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BOUNDEDNESS OF THE DISCRETE AHLFORS-BEURLING TRANSFORM ON DISCRETE MORREY SPACES

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Abstract. The Ahlfors–Beurling transform is one of the important operators in complex analysis. This transform plays an essential role in applications to the theory of quasiconformal mappings and to the Beltrami equation with discontinuous coefficients. The Ahlfors-Beurling transform has been well studied on classical Lebesgue, Morrey, Sobolev, Besov, Campanato, etc. spaces. But its discrete version has not been well studied. In this paper, we prove that the discrete Ahlfors-Beurling Transform is a bounded operator in discrete Morrey spaces.

1. Introduction

The Ahlfors-Beurling transform of a function $f \in L_p(C)$, $1 \leq p < \infty$ is defined as the following singular integral:

$$(Bf)(z) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{\{w \in C\, :\, |z-w| > \varepsilon\}} \frac{f(w)}{(z-w)^2} dm(w).$$

The Ahlfors–Beurling transform is one of the important operators in complex analysis. It has been shown in [2, 6, 10, 17, 26] that this transform plays an essential role in applications to the theory of quasiconformal mappings and to the Beltrami equation with discontinuous coefficients.

From the theory of singular integrals (see [8, 23]) it is known that the Ahlfors–Beurling transform is a bounded operator in the space L_p, $1 < p < \infty$. In the case $p = 1$ only the weak inequality holds.

Denote by $l_p := l_p(Z_C)$, $p \geq 1$, the class of sequences $h = \{h_n\}_{n \in Z_C}$ satisfying the condition

$$\|h\|_{l_p} := \left(\sum_{n \in Z_C} |h_n|^p \right)^{1/p} < \infty,$$

where $Z_C := \{m + in \in C : \quad m, n \in Z\}$ and Z is the set of integers.

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Let $h=\{h_n\}_{n\in Z_C}\in l_p$, $p\geq 1$. Namely, the sequence $\tilde{B}(h)=\{(\tilde{B}h)_n\}_{n\in Z_C}$ is called the Ahlfors-Beurling transform of the sequence h, where

$$(\tilde{B}h)_n=\sum_{m\in Z_C,\, m\neq n}\frac{h_m}{(n-m)^2}, n\in Z_C.$$

Note that if $h\in l_p$, $1\leq p<\infty$, then from the Hölder inequality it follows that the series $\sum_{m\in Z_C,\, m\neq n}\frac{h_m}{(n-m)^2}$ absolutely converges, and therefore the Ahlfors-Beurling transform of the sequence h exists. In [8], A.P.Calderon and A.Zygmund noted that the discrete Ahlfors-Beurling transform is of special interest among discrete analogs of singular integrals. In this work, it was also noted that the discrete analogues of singular integrals, including the discrete Ahlfors-Beurling transform, is bounded in l_p. In [3] the summability properties of the discrete Ahlfors-Beurling transform on discrete Lebesgue spaces are studied .

In [5, 7, 11, 12, 13, 16, 19, 24, 25] the boundedness of the Ahlfors-Beurling transform in other function spaces (in the spaces of Sobolev, Besov, Campanato, Morrey, etc.) was studied. But discrete version of the Ahlfors–Beurling transform has not been well studied. In this paper, we prove that the discrete Ahlfors-Beurling transform is a bounded operator in discrete Morrey spaces.

2. Discrete Morrey spaces

The classical Morrey spaces $M_{\lambda,p}$, $0\leq\lambda\leq\frac{1}{p}$, $1\leq p<\infty$ (see [1, 9, 18, 20, 21, 22]), consist of the functions $f\in L_{p,loc}$ for which the following norm is finite

$$\|f\|_{M_{\lambda,p}}=\sup_{z}\sup_{r>0}\left[|B(z;r)|^{-\lambda}\,\|f\|_{L_p(B(z;r))}\right].$$

