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ON SOME ASYMPTOTICALLY HALF-LINEAR EIGENVALUE PROBLEM FOR ORDINARY DIFFERENTIAL EQUATIONS OF FOURTH ORDER

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Abstract. In this paper, we consider a half-linearizable at infinity eigenvalue problem for ordinary differential equations of fourth order. We prove the existence of four families of global continua of nontrivial solutions of this problem in $\mathbb{R} \times C^3$ emanating from the points in $\mathbb{R} \times \{\infty\}$ and possessing the usual nodal properties in some neighborhoods of these points. Moreover, we will demonstrate the existence of nodal solutions of some boundary value problems that are half-linearizable at zero and infinity.

1. Introduction

We consider the following nonlinear eigenvalue problem

$$\begin{gathered}\ell y \equiv (p\,(x)\,y'')'' - (q(x)y')' + r(x)y = \lambda\tau(x)\,y + \\ \alpha(x)y^+(x) + \beta(x)y^-(x) + g(x, y, y', y'', y''', \lambda),\;\; x \in (0, l),\end{gathered} \tag{1.1}$$

$$y(0) = y'(0) = y(l) = y'(l) = 0, \tag{1.2}$$

where $\lambda \in \mathbb{R}$ is a spectral parameter, p is a twice continuously differentiable positive function on $[0, l]$, q is a continuously differentiable non-negative function on $[0, l]$, r is a continuous real-valued function on $[0, l]$, τ is a continuous positive function on $[0, l]$, α, β are continuous real-valued functions on $[0, l]$, $y^+ = \max\{y, 0\}$, $y^- = (-y)^+$. The nonlinear term g is a continuous real-valued function on $[0, l] \times \mathbb{R}^5$ and satisfies the following condition:

$$g(x, y, s, v, w, \lambda) = o(|y| + |s| + |v| + |w|) \;\text{ at }\; (y, s, v, w) = \infty \tag{1.3}$$

uniformly in $x \in [0, l]$ and in $\lambda \in \Lambda$, for any bounded interval $\Lambda \subset \mathbb{R}$.

Half-linear and half-linearizable Sturm-Liouville problems were first investigated by H. Berestycki [5]. He showed the existence of two sequences of half-eigenvalues of the half-linear Sturm-Liouville problem, corresponding to the usual nodal properties, but differing in sign of the eigenfunctions in the neighborhood of 0. Moreover, in [5], the author also proves that for a half-linearizable problem

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having different linearizations for $y \to 0^+$ and $y \to 0^-$, these half-eigenvalues correspond to bifurcation points in a global sense. The global bifurcation from infinity of nontrivial solutions to the asymptotically half-linear Sturm-Liouville problem was studied in [8], where for different asymptotic linearizations at $y = \pm\infty$ it is proved the existence of global continua of solutions which have the usual nodal properties in some neighborhoods of asymptotic bifurcation points. In [6], the authors showed the existence of nodal solutions of Sturm-Liouville boundary value problem (without potential) that are half-linearizable at zero and infinity.

Half-linear problem with jumping nonlinearity for a $2m$th-order, self-adjoint, disconjugate ordinary differential operator, together with appropriate boundary conditions was considered in [9]. In this paper the author shows that a sequence of half-eigenvalues to a half-linear eigenvalue problem exists, with certain properties, and proves various results regarding the existence and multiplicity of solutions of a half-linear boundary value problem. It should be noted that these results depend strongly on the location of the half-eigenvalues relative to the point $\lambda = 0$.

The present paper is devoted to the study of global bifurcation from infinity of nontrivial solutions of problem (1.1), (1.2).

The structure of this paper is as follows. Section 2 presents the classes of fixed oscillation count constructed in [2, § 3] and auxiliary results for the corresponding half-linear eigenvalue problem (1.1), (1.2) with $g \equiv 0$. In Section 3, we find the structure of asymptotic bifurcation points of problem (1.1), (1.2) with respect to the classes with fixed oscillation count, and using [3, Theorem 5.9], we establish a global bifurcation theorem for this problem. In Section 4, by applying this theorem, we prove the existence of nodal solutions for some boundary value problem that is asymptotically half-linearizable at zero and infinity.

