 0,1,...$, which are the zeros of the function $f_0(0,\lambda)$. The following equality holds:

$$f_0\left(x, \hat{\lambda}_n\right) = D_{2n+1}\left(\sqrt{2}x\right) = 2^{-\left(n+\frac{1}{2}\right)} e^{-\frac{x^2}{2}} H_{2n+1}(x),$$

where $H_n(x)$ is the Hermite polynomial. From the well-known properties of Hermite polynomials it follows that $\hat{\alpha}_n^2 \overset{def}{=} \int_0^\infty \left| f_0\left(x, \hat{\lambda}_n\right)\right|^2 dx = (2n+1)!\frac{\sqrt{\pi}}{2}$. Moreover, the system of functions $\left\{\frac{f_0\left(x,\hat{\lambda}_n\right)}{\hat{\alpha}_n}\right\}_{n=0}^\infty$ serves as an orthonormal basis in the space $L_2(0,\infty)$, i.e. the following equality holds

$$\sum_{n=0}^\infty \frac{f_0\left(x,\hat{\lambda}_n\right)}{\hat{\alpha}_n}\frac{f_0\left(y,\hat{\lambda}_n\right)}{\hat{\alpha}_n} = \delta(x-y), \tag{2.2}$$

where $\delta$is the Dirac delta function .

Now we introduce the solution $f(x,\lambda)$ of the perturbed equation (1.1) with asymptotics $f(x,\lambda) = f_0(x,\lambda)(1+o(1))$, $x\to\infty$. It follows from [14] that under the conditions (1.3) such solution exists and admits the following representation by means of a transformation operator:

$$f(x,\lambda) = f_0(x,\lambda) + \int_x^\infty K(x,t)\ f_0(t,\lambda)\,dt, \tag{2.3}$$

where the kernel $K(x,t)$ is a continuous function and satisfies the following relations:

$$|K(x,t)| \le C\sigma_0\left(\frac{x+t}{2}\right), \tag{2.4}$$

$$K(x,x) = \frac{1}{2}\int_x^\infty q\ (t)\ dt, \tag{2.5}$$

Here and everywhere below, one and the same symbol C denotes various constants which do not depend on x,λ and n. By using the well-known properties of transformation operators (see, e.g., [13]) and (2.3), we find

$$f_0(x,\lambda) = f(x,\lambda) + \int_x^\infty \hat{K}(x,t)f(t,\lambda)dt, \tag{2.6}$$

where the kernel $\hat{K}(x,y)$ satisfies the equation

$$\hat{K}(x,y) + K(x,y) + \int_x^y \hat{K}(x,t)\,K(t,y)\,dt = 0. \tag{2.7}$$

It follows from the last equation and (2.4) that

$$\left|\hat{K}(x,y)\right| \le C\sigma_0\left(\frac{x+y}{2}\right). \tag{2.8}$$

Let us now return to problem (1.1) - (1.2). It follows from [17] that under condition (1.3) the spectrum of this problem consists of simple eigenvalues and has the asymptotic expansion $\lambda_n = \hat{\lambda}_n + O\left(n^{-\frac{1}{2}}\right)$,$n\to\infty$. Moreover, the spectrum of problem (1.1), (1.2) coincides with the set of roots of the function $f(0,\lambda)$, i.e., the equalities $f(0,\lambda_n) = 0$, $n = 0,1,2,...$, are true.

Obviously, differential equation (1.1) together with boundary condition (1.2) defines in space $L_2(0,\infty)$ a self-adjoint operator, which can be obtained by closure the symmetric operator defined by equation (1.1) and boundary condition (1.2) on twice continuously differentiable finite functions. Therefore, the eigenfunctions

$\left\{\frac{f(x,\lambda_n)}{\alpha_n}\right\}_{n=0}^{\infty}$ of problem (1.1), (1.2), where $\alpha_n = \sqrt{\int\limits_0^{\infty} |f(x,\lambda_n)|^2 dx}$, form an orthonormal basis in the space $L_2(0,\infty)$, i.e., the following relation is true:

$$\sum_{n=0}^{\infty} \frac{f(x,\lambda_n)}{\alpha_n} \frac{f(y,\lambda_n)}{\alpha_n} = \delta(x-y). \tag{2.9}$$

The set of quantities $\{\lambda_n, \alpha_n > 0\}_{n=0}^{\infty}$ is called the spectral data of problem (1.1), (1.2). The inverse spectral problem for problem (1.1), (1.2) is to reconstruct the potential $q(x)$ of equation (1.1) from the spectral data.