We note that if $\lambda=0$, then $M_{\lambda,p}=L_p$; if $\lambda=\frac{1}{p}$, then $M_{\lambda,p}=L_\infty$ (see [1]). In case $p>1$, $0\leq\lambda<\frac{1}{p}$, F.Chiarenza and M.Frasca [9] showed the boundedness of the Hardy–Littlewood maximal operator, the fractional integral operator and a singular integral operator in the Morrey spaces. Hence, in particular, it implies the boundedness of the Ahlfors-Beurling transform in Morrey spaces. It means that, in case $p>1$, $0\leq\lambda<\frac{1}{p}$, for any $f\in M_{\lambda,p}$ we have $Bf\in M_{\lambda,p}$, and there exist $C_{\lambda,p}>0$ such that

$$\|Bf\|_{M_{\lambda,p}}\leq C_{\lambda,p}\cdot\|f\|_{M_{\lambda,p}}$$

holds for all $f\in M_{\lambda,p}$.

In [14], the authors introduced a discrete analogue of Morrey spaces and studied their inclusion properties. For $m\in Z_C$ and $n\in N\cup\{0\}$ define $S_{m,n}=\{k\in Z_C:\|k-m\|\leq n\}$, where $\|k\|:=\max\{|\Re(k)|,|\Im(k)|\}$. Following standard conventions, we denote the cardinality of a set S by $|S|$. Then we have $|S_{m,n}|=(2n+1)^2$ for all $m\in Z$ and each $n\in N\cup\{0\}$. Discrete Morrey spaces $m_{\lambda,p}$, $0\leq\lambda\leq\frac{1}{p}$, $1\leq p<\infty$, consist of the sequences $h=\{h_n\}_{n\in Z_C}$ for which the following norm is finite

$$\|h\|_{m_{\lambda,p}}=\sup_{m\in Z_C}\sup_{n\in N\cup\{0\}}\left[|S_{m,n}|^{-\lambda}\left(\sum_{k\in S_{m,n}}|h_k|^p\right)^{1/p}\right].$$

In [4, 15] it was proved the boundedness of the discrete Hardy-Littlewood maximal operators, discrete Riesz potentials and discrete Hilbert transforms on discrete Morrey spaces.

Note that if $h = \{h_n\}_{n\in Z_C} \in m_{\lambda,p}$, $1 \le p < \infty$, $0 \le \lambda < \frac{1}{p}$, then the series $\sum_{m\in Z_C,\, m\neq n} \frac{h_m}{(n-m)^2}$ absolutely converges. Indeed, for each $n \in Z_C$ we have

$$\sum_{m\in Z_C,\, m\neq n} \frac{|h_m|}{|n-m|^2} = \sum_{j=1}^{\infty} \sum_{2^{j-1}\le\|n-m\|<2^j} \frac{|h_m|}{|n-m|^2} \le \sum_{j=1}^{\infty} \frac{1}{2^{2j-2}} \sum_{\|n-m\|<2^j} |h_m|$$

$$\le \sum_{j=1}^{\infty} \frac{1}{2^{2j-2}} \left(\sum_{\|n-m\|<2^j} |h_m|^p \right)^{1/p} \cdot (2^{j+1}-1)^{2-2/p}$$

$$\le 16\|h\|_{m_{\lambda,p}} \cdot \sum_{j=1}^{\infty} 4^{(j+1)(\lambda-1/p)} \le \frac{16\|h\|_{m_{\lambda,p}}}{1-4^{\lambda-1/p}}. \tag{2.1}$$

It follows that if $h = \{h_n\}_{n\in Z_C} \in m_{\lambda,p}$, $1 \le p < \infty$, $0 \le \lambda < \frac{1}{p}$, then the Ahlfors-Beurling transform of the sequence h exists.

3. Boundedness of the discrete Ahlfors-Beurling transform on discrete Morrey spaces

We next present the main results of this paper.

Theorem 3.1. *Let $1 < p < \infty$, $0 \le \lambda < 1/p$. For any $h \in m_{\lambda,p}$ we have $\tilde{B}(h) \in m_{\lambda,p}$, and there exists $c_{\lambda,p} > 0$ such that*

$$\|\tilde{B}(h)\|_{m_{\lambda,p}} \le c_{\lambda,p} \cdot \|h\|_{m_{\lambda,p}}$$

holds for all $h \in m_{\lambda,p}$.

