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ON A NEW PARAMETERIZED BETA FUNCTION

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Abstract. In this paper we define a new parameterized Beta function which provides a generalization of standard Beta function and logarithmic mean. Several algebraic properties of this new function are explored. To highlight the utility of this new function, we present an application inspired from the probability area.

1. Introduction

In this introductive section, we collect some basic notions that will be needed throughout this paper.

The standard Beta function, also called the Euler's integral of the first kind, is defined for any $x, y > 0$ by

$$B(x,y) =: \int_0^1 t^{x-1}(1-t)^{y-1}\, dt. \tag{1.1}$$

Basic properties of this function as well as its applications in various mathematical areas can be found in the literature. See [1, 2, 6] for instance.

In 1997 Chaudhry et al. defined an extension of the beta function as follows: for any $x, y > 0$ and $p \geq 0$, they defined [4]

$$B(x,y;p) =: \int_0^1 t^{x-1}(1-t)^{y-1} \exp\left(\frac{-p}{t(1-t)}\right)\, dt. \tag{1.2}$$

In the same direction, Choi et al. in [5] gave another extension of $B(x,y)$ defined for any $x, y > 0$ and $p, q \geq 0$, by

$$\mathcal{B}(x,y;p,q) =: \int_0^1 t^{x-1}(1-t)^{y-1} \exp\left(-\frac{p}{t} - \frac{q}{1-t}\right)\, dt. \tag{1.3}$$

If $p = 0$ then (1.2) coincides with (1.1) and if $p = q$ then (1.3) is reduced to (1.2). To give more extension of the beta function, we need to recall the logarithmic mean which is given, for $a, b > 0$, by

$$L(a,b) = \int_0^1 a^{1-t}\, b^t\, dt = \frac{a-b}{\log a - \log b}, \text{ with } L(a,a) = a.$$

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The following chain of inequalities is well-known in the literature, see [7] for instance

$$\min(a,b) \le a\sharp b \le L(a,b) \le a\nabla b \le \max(a,b),$$

where $a\sharp b$ and $a\nabla b$ stand respectively for the geometric and arithmetic means of a and b.

The present paper will be organized in the following manner: in Section 2 we introduce the so-called Beta-logarithmic function and we study its properties. Such function extends simultaneously the Beta function $B(x,y)$ and the logarithmic mean $L(a,b)$. Section 3 deals with a parameterized random variable associated to the previous Beta-logarithmic function.

2. Beta-Logarithmic function

In this section, we introduce what we call the Beta-logarithmic function. For $a,b>0$ fixed, the function $t \longmapsto a^{1-t}b^t$ is continuous on $[0,1]$ and so it is bounded on $[0,1]$. It follows that, there exists $c>0$ such that for any $a,b,x,y>0$ we have

$$\forall t \in (0,1) \quad 0 \le a^{1-t}b^t t^{x-1}(1-t)^{y-1} \le c\, t^{x-1}(1-t)^{y-1}.$$

Thus, $t \longmapsto a^{1-t}b^t t^{x-1}(1-t)^{y-1}$ is integrable on $(0,1)$ and so, we can state the following definition.

Definition 2.1. Let $a,b;x,y>0$. We set

$$\mathcal{BL}(a,b;x,y) = \int_0^1 a^{1-t}b^t t^{x-1}(1-t)^{y-1}\,dt, \tag{2.1}$$

which we call the Beta-logarithmic function.

It is clear that $\mathcal{BL}(a,b;x,y)$ extends simultaneously the Beta function $B(x,y)$ and the logarithmic mean $L(a,b)$, in the sense that

$$\mathcal{BL}(1,1;x,y) = B(x,y), x,y>0 \text{ and } \mathcal{BL}(a,b;1,1) = L(a,b), a,b>0.$$

The elementary properties of $\mathcal{BL}(a,b;x,y)$ are embodied in the following result.