3. Derivation of the main integral equation of the inverse problem

An important role in the solution of the inverse problem is played by the Marchenko-type integral equation. We put

$$\Phi_N(x,y) = \sum_{n=0}^{N} \frac{f(x,\lambda_n) f(y,\lambda_n)}{\alpha_n^2} - \sum_{n=0}^{N} \frac{f_0\left(x,\hat{\lambda}_n\right) f_0\left(y,\hat{\lambda}_n\right)}{\hat{\alpha}_n^2} \tag{3.1}$$

$$F_N(x,y) = \sum_{n=0}^{N} \frac{f_0(x,\lambda_n) f_0(y,\lambda_n)}{\alpha_n^2} - \sum_{n=0}^{N} \frac{f_0\left(x,\hat{\lambda}_n\right) f_0\left(y,\hat{\lambda}_n\right)}{\hat{\alpha}_n^2}. \tag{3.2}$$

From the results of papers [8]-[10] it follows that the sequences $\Phi_N(x,y)$ and $F_N(x,y)$ is uniformly convergent in each finite range of the variables x and y. Put $\Phi(x,y) = \lim\limits_{N\to\infty} \Phi_N(x,y)$. Fix x and denote the smooth function with bounded support contained in the interval (x,∞) by $g(y)$. By using (2.2), (2.9), (3.1) we find that the sequence $\int\limits_x^{\infty} \Phi_N(x,y)\, g(y)\, dy$ converges to zero in quadratic mean. In our further considerations we will for brevity write this equality in the form

$$\underset{N\to\infty}{l\cdot i\cdot m\cdot}\ (L_2(x,\infty)) \int_x^{\infty} \Phi_N(x,y)\, g(y)\, dy = 0.$$

Since $g(y)$ is a function with bounded support, integration in the integral $\int\limits_x^{\infty} \Phi_N(x,y)\, g(y)\, dy$ is, in fact, between finite limits. Passing to the limit ($N \to \infty$), we obtain

$$\lim_{N\to\infty} \int_x^{\infty} \Phi_N(x,y)\, g(y)\, dy = \int_x^{\infty} \Phi(x,y)\, g(y)\, dy = 0.$$

Thus, we have proved the identity

$$\int_x^{\infty} \Phi(x,y)\, g(y)\, dy = 0,$$

in which $g(y)$ is an arbitrary function with bounded support. Therefore,

$$\Phi(x,y) = \lim_{N\to\infty} \Phi_N(x,y) = 0.$$

Let

$$F(x,y)=\sum_{n=0}^{\infty}\left\{\frac{f_0(x,\lambda_n)\,f_0(y,\lambda_n)}{\alpha_n^2}-\frac{f_0\left(x,\hat{\lambda}_n\right)f_0\left(y,\hat{\lambda}_n\right)}{\hat{\alpha}_n^2}\right\}. \tag{3.3}$$

Since the sequence $F_N(x,y)$ is uniformly convergent in each finite region of variation of the variables x and y, it follows from (3.2) that it is continuous in the entire plane (x,y).

Theorem 3.1. *For any fixed $x\geq 0$, the function $K(x,y)$ from representation* (2.3) *satisfies the integral equation*

$$F(x,y)+K(x,y)+\int_x^{\infty}K(x,t)\,F(t,y)\,dt=0,\ \ y>x. \tag{3.4}$$

Proof. Consider representation (2.6). For $y>x$, it follows from (3.1) that

$$\sum_{n=0}^{N}\frac{f(x,\lambda_n)}{\alpha_n}\frac{f_0(y,\lambda_n)}{\alpha_n}=\sum_{n=0}^{N}\frac{f(x,\lambda_n)}{\alpha_n}\frac{f(y,\lambda_n)}{\alpha_n}+$$

$$+\int_y^{\infty}\hat{K}(y,t)\left\{\sum_{n=0}^{N}\frac{f(x,\lambda_n)}{\alpha_n}\frac{f(t,\lambda_n)}{\alpha_n}\right\}dt=$$

$$=\sum_{n=0}^{N}\frac{f_0\left(x,\hat{\lambda}_n\right)}{\hat{\alpha}_n}\frac{f_0\left(y,\hat{\lambda}_n\right)}{\hat{\alpha}_n}+\Phi_N(x,y)+$$

$$+\int_y^{\infty}\hat{K}(y,t)\left\{\sum_{n=0}^{N}\frac{f(x,\lambda_n)}{\alpha_n}\frac{f(t,\lambda_n)}{\alpha_n}\right\}dt.$$