Proof. We define the function $f(z)$ to be $(-4\pi h_n)$ for $z \in P(n, 1/4)$, $n \in Z_C$, and 0 elsewhere, where

$$P(n,\delta) := \{w \in C :\ -\delta \le \Re(w-n) < \delta,\ -\delta \le \Im(w-n) < \delta\}.$$

We first show that $f \in M_{\lambda,p}$. Indeed, for any $z \in P(n, 1/2)$, $n \in Z_C$ we have:

if $r \in (0, 1/4]$, then

$$|B(z;r)|^{-\lambda}\|f\|_{L_p(B(z;r))} = |\pi r^2|^{-\lambda} \left(\int_{B(z;r)} |f(w)|^p dm(w) \right)^{1/p}$$

$$\le |\pi r^2|^{1/p-\lambda} 4\pi \cdot |h_n| \le 4\pi \cdot \|h\|_{m_{\lambda,p}}; \tag{3.1}$$

if $r \in (1/4, 1]$, then

$$|B(z;r)|^{-\lambda}\|f\|_{L_p(B(z;r))} \le |\pi r^2|^{-\lambda} \left(\frac{1}{4} \sum_{k\in S_{n,1}} |4\pi h_k|^p \right)^{1/p}$$

$$\le |\pi r^2|^{-\lambda} \cdot 4^{1-1/p} \cdot \pi \cdot 9^{\lambda} \|h\|_{m_{\lambda,p}} \le 36\pi \cdot \|h\|_{m_{\lambda,p}}; \tag{3.2}$$

if $r \in (m, m+1]$, $m \in N$, then

$$|B(z;r)|^{-\lambda} \|f\|_{L_p(B(z;r))} \le |\pi r^2|^{-\lambda} \left(\frac{1}{4} \sum_{k \in S_{n,m+1}} |4\pi h_k|^p \right)^{1/p}$$

$$\le |\pi r^2|^{-\lambda} \cdot 4^{1-1/p} \cdot \pi \cdot (2m+1)^{2\lambda} \|h\|_{m_{\lambda,p}} \le 12\pi \cdot \|h\|_{m_{\lambda,p}}. \tag{3.3}$$

From inequalities (3.1), (3.2), (3.3) it follows that $f \in M_{\lambda,p}$, and

$$\|f\|_{M_{\lambda,p}} = \sup_z \sup_{r>0} \left[|B(z;r)|^{-\lambda} \|f\|_{L_p(B(z;r))} \right] \le 12\pi \cdot \|h\|_{m_{\lambda,p}}.$$

Therefore, $Bf \in M_{\lambda,p}$ and there exist $d_0 > 0$ such that

$$\|Bf\|_{M_{\lambda,p}} \le d_0 \|h\|_{m_{\lambda,p}}. \tag{3.4}$$

Define the function $F(z)$ to be $(\tilde{B}h)_n$ for $z \in P(n, 1/2)$, $n \in Z_C$ and

$$G(z) = (Bf)(z) - F(z). \tag{3.5}$$

We first show that $G \in M_{\lambda,p}$. For any $z \in P(n, 1/2)$, $n \in Z_C$ we have

$$G(z) = \sum_{m \in Z_C} 4h_m \int_{P(m,1/4)} \frac{dm(w)}{(z-w)^2} - \sum_{m \in Z_C,\, m \ne n} \frac{h_m}{(n-m)^2}$$

$$= \sum_{m \in Z_C,\, m \ne n} 4h_m \int_{P(m,1/4)} \left(\frac{1}{(z-w)^2} - \frac{1}{(n-m)^2} \right) dm(w)$$

$$+4h_n \int_{P(n,1/4)} \frac{dm(w)}{(z-w)^2} = G_1(z) + G_2(z), \tag{3.6}$$

where

$$\int_{P(n,1/4)} \frac{dm(w)}{(z-w)^2} := \lim_{\varepsilon \to 0+} \int_{\{w \in P(n,1/4):\ |w-z| \ge \varepsilon\}} \frac{dm(w)}{(z-w)^2}.$$