Proposition 2.1. *Let $a,b,x,y,\alpha>0$. Then the following assertions hold true:*

$$\mathcal{BL}(a,b;x,y) = \mathcal{BL}(b,a;y,x). \tag{2.2}$$

$$\mathcal{BL}(a,a;x,y) = a\,B(x,y) \text{ and } \mathcal{BL}(\alpha\, a,\alpha\, b;x,y) = \alpha\,\mathcal{BL}(a,b;x,y). \tag{2.3}$$

$$\mathcal{BL}(a,b;x+1,y) + \mathcal{BL}(a,b;x,y+1) = \mathcal{BL}(a,b;x,y). \tag{2.4}$$

Proof. The assertion (2.2) follows by the change of variables $t=1-u$ in (2.1). On the other hand, (2.3) and (2.4) can be deduced by simple computations from (2.1). □

Remark 2.1. Taking $a=b=1$ in (2.4), we get the following well known relation,

$$B(x+1,y) + B(x,y+1) = B(x,y). \tag{2.5}$$

The following proposition gives a bounding for the Beta-logarithmic function.

Proposition 2.2. *Let $a,b,x,y>0$. We have*

$$\min(a,b)\;B(x,y) \le \mathcal{BL}(a,b;x,y) \le a\,B(x,y+1) + b\,B(x+1,y) \le \max(a,b)\;B(x,y).$$

Proof. From the inequalities $\min(a,b) \le L(a,b) \le \max(a,b)$ and $B(x,y) > 0$, we get

$$\min(a,b)\ B(x,y) \le \mathcal{BL}(a,b;x,y).$$

By using the following well known Young's inequality

$$a^{1-t}b^t \le (1-t)a + tb, \text{ for all } t \in [0,1],$$

we obtain

$$\begin{aligned}\mathcal{BL}(a,b;x,y) &\le a\,B(x,y+1) + b\,B(x+1,y)\\ &\le \max(a,b)\big(B(x,y+1) + B(x+1,y)\big).\end{aligned}$$

Employing the relation (2.5), the proof is achieved. □

The following result provides a generalization of the relation (2.4).

Proposition 2.3. *For any $a,b;x,y > 0$, we have the following formula*

$$\mathcal{BL}(a,b;x,y) = \sum_{n,m=0}^{\infty} \mathcal{BL}(a,b;x+n+1,y+m+1) \tag{2.6}$$

Proof. Using the series representation, $(1-t)^{-1} = \sum_{n=0}^{\infty} t^n$ for $t \in (0,1)$, with an argument of uniform convergence of this power series, we can write

$$\mathcal{BL}(a,b;x,y) = \sum_{n=0}^{\infty} \int_0^1 a^{1-t}b^t t^{x+n-1}(1-t)^y\, dt.$$

Now, by the power series $t^{-1} = \sum_{m=0}^{\infty}(1-t)^m$ for $t \in (0,1)$, with similar argument as previous, we get

$$\mathcal{BL}(a,b;x,y) = \sum_{n,m=0}^{\infty} \int_0^1 a^{1-t}b^t t^{x+n}(1-t)^{y+m}\, dt.$$

Whence the desired result. □

From (2.6) we immediately deduce the following corollary.

Corollary 2.1. *For any $a,b;x,y > 0$ and any integers $n,m \ge 1$ one has*

$$\mathcal{BL}(a,b;x,y) > \mathcal{BL}(a,b;x+n,y+m).$$

We have the following result as well.

Theorem 2.1. *Let $a,b;x,y > 0$. Then we have the following representation*

$$\mathcal{BL}(a,b;x,y) = \sum_{n,m=0}^{\infty} \frac{B(x+n,y+m)}{n!\,m!}\big(\log a\big)^m\big(\log b\big)^n. \tag{2.7}$$

Proof. The following power series

$$a^{1-t} = \sum_{m=0}^{\infty} \frac{(\log a)^m}{m!}(1-t)^m, \quad b^t = \sum_{n=0}^{\infty} \frac{(\log b)^n}{n!} t^n,$$

when substituted in (2.1) gives

$$\mathcal{BL}(a,b;x,y) = \int_0^1 \sum_{m,n=0}^{\infty} \frac{(\log a)^m(\log b)^n}{n!m!} t^{x+n-1}(1-t)^{y+m-1} dt.$$

This, with the fact that the involved power series is uniformly convergent, allows us to interchange the order of the integral with the infinite sum and then (2.7) is obtained. □

The following result may be also stated.