Further, using formulas (2.3) and (2.2), we obtain

$$\sum_{n=0}^{N}\frac{f(x,\lambda_n)}{\alpha_n}\frac{f_0(y,\lambda_n)}{\alpha_n}=\sum_{n=0}^{N}\frac{f_0(x,\lambda_n)}{\alpha_n}\frac{f_0(y,\lambda_n)}{\alpha_n}+$$

$$+\int_x^{\infty}K(x,t)\left\{\sum_{n=0}^{N}\frac{f_0(t,\lambda_n)}{\alpha_n}\frac{f_0(y,\lambda_n)}{\alpha_n}\right\}dt=\sum_{n=0}^{N}\frac{f_0\left(x,\hat{\lambda}_n\right)}{\hat{\alpha}_n}\frac{f_0\left(y,\hat{\lambda}_n\right)}{\hat{\alpha}_n}+$$

$$+\sum_{n=0}^{N}\left\{\frac{f_0(x,\lambda_n)}{\alpha_n}\frac{f_0(y,\lambda_n)}{\alpha_n}-\frac{f_0\left(x,\hat{\lambda}_n\right)}{\hat{\alpha}_n}\frac{f_0\left(y,\hat{\lambda}_n\right)}{\hat{\alpha}_n}\right\}+$$

$$+\int_x^{\infty}K(x,t)\left\{\sum_{n=0}^{N}\frac{f_0\left(t,\hat{\lambda}_n\right)}{\hat{\alpha}_n}\frac{f_0\left(y,\hat{\lambda}_n\right)}{\hat{\alpha}_n}\right\}dt+$$

$$+\int_x^{\infty}K(x,t)\left\{\sum_{n=0}^{N}\left\{\frac{f_0(t,\lambda_n)}{\alpha_n}\frac{f_0(y,\lambda_n)}{\alpha_n}-\frac{f_0\left(t,\hat{\lambda}_n\right)}{\hat{\alpha}_n}\frac{f_0\left(y,\hat{\lambda}_n\right)}{\hat{\alpha}_n}\right\}\right\}dt=$$

$$=\sum_{n=0}^{N}\frac{f_0\left(x,\hat{\lambda}_n\right)}{\hat{\alpha}_n}\frac{f_0\left(y,\hat{\lambda}_n\right)}{\hat{\alpha}_n}+F_N(x,y)+$$

$$+\int_x^\infty K(x,t)\left\{\sum_{n=0}^N \frac{f_0\left(t,\hat\lambda_n\right)}{\hat\alpha_n}\frac{f_0\left(y,\hat\lambda_n\right)}{\hat\alpha_n}\right\}dt+\int_x^\infty K(x,t)\,F_N(t,y)\,dt.$$

Comparing the last two equalities, we get

$$\Phi_N(x,y)+\int_y^\infty \hat K(y,t)\left\{\sum_{n=0}^N \frac{f(x,\lambda_n)}{\alpha_n}\frac{f(t,\lambda_n)}{\alpha_n}\right\}dt=$$

$$=\int_x^\infty K(x,t)\left\{\sum_{n=0}^N \frac{f_0\left(t,\hat\lambda_n\right)}{\hat\alpha_n}\frac{f_0\left(y,\hat\lambda_n\right)}{\hat\alpha_n}\right\}dt+$$

$$+\int_x^\infty K(x,t)\,F_N(t,y)\,dt+F_N(x,y)\,.$$

Multiplying both sides of the latter identity by $g(y)$, integrating with respect to y, gives

$$\int_x^\infty \Phi_N(x,y)\,g(y)\,dy+$$

$$+\int_x^\infty\left[\sum_{n=0}^N\left(\int_y^\infty \hat K(y,t)\,\frac{f(t,\lambda_n)}{\alpha_n}dt\right)\frac{f(x,\lambda_n)}{\alpha_n}\right]g(y)\,dy=$$

$$=\int_x^\infty F_N(x,y)\,g(y)\,dy+$$

$$+\int_x^\infty\left[\sum_{n=0}^N\left(\int_y^\infty K(x,t)\,\frac{f_0\left(t,\hat\lambda_n\right)}{\hat\alpha_n}dt\right)\frac{f_0\left(y,\hat\lambda_n\right)}{\hat\alpha_n}\right]g(y)\,dy+$$

$$+\int_x^\infty\left[\int_x^\infty K(x,t)\,F_N(t,y)\,dt\right]g(y)\,dy.$$

Passing to the limit ($N\to\infty$), for each smooth function $g(y)$ with bounded support, we obtain

$$\int_x^\infty \Phi(x,y)\,g(y)\,dy+\int_x^\infty \hat K(y,x)\,g(y)\,dy=$$

$$=\int_x^\infty F(x,y)\,g(y)\,dy+\int_x^\infty K(x,y)\,g(y)\,dy+$$

$$+\lim_{N\to\infty}\int_x^\infty\left[\int_x^\infty K(x,t)\,F_N(t,y)\,dt\right]g(y)\,dy.$$