Let $m \ne n$. Since for every $z \in P(n, 1/2)$ and $w \in P(m, 1/4)$

$$|n-m| - 3/4 \le |z-w| \le |n-m| + 3\sqrt{2}/4,$$

then we get

$$\left| \frac{1}{(z-w)^2} - \frac{1}{(n-m)^2} \right| = \frac{|n-m-z+w| \cdot |n-m+z-w|}{|z-w|^2 \cdot |n-m|^2}$$

$$\le \frac{3/4(2|n-m| + 3\sqrt{2}/4)}{|n-m|^2 \cdot (|n-m| - 3/4)^2} \le \frac{39}{|n-m|^3}.$$

Therefore, for every $z \in P(n, 1/2)$

$$|G_1(z)| \le \sum_{m \in Z_C,\, m \ne n} 4|h_m| \int_{P(m,1/4)} \left| \frac{1}{(z-w)^2} - \frac{1}{(n-m)^2} \right| dm(w)$$

$$\le \sum_{m \in Z_C,\, m \ne n} \frac{39|h_m|}{|n-m|^3}. \tag{3.7}$$

From (3.7) and (2.1) it follows that

$$|G_1(z)| \le \sum_{m \in Z_C,\, m \ne n} \frac{39|h_m|}{|n-m|^3} \le \frac{624\|h\|_{m_{\lambda,p}}}{1 - 4^{\lambda - 1/p}}. \tag{3.8}$$

If $r \in (0,1)$, then we have from (3.8)

$$|B(z;r)|^{-\lambda}\|G_1\|_{L_p(B(z;r))} = |\pi r^2|^{-\lambda}\left(\int_{B(z;r)}|G_1(w)|^p dm(w)\right)^{1/p}$$
$$\leq |\pi r^2|^{-\lambda}\frac{624\|h\|_{m_{\lambda,p}}}{1-4^{\lambda-1/p}}\cdot(\pi r^2)^{1/p} \leq \frac{624\cdot\pi^{1/p-\lambda}}{1-4^{\lambda-1/p}}\|h\|_{m_{\lambda,p}}; \tag{3.9}$$

if $r \in [k-1,k)$, $k \in N$, $k \geq 2$, then from (3.7) and from the Hölder's inequality we have

$$|B(z;r)|^{-\lambda}\|G_1\|_{L_p(B(z;r))} = |\pi r^2|^{-\lambda}\left(\int_{B(z;r)}|G_1(w)|^p dm(w)\right)^{1/p}$$
$$\leq |\pi r^2|^{-\lambda}\left(\int_{P(n;k+1/2)}|G_1(w)|^p dm(w)\right)^{1/p}$$
$$= |\pi r^2|^{-\lambda}\left(\sum_{s\in Z_C\colon \|s-n\|\leq k}\int_{P(s;1/2)}|G_1(w)|^p dm(w)\right)^{1/p}$$
$$\leq |\pi r^2|^{-\lambda}\left(\sum_{s\in Z_C\colon \|s-n\|\leq k}\left(\sum_{m\in Z_C,\, m\neq s}\frac{39|h_m|}{|s-m|^3}\right)^p\right)^{1/p}$$
$$\leq 39|\pi r^2|^{-\lambda}\left(\sum_{s\in Z_C\colon \|s-n\|\leq k}\left(\sum_{m\neq s}\frac{|h_m|^p}{|s-m|^3}\right)\cdot\left(\sum_{m\neq s}\frac{1}{|s-m|^3}\right)^{p-1}\right)^{1/p}$$
$$\leq 39 d_1^{1-1/p}|\pi r^2|^{-\lambda}\left(\sum_{s\in Z_C\colon \|s-n\|\leq k}\sum_{m\neq s}\frac{|h_m|^p}{|s-m|^3}\right)^{1/p}$$
$$= 39 d_1^{1-1/p}|\pi r^2|^{-\lambda}\left(\sum_{m\in Z_C}|h_m|^p\sum_{\|s-n\|\leq k, s\neq m}\frac{1}{|s-m|^3}\right)^{1/p}$$
$$\leq 39 d_1^{1-1/p}|\pi r^2|^{-\lambda}\left(\sum_{\|m-n\|\leq 2k}|h_m|^p\cdot d_1 + \sum_{i=1}^{\infty}\sum_{2^i k<\|n-m\|\leq 2^{i+1}k}\frac{|h_m|^p}{2^{3i-7}k}\right)^{1/p}$$
$$\leq 39 d_1^{1-1/p}|\pi r^2|^{-\lambda}\left(d_1(2k+1)^{2p\lambda}\|h\|^p_{m_{\lambda,p}} + \sum_{i=1}^{\infty}\frac{(2^{i+1}k)^{2p\lambda}\|h\|^p_{m_{\lambda,p}}}{2^{3i-7}k}\right)^{1/p}$$
$$\leq 39 d_1^{1-1/p}\pi^{-\lambda}(k-1)^{-2\lambda}k^{2\lambda}\|h\|_{m_{\lambda,p}}\left(d_1\cdot 3^{2p\lambda} + \frac{256}{2^{3-2p\lambda}-1}\right)^{1/p}$$
$$\leq 78 d_1^{1-1/p}\pi^{-\lambda}\|h\|_{m_{\lambda,p}}\left(9d_1 + \frac{256}{2^{3-2p\lambda}-1}\right)^{1/p}, \tag{3.10}$$