Theorem 2.2. *Let $a, b > 0$ with $a \neq b$ and $x, y > 1$. Then we have*

$$\mathcal{BL}(a,b;x,y) = \frac{1}{\log a - \log b}\Big((x-1)\mathcal{BL}(a,b;x-1,y) - (y-1)\mathcal{BL}(a,b;x,y-1)\Big). \tag{2.8}$$

Proof. For $t \in (0,1)$, we consider the following formulas

$$u'(t) =: a^{1-t}b^t \Longleftarrow u(t) = \frac{a^{1-t}b^t}{\log b - \log a},$$

$$v(t) =: t^{x-1}(1-t)^{y-1} \Longrightarrow v'(t) = (x-1)t^{x-2}(1-t)^{y-1} - (y-1)t^{x-1}(1-t)^{y-2}.$$

These, with the principle of integration by parts applied for the integral representation (2.1) of $\mathcal{BL}(a,b;x,y)$, imply the following

$$\mathcal{BL}(a,b;x,y) = \frac{1}{\log a - \log b}\Bigg[(x-1)\int_0^1 a^{1-t}b^t t^{x-2}(1-t)^{y-1}\,dt - (y-1)\int_0^1 a^{1-t}b^t t^{x-1}(1-t)^{y-2}\,dt\Bigg].$$

Whence, by the use of (2.1), we get the desired result. □

This last theorem, enable us to state the following interesting result.

Corollary 2.2. *Let $\alpha > 0$ be given. The functional equation: find a positive function $f_\alpha(x,y)$ such that*

$$\forall x, y > 1 \quad f_\alpha(x,y) = \alpha\Big((x-1)f_\alpha(x-1,y) - (y-1)f_\alpha(x,y-1)\Big)$$

has at least one nontrivial solution given by

$$f_\alpha(x,y) = \int_0^1 e^{t/\alpha}t^{x-1}(1-t)^{y-1}\,dt.$$

Proof. Setting $a/b = e^{1/\alpha}$ in (2.8), with the definition of $\mathcal{BL}(a,b;x,y)$, we get the desired result. □

The following result holds true.

Theorem 2.3. *Let $a, b, a', b', x, y, x', y' > 0$. For any positive numbers α and β such that $\alpha + \beta = 1$ we have,*

$$\mathcal{BL}\big(a^\alpha a'^\beta,\, b^\alpha b'^\beta;\; \alpha x + \beta x', \alpha y + \beta y'\big) \le \Big(\mathcal{BL}(a,b;x,y)\Big)^\alpha \Big(\mathcal{BL}(a',b';x',y')\Big)^\beta. \tag{2.9}$$

Proof. For the sake of simplicity, we set

$$\Phi =: \mathcal{BL}\big(a^\alpha a'^\beta, b^\alpha b'^\beta;\; \alpha x + \beta x', \alpha y + \beta y'\big).$$

By definition, we have

$$\begin{aligned}\Phi &= \int_0^1 (a^\alpha a'^\beta)^{1-t}(b^\alpha b'^\beta)^t t^{\alpha x+\beta x'-1}(1-t)^{\alpha y+\beta y'-1}\,dt\\ &= \int_0^1 (a^{1-t}\,b^t)^\alpha\,(a'^{1-t}\,b'^t)^\beta\,t^{\alpha(x-1)+\beta(x'-1)}(1-t)^{\alpha(y-1)+\beta(y'-1)}\,dt\\ &= \int_0^1 \Big(a^{1-t}\,b^t\,t^{x-1}\,(1-t)^{y-1}\Big)^\alpha \Big(a'^{1-t}\,b'^t\,t^{x'-1}\,(1-t)^{y'-1}\Big)^\beta\,dt.\end{aligned}$$

So, employing Hölder's integral inequality, we obtain

$$\Phi \le \left(\int_0^1 a^{1-t}\,b^t\,t^{x-1}\,(1-t)^{y-1}\,dt\right)^\alpha \left(\int_0^1 a'^{1-t}\,b'^t\,t^{x'-1}\,(1-t)^{y'-1}\,dt\right)^\beta,$$

which is equivalent to (2.9). □

Taking $\alpha = \beta = 1/2$ in the previous theorem we immediately obtain the following.

Corollary 2.3. *Let $a, b, a', b', x, y, x', y' > 0$. Then we have*

$$\Big(\mathcal{BL}\big(\sqrt{aa'}, \sqrt{bb'};\ \frac{x+x'}{2}, \frac{y+y'}{2}\big)\Big)^2 \le \Big(\mathcal{BL}(a,b;x,y)\Big)\Big(\mathcal{BL}(a',b';x',y')\Big).$$

3. The Beta-logarithmic random variable

In this section, we define a random variable associated to $\mathcal{BL}(a,b;x,y)$ and we provide some of its properties.