Since $y>x$, then $\hat K(y,x)=0$. Moreover, it shall be shown above, the identity $\Phi(x,y)=0$ is true. Therefore, the relation

$$\int_x^\infty F(x,y)\,g(y)\,dy+\int_x^\infty K(x,y)\,g(y)\,dy+$$

$$+\lim_{N\to\infty}\int_x^\infty\left[\int_x^\infty K(x,t)\,F_N(t,y)\,dt\right]g(y)\,dy=0 \tag{3.5}$$

holds. We now show that

$$\lim_{N\to\infty}\int_x^\infty\left[\int_x^\infty K(x,t)\,F_N(t,y)\,dt\right]g(y)\,dy=$$

$$= \int_x^\infty \left[\int_x^\infty K(x,t) F(t,y)\, dt \right] g(y)\, dy. \tag{3.6}$$

Note that for each $b > x$, uniformly with respect to $y \in (x, b)$, the following equality holds:

$$\lim_{N\to\infty} \int_x^b K(x,t) F_N(t,y)\, dt = \int_x^b K(x,t) F(t,y)\, dt. \tag{3.7}$$

Further, it follows from the definition of the function $F_N(x,y)$ and (2.2), (2.9) that the sequence $\int_x^\infty K(x,t) F_N(t,y)\, dt$ converges to limit $K_0(x,y) = \left(I + \hat{K}\right)\left(I + \hat{K}^*\right) K(x,y) - K(x,y)$ in quadratic mean, where I is the unit operator, and the operator $\hat{K}$ is defined by the formula $\hat{K}h(y) = \int_y^\infty \hat{K}(y,s) h(s)\, ds$. From the last relations we find that

$$K_0(x,y) = \int_y^\infty \hat{K}(y,t) K(x,t)\, dt + \int_x^y \hat{K}(s,y) K(x,s)\, ds +$$

$$+ \int_y^\infty \hat{K}(y,t) \int_x^t \hat{K}(s,t) K(x,s)\, ds dt. \tag{3.8}$$

If we consider the function

$$K_b(x,t) = \begin{cases} K(x,t), x \le t \le b, \\ 0, t > b, \end{cases}$$

then it can be shown similarly, that the equality

$$\underset{N\to\infty}{l \cdot i \cdot m \cdot} (L_2(x,b)) \int_x^b K(x,t) F_N(t,y)\, dt = G(y,b),$$

is valid, where

$$G(y,b) = \int_y^b \hat{K}(y,t) K(x,t)\, dt + \int_x^y \hat{K}(s,y) K(x,s)\, ds +$$
$$+ \int_y^b \hat{K}(y,t) \int_x^t \hat{K}(s,t) K(x,s)\, ds dt + \int_b^\infty \hat{K}(y,t) \int_x^b \hat{K}(s,t) K(x,s)\, ds dt,$$

when $x \le y \le b$.

Further, using (2.4), (2.8), (3.7), we obtain

$$G(y,b) \to K_0(x,y), b \to \infty.$$

Moreover, the last relation is true uniformly with respect to y. Indeed, due to formulas (2.4), (2.8), (3.7)

$$|G(y,b) - K_0(x,y)| \le C\sigma_0(b) \to 0, b \to \infty.$$

On other hand, taking into account (3.7), we obtain

$$\int_x^b K(x,t) F(t,y)\, dt = G(y,b)$$

for $y \le b$. Therefore, we get

$$\lim_{b\to\infty} \int_x^b K(x,t) F(t,y)\, dt = K_0(x,y)$$

and this equality is true uniformly with respect to y taken from each finite interval (x,a). Thus, we have proved that the improper integral $\int\limits_x^\infty K(x,t)\,F(t,y)\,dt$ converges and the equality

$$\int_x^\infty K(x,t)\,F(t,y)\,dt = K_0(x,y)$$

is true. Taking into account that $\underset{N\to\infty}{l\cdot i\cdot m}\cdot\ (L_2(x,\infty))\int_x^\infty K(x,t)\,F_N(t,y)\,dt = K_0(x,y)$, we have

$$\underset{N\to\infty}{l\cdot i\cdot m}\cdot\ (L_2(x,\infty))\int_x^\infty K(x,t)\,F_N(t,y)\,dt = \int_x^\infty K(x,t)\,F(t,y)\,dt.$$

The last equality implies (3.6). Since $g(y)$ is arbitrary function with bounded support, from (3.5) we finally obtain equation (3.4).

The theorem is proved. □

Equation (3.4) is called the main integral equation of the Marchenko- type. This equation allows us to formally solve the inverse problem. Indeed, let the spectral data $\{\lambda_n, \alpha_n > 0\}_{n=0}^\infty$ be given. Then calculate the function $F(x,y)$ by (3.3). Find $K(x,y)$ by solving the main equation (3.4). Construct $q(x)$ by (2.5).

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