where

$$d_1 = \sum_{m\in Z_C\colon m\neq 0}\frac{1}{|m|^3} \leq 4\sum_{n\in N}\frac{2n+1}{n^3} \leq 2\pi^2.$$

It follows from (3.9), (3.10) that $G_1 \in M_{\lambda,p}$ and

$$\|G_1\|_{M_{\lambda,p}} = \sup_z \sup_{r>0} \left[|B(z;r)|^{-\lambda} \|G_1\|_{L_p(B(z;r))} \right] \le d_2 \|h\|_{m_{\lambda,p}}, \tag{3.11}$$

where

$$d_2 := \max \left\{ \frac{624 \cdot \pi^{1/p-\lambda}}{1 - 4^{\lambda - 1/p}},\ 78 d_1^{1-1/p} \pi^{-\lambda} \left(9d_1 + \frac{256}{2^{3-2p\lambda} - 1} \right)^{1/p} \right\}.$$

Let us show that $G_2 \in M_{\lambda,p}$.

For every $n \in Z_C$ and $a > 0$ denote

$$\Gamma(n;a) = \{ w \in C : \|w - n\| = a \},$$

and for every $z \in P(n, 1/2)$ denote

$$\delta_z = \rho(z; \Gamma(n;1/4)) = \inf\{ |z - w| : w \in \Gamma(n;1/4) \}.$$

For every $z \in P(n,1/2) \setminus \Gamma(n;1/4)$ we have

$$|G_2(z)| = 4|h_n| \left| \int_{P(n,1/4)} \frac{dm(w)}{(z-w)^2} \right|$$

$$\le 4|h_n| \int_{B(z,2) \setminus B(z;\delta_z)} \frac{dm(w)}{|z-w|^2} = 8\pi |h_n| \ln \frac{2}{\delta_z} \tag{3.12}$$

Since

$$\int_{P(n,1/2)} \left| \ln \frac{2}{\delta_z} \right|^p dm(z) \le 8 \int_0^{1/4} \left| \ln \frac{2}{t} \right|^p dt,$$

then for every $n \in Z_C$

$$\int_{P(n,1/2)} |G_2(z)|^p \, dm(z) \le (8\pi)^p |h_n|^p \int_{P(n,1/2)} \left| \ln \frac{2}{\delta_z} \right|^p dm(z) \le d_3 |h_n|^p, \tag{3.13}$$

where

$$d_3 = 8 \cdot (8\pi)^p \int_0^{1/4} \left| \ln \frac{2}{t} \right|^p dt.$$

Let $z \in P(n,1/2)$. If $r \in (0,1)$, then it follows from (3.12) that

$$|B(z;r)|^{-\lambda} \|G_2\|_{L_p(B(z;r))} = |\pi r^2|^{-\lambda} \left(\int_{B(z;r)} |G_2(w)|^p dm(w) \right)^{1/p}$$