2. Preliminary

Denote by $(b.c.)$ the set of differentiable functions on $[0, l]$ satisfying the boundary conditions (1.2).

Let E be the Banach space $C^3[0, l] \cap (b.c.)$ equipped with usual norm $||u||_3 = ||u||_\infty + ||u'||_\infty + ||u''||_\infty + ||u'''||_\infty$, where $||u||_\infty = \max\limits_{x\in[0,\, l]} |u(x)|$

A pair $(\lambda, y) \in \mathbb{R} \times C^4[0, l]$ satisfying (1.1), (1.2) is called a solution of problem (1.1), (1.2).

The Green's function of the differential expression $(p\,(x)\,y'')'' - (q(x)y')'$ together with boundary conditions (1.2) can be used to convert (1.1), (1.2) to an equivalent equation in $\mathbb{R} \times E$ (see [2, § 3.3]). Thus we may consider the structure of the set of solutions of problem (1.1), (1.2) in the space $\mathbb{R} \times E$.

In this section we introduce subsets of E with fixed oscillation count, the construction of which is presented in [2, § 3.1] under more general boundary conditions.

Let's introduce the notation: $Ty \equiv (py'')' - qy'$.

By S we denote the subset of E defined as $S = S_1 \cup S_2$, where

$$S_1 = \{u \in E : u^{(i)}(x) \neq 0, Tu(x) \neq 0,\, x \in (0, l),\, i = 0,\, 1,\, 2\,\}$$

and

$S_2 = \{u \in E:$ there exists $i_0 \in \{0, 1, 2\}$ and $x_0 \in (0, l)$ such that $u^{(i_0)}(x_0) = 0$, or $Tu(x_0) = 0$ and if $u(x_0)u''(x_0) = 0$, then $u'(x)Tu(x) < 0$ in a neighborhood of x_0, and if $u'(x_0)Tu(x_0) = 0$, then $u(x)u''(x) < 0$ in a neighborhood of $x_0\}$.

It follows from the definition of the set S that if $u \in S$, then the Jacobian $J = \rho^3 \cos\psi \sin\psi$ of the Prüfer-type transformation

$$\begin{cases} y(x) = \rho(x) \sin\psi(x) \cos\theta(x), \\ y'(x) = \rho(x) \cos\psi(x) \sin\varphi(x), \\ (py'')(x) = \rho(x) \cos\psi(x) \cos\varphi(x), \\ Ty(x) = \rho(x) \sin\psi(x) \sin\theta(x), \end{cases} \tag{2.1}$$

does not vanish in $(0, l)$ (see [2, 4]).

For every $y \in S$ we define $\rho(y, x)$, $\theta(y, x)$, $\varphi(y, x)$ and $w(y, x)$ to be the continuous functions on $[0, l]$ satisfying

$$\rho(y, x) = y^2(x) + y'^2(x) + (p(x)y''(x))^2 + (Ty(x))^2,$$

$$\theta(y, x) = \operatorname{arctg} \frac{Ty(x)}{y(x)}, \ \theta(y, 0) = -\pi/2,$$

$$\varphi(y, x) = \operatorname{arctg} \frac{y'(x)}{(py'')(x)}, \ \varphi(y, 0) = 0,$$

$$w(y, x) = \operatorname{ctg} \psi(y, x) = \frac{(py'')(x) \sin\theta(y, x)}{Ty(x) \cos\varphi(y, x)}, \ w(y, 0) = -\frac{(py'')(0)}{Ty(0)},$$

and $\psi(y, x) \in (0, \pi/2)$, $x \in (0, l)$.

For each $k \in \mathbb{N}$ and each $\nu \in \{+, -\}$ let S_k^ν be the set of functions $y \in S$ that satisfy the following conditions:

1) $\theta(y, l) = (2k - 1)\pi/2$;

2) $\varphi(y, l) = k\pi$ or $\varphi(y, l) = (k + 1)\pi$;

3) for fixed y, as x increases from 0 to l, the function $\theta(y, x)$ (respectively $\varphi(y, x)$) strictly increasing takes values of $m\pi/2$, $m \in \mathbb{Z}$ ($s\pi$, $s \in \mathbb{Z}$); as x decreases, the function $\theta(y, x)$ (respectively $\varphi(y, x)$), strictly decreasing takes values of $m\pi/2$, $m \in \mathbb{Z}$ (respectively $s\pi$, $s \in \mathbb{Z}$);

4) the function $\nu y(x)$ is positive in a deleted neighborhood of $x = 0$.