Definition 3.1. Let $a, b, x, y > 0$. The Beta-logarithmic random variable with parameters $(a,b;x,y)$ is a random variable with the following probability density function,

$$\mathcal{F}(t) := \begin{cases} \dfrac{a^{1-t}\,b^t\,t^{x-1}\,(1-t)^{y-1}}{\mathcal{BL}(a,b;x,y)} & \text{if} \quad 0<t<1 \\ \\ 0 & \text{otherwise} \end{cases}. \tag{3.1}$$

We have the following result.

Proposition 3.1. *Let X be a Beta-logarithmic random variable with parameters $(a,b;x,y)$. For any $r>0$ we have*

$$E(X^r) = \frac{\mathcal{BL}(a,b;x+r,y)}{\mathcal{BL}(a,b;x,y)}. \tag{3.2}$$

In particular the mean $E(X)$ of the distribution $\mathcal{F}$ is given by

$$E(X) = \frac{\mathcal{BL}(a,b;x+1,y)}{\mathcal{BL}(a,b;x,y)}. \tag{3.3}$$

Proof. For any $r > 0$ we have,

$$\begin{aligned} E(X^r) &= \int_0^1 x^r\, \mathcal{F}(t)\, dt \\ &= \frac{1}{\mathcal{BL}(a,b;x,y)} \int_0^1 a^{1-t}\, b^t\, t^{x+r-1}\, (1-t)^{y-1}\, dt \\ &= \frac{\mathcal{BL}(a,b;x+r,y)}{\mathcal{BL}(a,b;x,y)}. \end{aligned}$$

By setting $r = 1$ we get the mean in (3.3). □

We recall the following lemma, [3].

Lemma 3.1. *Let Y be a random variable taking values in the finite interval $[a;b]$. Then we have for all $\varepsilon \in [a,b]$,*

$$\left| P(Y \le \varepsilon) - \frac{b - E(Y)}{b-a} \right| \le \frac{1}{2} + \frac{\left|\varepsilon - \frac{a+b}{2}\right|}{b-a}. \tag{3.4}$$

Applying this lemma, we will show the following result.

Proposition 3.2. *Let X be the Beta-logarithmic random variable with parameters $(a,b;x,y)$. Then, the following estimations hold true for every $r, \varepsilon > 0$,*

$$\left| P(X \le \varepsilon) - \frac{\mathcal{BL}(a,b;x,y+1)}{\mathcal{BL}(a,b;x,y)} \right| \le \frac{1}{2} + |\varepsilon - \frac{1}{2}| \tag{3.5}$$

and

$$P(X^r \ge \varepsilon) \le \frac{\mathcal{BL}(a,b;x+r,y)}{\varepsilon\, \mathcal{B}(a,b;x,y)}. \tag{3.6}$$

Proof. By (3.3) we have $E(X) = \dfrac{\mathcal{BL}(a,b;x+1,y)}{\mathcal{BL}(a,b;x,y)}$, which when combined with the relation (2.5) allows us to write,

$$E(X) = 1 - \frac{\mathcal{BL}(a,b;x,y+1)}{\mathcal{BL}(a,b;x,y)}$$

This, with the inequality (3.4) in Lemma 3.1, yields (3.5).

The second inequality (3.6) can be easily deduced by an application of the Markov's inequality, namely

$$P(X^r \ge \varepsilon) \le \frac{E(X^r)}{\varepsilon}.$$

□

Finally, we will state another probability estimation for the Beta-logarithmic random variable. For this, we need to recall the following result, [6, Theorem 27].

Lemma 3.2. *Let X be a random variable with the probability density $f : [a,b] \longrightarrow [0,\infty)$. If $f \in L_r\big([a,b]\big)$ for some $r > 1$, then the inequalities*

$$\begin{aligned} \left| P(X \le \varepsilon) - \frac{b - E(X)}{b-a} \right| &\le \frac{s}{s+1}\, (b-a)^{1/s}\, \parallel f \parallel_r \left[\left(\frac{\varepsilon - a}{b-a}\right)^{\frac{1+s}{s}} + \left(\frac{b-\varepsilon}{b-a}\right)^{\frac{1+s}{s}} \right] \\ &\le \frac{s}{1+s} \parallel f \parallel_r\, (b-a)^{1/s}, \end{aligned} \tag{3.7}$$

hold for all $\varepsilon \in [a, b]$ *and* $s > 1$ *such that* $\frac{1}{r} + \frac{1}{s} = 1$*.*

Now, our last result reads as follows.