$$\le |\pi r^2|^{-\lambda} \cdot 8\pi \cdot \sum_{\|s-n\| \le 1} |h_s| \left[8 \int_0^r \left| \ln \frac{2}{t} \right|^p dt \right]^{1/p}$$

$$\le 72 \pi^{1-\lambda} r^{-2\lambda} 8^{1/p} \left[\int_0^r \left| \ln \frac{2}{t} \right|^p dt \right]^{1/p} \|h\|_{m_{\lambda,p}} \le d_4 \|h\|_{m_{\lambda,p}}, \tag{3.14}$$

where

$$d_4 = 72 \pi^{1-\lambda} 8^{1/p} \sup_{0<r<1} r^{-2\lambda} \left[\int_0^r \left| \ln \frac{2}{t} \right|^p dt \right]^{1/p} < \infty;$$

if $r \in [k-1, k)$, $k \in N$, $k \ge 2$, then it follows from (3.13) that

$$|B(z;r)|^{-\lambda} \|G_2\|_{L_p(B(z;r))} = |\pi r^2|^{-\lambda} \left(\int_{B(z;r)} |G_2(w)|^p dm(w) \right)^{1/p}$$

$$\leq |\pi r^2|^{-\lambda} \left(\int_{P(n;k+1/2)} |G_2(w)|^p dm(w) \right)^{1/p}$$

$$= |\pi r^2|^{-\lambda} \left(\sum_{s \in Z_C \colon \|s-n\| \leq k} \int_{P(s;1/2)} |G_2(w)|^p dm(w) \right)^{1/p}$$

$$\leq |\pi r^2|^{-\lambda} \left(\sum_{s \in Z_C \colon \|s-n\| \leq k} d_3 |h_n|^p \right)^{1/p}$$

$$\leq \pi^{-\lambda} d_3^{1/p} (k-1)^{-2\lambda} (2k+1)^{2\lambda} \|h\|_{m_{\lambda,p}} \leq \pi^{-\lambda} 5^{2\lambda} d_3^{1/p} \|h\|_{m_{\lambda,p}} \tag{3.15}$$

By (3.14), (3.15) we find that $G_2 \in M_{\lambda,p}$ and

$$\|G_2\|_{M_{\lambda,p}} = \sup_z \sup_{r>0} \left[|B(z;r)|^{-\lambda} \|G_2\|_{L_p(B(z;r))} \right] \leq d_5 \|h\|_{m_{\lambda,p}}, \tag{3.16}$$

where

$$d_5 = \max\{\pi^{-\lambda} 5^{2\lambda} d_3^{1/p}, d_4\}.$$

Then it follows from (3.6), (3.11) and (3.16) that $G \in M_{\lambda,p}$ and

$$\|G\|_{M_{\lambda,p}} \leq (d_2 + d_5) \|h\|_{m_{\lambda,p}}. \tag{3.17}$$

Since $F(z) = G(z) + (Bf)(z)$, by (3.4) and (3.17) we get that $F \in M_{\lambda,p}$ and

$$\|F\|_{M_{\lambda,p}} \leq (d_0 + d_2 + d_5) \|h\|_{m_{\lambda,p}}.$$

Therefore, for every $m \in Z_C$ and $n \in N \cup \{0\}$ we have

$$|S_{m,n}|^{-\lambda} \left(\sum_{k \in S_{m,n}} |(\tilde{B}h)_k|^p \right)^{1/p} = (2n+1)^{-2\lambda} \left(\int_{P(m;n+1/2)} |F(w)|^p dm(w) \right)^{1/p}$$

$$\leq (2n+1)^{-2\lambda} \|F\|_{L_p(B(m,2n+1))} \leq (2n+1)^{-2\lambda} |B(m,2n+1)|^{\lambda} \|F\|_{M_{\lambda,p}}$$

$$\leq \pi^{\lambda} (d_0 + d_2 + d_5) \|h\|_{m_{\lambda,p}}.$$

It follows that $\tilde{B}(h) \in m_{\lambda,p}$ and

$$\|\tilde{B}(h)\|_{m_{\lambda,p}} \leq \pi^{\lambda} (d_0 + d_2 + d_5) \|h\|_{m_{\lambda,p}}.$$

This completes the proof of Theorem 3.1.

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