For each $k \in \mathbb{N}$ let $S_k = S_k + \cup S_k^-$. It follows directly from the definitions of the sets S_k^+, S_k^-, S_k, $k \in \mathbb{N}$, that they are open in E. Moreover, if $y \in \partial S_k^\nu$, $k \in \mathbb{N}$, $\nu \in \{+, -\}$, then by [1, Lemma 2.4] there exists $\tau \in [0, l]$ such that $y(\tau) = y'(\tau) = y''(\tau) = y'''(\tau) = 0$.

It follows from [2, Theorem 1.2] that the eigenvalues of the problem

$$\begin{cases} \ell(y)(x) = \lambda\tau(x)y(x), \ x \in (0, l), \\ y \in (b.c.), \end{cases} \tag{2.2}$$

are real, simple, and form an infinitely increasing sequence $\{\lambda_k\}_{k=1}^\infty$. Moreover, the eigenfunction $y_k(x)$, $k \in \mathbb{N}$, corresponding to the eigenvalue λ_k, lies in S_k.

Putting $g \equiv 0$ from (1.1)-(1.2) we get the following half-linear eigenvalue problem

$$\begin{cases} \ell(y)(x) = \lambda\tau(x)y(x) + \alpha(x)y^+(x) + \beta(x)y^-(x), \ x \in (0, l), \\ y \in (b.c.), \end{cases} \tag{2.3}$$

Obviously, the problem (2.3) is positively homogeneous and linear in the cones $y > 0$ and $y < 0$. Therefore, it is called a half-linear problem.

We present the following definitions, which are given in [5, 8, 9]. If there exists a nontrivial solution (λ, y_λ) to problem (2.3), then the number λ is called the half-eigenvalue of this problem, and y_λ is called the corresponding half-eigenfunction. In this case the set $\{(\lambda, ty_\lambda) : t > 0\}$ is a half-line of non-trivial solutions of problem (2.3). Note that there may exist other half-lines of solutions (λ, v_λ). A half-eigenvalue λ is said to be simple if there is only one such half-line or there are exactly two such half-lines $\{(\lambda, ty_\lambda) : t > 0\}$ and $\{(\lambda, tv_\lambda) : t > 0\}$ with y_λ and v_λ having opposite signs on a deleted neighborhood of $x = 0$, and all solutions (λ, y_λ) of problem (2.3) lie on these two half-lines.

By following the arguments in Theorem 3.3 of [9] and taking into account [2, Theorem 1.3] we verify the validity of the following theorem for problem (2.1).

Theorem 2.1. *There exist two unbounded sequences of simple half-eigenvalues of problem* (2.1),

$$\lambda_1^+ < \lambda_2^+ < \ldots < \lambda_k^+ < \ldots,$$

and

$$\lambda_1^- < \lambda_2^- < \ldots < \lambda_k^- < \ldots,$$

The half-eigenfunction y_k^ν, corresponding to the half-eigenvalue λ_k^ν, lies in S_k^ν. Furthermore, aside from solutions on the collection of the half-lines $\{(\lambda_k^\nu, ty_k^\nu) : t > 0\}$ and trivial ones, problem (2.1) *has no other solutions.*

In the next lemma, the distances between the corresponding eigenvalues of problems (2.3) and (2.2) are found.

Lemma 2.1. *For each $k \in \mathbb{N}$ and each $\nu \in \{+, -\}$ the following relation holds:*

$$|\lambda_k^\nu - \lambda_k| \le \frac{M}{\tau_0},$$

where

$$M = \max_{x\in[0,\,l]} |\alpha(x)| + \max_{x\in[0,\,l]} |\beta(x)| \ \text{ and } \ \tau_0 = \min_{x\in[0,\,l]} |\tau(x)|. \tag{2.4}$$