Proposition 3.3. *Let* $a, b > 0$*,* $r > 1$ *and* x, y *with* $x > 1 - 1/r$ *and* $y > 1 - 1/r$*. Let* X *be the Beta-logarithmic random variable with parameters* (a, b, x, y)*. Then the inequalities*

$$\left| P(X \le \varepsilon) - \frac{\mathcal{BL}(a, b; x, y + 1)}{\mathcal{BL}(a, b; x, y)} \right|$$
$$\le \frac{s}{s+1} \frac{\Big(\mathcal{BL}(a^r, b^r; r(x-1)+1, r(y-1)+1)\Big)^{1/r}}{\mathcal{BL}(a, b; x, y)} \Big[\varepsilon^{\frac{1+s}{s}} + (1-\varepsilon)^{\frac{1+s}{s}}\Big]$$
$$\le \frac{s}{1+s} \frac{\Big(\mathcal{BL}(a^r, b^r; r(x-1)+1, r(y-1)+1)\Big)^{1/r}}{\mathcal{BL}(a, b; x, y)} \tag{3.8}$$

hold true for all $\varepsilon \in [0, 1]$ *and* $s > 1$ *with* $\frac{1}{r} + \frac{1}{s} = 1$*.*

Proof. We have for every $r > 1$,

$$\| \mathcal{F} \|_r := \left(\int_0^1 \big(\mathcal{F}(t)\big)^r \, dt \right)^{1/r}$$
$$= \frac{1}{\mathcal{BL}(a, b; x, y)} \left(\int_0^1 a^{r(1-t)} \, b^{rt} \, t^{r(x-1)} (1-t)^{r(y-1)} \, dt \right)^{1/r}.$$

Since $r(x-1) + 1 > 0$ and $r(y-1) + 1 > 0$ then we get

$$\| \mathcal{F} \|_r = \frac{\Big(\mathcal{BL}(a^r, b^r; r(x-1)+1, r(y-1)+1)\Big)^{1/r}}{\mathcal{BL}(a, b; x, y)} < \infty. \tag{3.9}$$

This enable us to apply the inequalities (3.7) for obtaining

$$|P(X \le \varepsilon) - (1 - E(X))| \le \frac{s}{s+1} \| f \|_r \Big[\varepsilon^{\frac{1+s}{s}} + (1-\varepsilon)^{\frac{1+s}{s}}\Big] \le \frac{s}{1+s} \| f \|_r,$$

or, equivalently, (3.8) after substituting $\| f \|_r$ by its expression in (3.9). □

4. Conclusions

In this paper, a new parameterized Beta function which links between the classical beta function and the logarithmic mean is stated. It provides a common generalization of these two last mathematical objects. Some algebraic properties of this new extended function are developed as well as an application related to the probability area.

References

[1] M. Abramowitz and L. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Dover, New York, 1965.

[2] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Cambridge Univ. Press, Cambridge, 1999.

[3] N. S. Barnett and S. S. Dragomir, An inequality of Ostrowski's type for cumulative distribution functions, *RGMIA Research Report Collection*, **1**(1) (1998), 3-12.

[4] M. A. Chaudhry, A. Qadir, M. Rafique, and S. M. Zubair, Extension of Eulers Beta function, *J. Comput. Appl. Math.* **78**(1) (1997), 19-32.

[5] J. Choi, A. K. Rathie, and R. K. Parmar, Extension of extended Beta, hypergeometric and confluent hypergeometric functions, *Honam Mathematical J.* **36**(2) (2014), 357-385.

[6] S. S. Dragomir, R. P. Agarwal, and N. S. Barnett, Inequalities for Beta and Gamma functions via some classical and new integral inequalities, *J. Inequal. Appl.* **5**(2) (2000), 103-165.

[7] M. Raïssouli, On an approach in service of mean-inequalities, *J. Math. Inequal.*, **10**(1) (2016), 83-99.

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