Proof. For any $y \in E$ we denote by $\chi_{\{y>0\}}(x)$ (respectively, $\chi_{\{y<0\}}(x)$), $x \in [0, l]$, the characteristic function of the set $\{x \in [0, l] : y(x) > 0\}$ (respectively, $\{x \in [0, l] : y(x) < 0\}$). Since $y_k^\nu \in S_k = S_k^\nu$ it follows that λ_k^ν is the kth eigenvalue of the linear problem

$$\begin{cases} \ell(y)(x) + \varphi_k^\nu(x)y(x) = \lambda\tau(x)y(x), \ x \in (0, l), \\ y \in (b.c.), \end{cases}$$

where

$$\varphi_k^\nu(x) = \alpha(x)\chi_{\{y_k^\nu>0\}}(x) + \beta(x)\chi_{\{y_k^\nu<0\}}(x), \ x \in [0, l].$$

It is obvious that

$$|\varphi_k^\nu(x)| \le |\alpha(x)|\,|\chi_{\{y_k^\nu>0\}}(x)| + |\beta(x)|\,|\chi_{\{y_k^\nu<0\}}(x)| \le$$
$$|\alpha(x)| + |\beta(x)| \le M, \ x \in [0, l].$$

By following the arguments in Lemma 4.1 of [2] we get

$$\lambda_k - \frac{M}{\tau_0} \le \lambda_k^\nu \le \lambda_k + \frac{M}{\tau_0}.$$

The proof of this lemma is complete.

Let us introduce the following notations:

$$f(x,y,s,v,w,\lambda) = \alpha(x)y^+ + \beta(x)y^-,\ (x,y,s,v,w,\lambda) \in [0,l] \times \mathbb{R}^5,$$

and

$$I_k = [\lambda_k - M/\tau_0, \lambda_k + M/\tau_0]\,.$$

Then, we have

$$|f(x,y,s,v,w,\lambda)| \leq M\,|y|,\ (x,y,s,v,w,\lambda) \in [0,l] \times \mathbb{R}^5. \tag{2.5}$$

Moreover, problem (1.1), (1.2) can be rewritten in the following form

$$\begin{array}{l} \ell y = \lambda\tau(x)\,y + f(x,y,y',y'',y''',\lambda) + g(x,y,y',y'',y''',\lambda),\ x \in (0,l), \\ y \in (b.c.), \end{array} \tag{2.6}$$

which has the same form as (1.1), (1.2) of [3]. Lemma 2.1 and conditions (1.3), (2.4) show that conditions (5.1) and (5.2) of [3] are satisfied for problem (2.6). Therefore, the statement of Corollary 5.7 in [3] holds for this problem. Hence we have the following results.

Lemma 2.2. *The set of asymptotic bifurcation points of problem* (1.1), (1.2) *with respect to the set* $\mathbb{R} \times S_k^\nu$ *is nonempty, and if* (λ,∞) *is a bifurcation point to this problem with respect to* $\mathbb{R} \times S_k^\nu$*, then* $\lambda \in I_k$.

Remark 2.1. Note that Lemma 2.2 does not give an answer to the question of what structure the bifurcation points respect to the set $\mathbb{R} \times S_k^\nu$, $k \in \mathbb{N}$, $\nu \in \{+\,,-\}$, have in the interval $I_k \times \{\infty\}$ (in fact, the question of how many such points are contained in the interval $I_k \times \{\infty\}$ is of interest).

In the next section, we will give an answer to the question expressed in Remark 2.1.

3. The structure of asymptotic bifurcation points and global bifurcation of solutions to problem (1.1)-(1.2)

Lemma 3.1. *Let* $\lambda^* \in I_k$ *and* (λ^*,∞)*,* $k \in \mathbb{N}$*,* $\nu \in \{+\,,-\}$*, be an asymptotic bifurcation point of problem* (1.1), (1.2) *with respect to the set* $\mathbb{R} \times S_k^\nu$*. Then* $\lambda^* = \lambda_k^\nu$.

Proof. Let $k \in \mathbb{N}$ and $\nu \in \{+\,,-\}$ are arbitrary and fixed. Assume that (λ,∞), $\lambda \in I_k$, is an asymptotic bifurcation point with respect to the set $\mathbb{R} \times S_k^\nu$ of problem (1.1), (1.2). Then there exists a sequence $\{(\lambda_n^*, y_n^*)\}_{n=1}^\infty \in \mathbb{R} \times E$ such that

$$\begin{cases} \ell(y_n^*) = \lambda_n^*\tau(x)y_n^* + \alpha(x)(y_n^*)^+ + \beta(x)(y_n^*)^- + \\ \qquad\qquad g(x,\,y_n^*,(y_n^*)',(y_n^*)'',(y_n^*)''',\lambda_n^*),\ x \in (0,l), \\ y_n^* \in (b.c.), \end{cases} \tag{3.1}$$

Setting $w_n^* = \frac{y_n^*}{||y_n^*||_3}$, we see that (λ_n^*, w_n^*) satisfies the following relations

$$\begin{cases} \ell(w_n^*) = \lambda_n^*\tau(x)w_n^* + \alpha(x)(w_n^*)^+ \ + \ \beta(x)(w_n^*)^- + \\ \qquad\qquad \frac{g(x,y_n^*,(y_n^*)',(y_n^*)'',(y_n^*)''',\lambda_n^*)}{||y_n^*||_3},\ x \in (0,l),\ w_n^* \in (b.c.). \end{cases} \tag{3.2}$$

We rewrite the first relation in (3.2) in the following form

$$(w_n^*)''''(x) = (p(x))^{-1}\{\lambda\tau(x)w_n^*(x) + \alpha(x)(w_n^*)^+(x) + \beta(x)(w_n^*)^-(x) +$$

$$-2p'(x)(w_n^*)'''(x) - p''(x)(w_n^*)''(x) + q(x)(w_n^*)''(x) + q'(x)(w_n^*)'(x) -$$

$$r(x)w_n^*(x) + \tfrac{g(x, y_n^*(x), (y_n^*)'(x), (y_n^*)''(x), (y_n^*)'''(x), \lambda_n^*)}{||y_n^*||_3}\Big\}. \tag{3.3}$$

It follows from [3, Lemma 5.5] that we can choose the number n so large enough to satisfy the inequality

$$\frac{|g(x,\, y_n^*(x), (y_n^*)'(x), (y_n^*)''(x), (y_n^*)'''(x), \lambda_n^*)|}{||y_n^*||_3} < 1.$$

Since $\lambda_n^* \to \lambda$ as $n \to \infty$ taking into account the relation $||w_n^*||_3 = 1$ and the conditions imposed on the functions p, q, r, τ, α, β equality (3.3) implies that there exists a constant $C_1 > 0$ such that

$$|(w_n^*)''''(x)| \le C_1,\ x \in [0, 1].$$

Therefore, by the Arzela-Ascoli theorem, there exists a subsequence $\{w_{n_m}^*\}_{m=1}^\infty$ of the sequence $\{(\lambda_n^*, w_n^*)\}_{n=1}^\infty$ which converges in $\mathbb{R} \times E$ to (λ, w^*) for some w^* with $||w^*||_3 = 1$. Then, it is seen from (3.2) (or (3.3)) that this subsequence $\{w_{n_m}^*\}_{m=1}^\infty$ converges to (λ, w^*) also in $\mathbb{R} \times C^4[0, l]$. Moreover, it follows from [3, Lemma 5.5] that

$$\frac{||g(x,\, y_{n_m}^*(x), (y_{n_m}^*)'(x), (y_{n_m}^*)''(x), (y_{n_m}^*)'''(x), \lambda_{n_m}^*)||_\infty}{||y_{n_m}^*||_3} \to 0 \text{ as } m \to \infty.$$

Then, passing to the limit as $m \to \infty$ in the relations

$$\begin{cases} \ell(w_{n_m}^*) = \lambda_{n_m}^*\tau(x)w_{n_m}^* + \alpha(x)(w_{n_m}^*)^+ + \beta(x)(w_{n_m}^*)^- + \\ \qquad \tfrac{g(x, y_{n_m}^*, (y_n^*)', (y_{n_m}^*)'', (y_{n_m}^*)''', \lambda_{n_m}^*)}{||y_{n_m}^*||_3},\ x \in (0, l),\ w_{n_m}^* \in (b.c.), \end{cases}$$

we get

$$\begin{cases} \ell(w^*) = \lambda^*\tau(x)w^* + \alpha(x)(w^*)^+ + \beta(x)(w^*)^-,\ x \in (0, l), \\ w^* \in (b.c.). \end{cases}$$

Since $w_{n,m}^* \in S_k^\nu$ it follows that $w^* \in S_k^\nu \cup \partial S_k^\nu$. If $w^* \in \partial S_k^\nu$, then by [2, Lemma 1.1] we have $w^* \equiv 0$ which contradicts to the condition $||w^*||_3 = 1$. Therefore, $w^* \in S_k^\nu$, and consequently, by Theorem 2.1 we get $\lambda^* = \lambda_k^\nu$ and $w^* = \frac{y_k^\nu}{||y_k^\nu||_3}$. The proof of this lemma is complete.

Let $\mathcal{D} \subset \mathbb{R} \times E$ be the set of nontrivial solutions to problem (1.1), (1.2). For each $k \in \mathbb{N}$ and each $\nu \in \{+, -\}$ by $D_k^\nu \subset \mathcal{D}$ we denote the union of all the components of $\mathcal{D}$ which meet (λ_k^ν, ∞) with respect to $\mathbb{R} \times S_k^\nu$ (this set is nonempty in view of Lemma 3.1 and [3, Theorem 5.9]). Note that the set D_k^ν may not be connected in the space $\mathbb{R} \times E$ but, by adding the points $\{(\lambda, \infty) : \lambda \in \mathbb{R}\}$ to this space and defining the corresponding topology on the resulting set, the set $D_k^\nu \cup \{(\lambda_k^\nu, \infty)\}$ is connected.

By Lemma 3.1 it follows from [3, Theorem 5.9] the following result.

Theorem 3.1. *For each $k \in \mathbb{N}$ and each $\nu \in \{+\,,\,-\}$ for the set D_k^ν one of the following assertions holds:*

(i) *D_k^ν meets $(\lambda_{k'}^{\nu'}, \infty)$ with respect to the set $\mathbb{R} \times S_{k'}^{\nu'}$ for some $(k', \nu') \neq (k, \nu)$;*

(ii) *D_k^ν meets $\mathcal{R} = \mathbb{R} \times \{0\}$ for some $\lambda \in \mathbb{R}$;*

(iii) *The projection $P_R(D_k^\nu)$ of the set D_k^ν onto $\mathcal{R}$ is unbounded.*

In addition, if the union $D_k = D_k^+ \cup D_k^-$ does not satisfy (ii) or (iii), then it must satisfy (i) with $k' \neq k$.

Remark 3.1. Let $k \in \mathbb{N}$ and $\nu \in \{+\,,\,-\}$ be arbitrary and fixed. Then it follows from Theorem 2.1 that $\lambda_{k'}^\nu \neq \lambda_k^\nu$ for any $k' \in \mathbb{N}$, $k' \neq k$. While for $\lambda_{k'}^\nu$ and λ_k^ν the following cases are possible: either (i) $\lambda_{k'}^{-\nu} \neq \lambda_k^\nu$ for any k', or (ii) $\lambda_{k'}^{-\nu} = \lambda_k^\nu$ for some k'. By Lemma 5.6 of [3], in the case (i) there exists an open neighborhood Q_k^ν of (λ_k^ν, ∞) such that

$$D_k^\nu \cap Q_k^\nu \subset \mathbb{R} \times S_k^\nu,$$

and in a case (ii) there exists an open neighborhood $\tilde{Q}_k^\nu$ of (λ_k^ν, ∞) such that

$$D_k^\nu \cap \tilde{Q}_k^\nu \cap (\mathbb{R} \times S_{k'}^{-\nu}) \neq \emptyset.$$

In the latter case, in a sense, D_k^ν contains a "closed loop" that meets the point (λ_k^ν, ∞) from two different directions.

4. Existence of nodal solutions to some half-linearizable problem

In this section, we consider the following nonlinear problem

$$\begin{cases} \ell(y)(x) = d\,\tau(x)h(y(x)) + \alpha(x)y^+(x) + \beta(x)y^-(x), \ x \in (0, l), \\ y \in (b.c.), \end{cases} \tag{4.1}$$

where $d \neq 0$ is a parameter, $h(s)$ is a continuous function on $\mathbb{R}$ that satisfies the following conditions:

$$uh(u) > 0, \ u \in \mathbb{R}\backslash\{0\}; \tag{4.2}$$

there exists $h_0, h_\infty \in (0, +\infty)$ such that

$$h_0 = \lim_{|u| \to 0} \frac{h(u)}{u} \ \text{ and } \ h_\infty = \lim_{|u| \to +\infty} \frac{h(u)}{u}. \tag{4.3}$$

We will determine the values of d for which there are solutions to problem (4.1) contained in $\bigcup\limits_{k=1}^{\infty} S_k^\nu$.

Theorem 4.1 *Suppose that for some $k \in \mathbb{N}$ and $\nu \in \{+\,,\,-\}$, either condition $\frac{\lambda_k^\nu}{h_\infty} < d < \frac{\lambda_k^\nu}{h_0}$ or $\frac{\lambda_k^\nu}{h_0} < d < \frac{\lambda_k^\nu}{h_\infty}$ holds. Then there exists a nontrivial solution of problem* (4.1) *which lies in S_k^ν.*

Proof. Consider the following nonlinear eigenvalue problem

$$\begin{cases} \ell(y)(x) = \lambda\,\tau(x)h(y(x)) + \alpha(x)y^+(x) + \beta(x)y^-(x), \ x \in (0, l), \\ y \in (b.c.). \end{cases} \tag{4.4}$$

By the second condition of (4.3) we get

$$h(u) = h_\infty u + \gamma(u), \tag{4.5}$$

where
$$\frac{\gamma(u)}{u} \to 0 \text{ as } |u| \to \infty.$$
Let $\tilde{\gamma} : [0, +\infty) \to [0, +\infty)$ be the continuous function defined by
$$\tilde{\gamma}(u) = \max_{0 \le |t| \le u} |\gamma(t)|.$$
It is obvious that if $0 < u_1 < u_2$, then
$$\tilde{\gamma}(u_1) \le \tilde{\gamma}(u_2).$$
Moreover, we have
$$\frac{\tilde{\gamma}(u)}{u} = \frac{\max\limits_{0 \le |t| \le u} |\gamma(t)|}{u} = \frac{|\gamma(t^*(u))|_{(|t^*(u)| \le u)}}{u} = \frac{|\gamma(t^*(u))|}{|t^*(u)|} \frac{|t^*(u)|}{u}. \tag{4.6}$$
In this case, either
$$|t^*(u)| \to +\infty \text{ as } u \to +\infty,$$
or there exists positive number m_0 such that
$$|t^*(u)| \le m_0 \ \text{ for } \ u \in [0, +\infty).$$
In both cases, it follows from (4.6) that
$$\frac{\tilde{\gamma}(u)}{u} \to 0 \text{ as } u \to +\infty. \tag{4.7}$$
We have the following relation
$$\frac{\gamma(u)}{||u||_3} \le \frac{\tilde{\gamma}(|u|)}{||u||_3} \le \frac{\tilde{\gamma}(||u||_3)}{||u||_3}.$$
which, by (4.7), implies that
$$||\gamma(u)||_\infty = o(||u||_3) \ \text{ as } \ ||u||_3 \to +\infty. \tag{4.8}$$
By (4.5) we can rewrite (4.4) as follows:
$$\begin{cases} \ell(y) = \lambda\,\tau(x) h_\infty y + \alpha(x) y^+ + \beta(x) y^- + \lambda\,\tau(x)\gamma(y), \ x \in (0, l), \\ y \in (b.c.). \end{cases} \tag{4.9}$$
In view of (4.8) for (4.9) Theorem 3.1 holds. Then there exists a component $\mathcal{D}_k^\nu$ of the set of nontrivial solutions of (4.9) for which one of the statements (i), (ii), and (iii) of this theorem holds.

By first condition of (4.3) we represent h in the following form
$$h(u) = h_0 u + \gamma_1(u)$$
where
$$\frac{\gamma_1(u)}{u} \to 0 \ \text{ as } \ |u| \to 0.$$
Hence we can rewrite (4.5) also in the following form
$$\begin{cases} \ell(y) = \lambda\,\tau(x) h_0 y + \alpha(x) y^+ + \beta(x) y^- + \lambda\,\tau(x)\gamma_1(y), \ x \in (0, l), \\ y \in (b.c.). \end{cases} \tag{4.10}$$
Following the above reasoning, we can show that
$$||\gamma_1(u)||_\infty = o(||u||_3) \ \text{ as } \ ||u||_3 \to 0. \tag{4.11}$$

Then, by Corollaries 5.2 and 5.3 of [2] the set of bifurcation points of (4.10) with respect to the set $\mathbb{R} \times S_k^\nu$ is nonempty. Hence following the arguments in Lemma 3.1 we can show that for each $k \in \mathbb{N}$ and $\nu \in \{+, -\}$ the point $(\frac{\lambda_k^\nu}{h_0}, 0)$ is an unique bifurcation point of (4.10) with respect to the set $\mathbb{R} \times S_k^\nu$. Moreover, it is clear from the proof of [3, Theorem 4.1] that $\mathcal{D}_k^\nu \subset \mathbb{R} \times S_k^\nu$, and consequently, the alternative (i) of Theorem 3.1 cannot hold. Moreover, $\mathcal{D}_k^\nu$ can meets $\mathbb{R} \times \{0\}$ for $\lambda = \frac{\lambda_k^\nu}{h_0}$.

Now, to complete the proof of the theorem, it only remains to prove that alternative (iii) of Theorem 3.1 does not hold for $\mathcal{D}_k^\nu$. Indeed, if the projection $P_R(\mathcal{D}_k^\nu)$ of the set $\mathcal{D}_k^\nu$ onto $\mathcal{R}$ is unbounded, then there exists the sequence $\{(\mu_n, u_n)\}_{n=1}^\infty \subset \mathcal{D}_k^\nu$ such that

$$\mu_n \to \infty \text{ as } n \to \infty.$$

Note that for each $k \in \mathbb{N}$ the pair (μ_n, u_n) satisfies the following relations

$$\begin{aligned} &\ell(u_n)(x) = \mu_n\, \tau(x) h(u_n)(x) + \alpha(x)\, u_n^+(x) + \beta(x) u_n^-(x),\ x \in (0, l), \\ &u_n \in (b.c.). \end{aligned} \tag{4.12}$$

We introduce the notation:

$$\varphi_n(x) = \begin{cases} \frac{h(u_n(x))}{u_n(x)} & \text{for } u_n(x) \neq 0, \\ h_0 & \text{for } u_n(x) = 0. \end{cases}$$

Then (μ_n, u_n) solves the problem

$$\begin{cases} \ell(y)(x) = \lambda\, \tau(x) \varphi_n(x) y(x) + \alpha(x) y^+(x) + \beta(x) y^-(x),\ x \in (0, l), \\ y \in (b.c.). \end{cases} \tag{4.13}$$

It follows from (4.2) and (4.3) that there exists a constant $\rho > 0$ such that

$$\frac{h(u)}{u} \geq \rho > 0 \text{ for any } u \neq 0,$$

which implies that

$$\varphi_n(x) \geq \max\{\rho, h_0\} \text{ for } x \in [0, l] \text{ and } n \in \mathbb{N}.$$

Consequently, we have

$$\mu_n\, \tau(x)\, \varphi_n(x) \to \pm\infty \text{ for any } x \in [0, l].$$

Since the half-eigenvalues of problem (4.13) are bounded from below in view of Theorem 2.1 it follows that

$$\mu_n\, \tau\, \varphi_n \to -\infty$$

is not possible. Note that the relation

$$\mu_n\, \tau\, \varphi_n \to +\infty$$

is also impossible, since for a sufficiently large n, by Theorem 2.1, the number of zeros of the function u_n will be large enough, which contradicts the condition $u_n \in S_k^\nu$.

Therefore, the alternatives (i) and (iii) of Theorem 3.1 cannot hold for (4.9). Then by alternative (ii) of this theorem $\mathcal{D}_k^\nu$ meet $(\frac{\lambda_k^\nu}{h_0}, 0)$ and $(\frac{\lambda_k^\nu}{h_\infty}, \infty)$, whence the assertion of the theorem follows immediately. The proof of this theorem is complete.

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