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OPTIMIZATION OF SOURCE PARAMETERS IN NON-LOCAL BOUNDARY CONDITIONS OF A LARGE SYSTEM OF ODE

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Abstract. We solve the problem of optimizing the values of source parameters in nonlocal boundary conditions of a system of ordinary differential equations. The system consists of a large number of ODE subsystems with unseparated boundary conditions. We obtain necessary conditions for optimality with respect to source parameters and present the results of the conducted numerical experiments on the test problem.

1. Introduction

The paper proposes a numerical solution to the problem of optimizing the values of the parameters of external sources [4, 6] involved in non-local boundary conditions [3, 5, 10] of a system of ODE consisting of a large number of subsystems. Subsystems are interconnected in an arbitrary order only by their initial or final values. The values of external sources, affecting the functioning of the entire system as a whole are present in linear non-local conditions. Source parameters need to be optimized based on the given objective functional of the problem. So, the considered optimization problem is described by a large system of ODE with nonlocal boundary conditions.

We show the convexity and differentiability of the functional with respect to the optimized source parameters and obtain the necessary conditions for optimality with respect to these parameters. Two-level scheme is used for the numerical solution of the problem. The upper level is intended to solve the optimization problem [13, 15] using formulas for the components of the gradient of the objective functional with respect to the optimized parameters. The lower level is intended to solve direct and corresponding adjoint systems, which are large ODEs with boundary conditions, including unseparated initial and final values of phase variables of adjacent subsystems and values of parameters of external sources. The method for solving these boundary value problems is based on the idea of the approach proposed in [7] and is analogous to the sweep method [2, 5, 14, 16]. This approach and calculation formulas allow to carry out the sweep procedure for each condition of each ODE subsystem separately, regardless of other conditions and subsystems. This makes it possible to essentially parallelize computations in

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the case of multiple solution of the direct and adjoint boundary value problems at the stage of solving the optimization problem [1, 9].

2. Problem statement

We consider a large ODE system consisting of K number subsystems connected in an arbitrary order. The subsystems are interconnected by unseparated boundary conditions, which include only the states at their initial or final points. At some or all final (initial) points, external sources affect the functioning of the entire system as a whole.

We will designate the set of all initial and final points (we will call them nodes) of all subsystems by I, and their total number by N. Let $\underline{I}$ and $\bar{I}$ be the set of nodes adjacent to the i-th node from I, which are, the initial and final nodes of the corresponding subsystems, respectively and we will designate their number by $\underline{n}_i$ and $\bar{n}_i$, respectively, $\bar{n}_i + \underline{n}_i = n_i$, $i \in I$

It is obvious that

$$\sum_{i \in I} \underline{n}_i = \underline{n}, \quad \sum_{i \in I} \bar{n}_i = \bar{n}, \quad \underline{n} + \bar{n} = 2K, \quad \sum_{i \in I} n_i = 2K.$$

In practice, as a rule, the relation $n_i << N$, $i \in I$ is true, i.e. the number of subsystems adjacent to any subsystem is much less than the total number of subsystems of the entire system as a whole.

Suppose each of the subsystems be described by m linear ordinary differential equations

$$\frac{du^k(x)}{dx} = A^k(x)u^k(x) + f^k(x), \; x \in (0, l^k), \quad k = 1, 2, ..., K, \tag{2.1}$$

with M_i, $M_i \leq n_i \cdot m$ linearly independent boundary conditions given in the following unseparated form

$$\sum_{k \in \underline{I}} \underline{g}_j^k u^k(0) + \sum_{k \in \bar{I}} \bar{g}_j^k u^k(l^k) = \mathrm{v}_j^i, \quad j = \overline{1, M_i}, \quad i \in I. \tag{2.2}$$

Here the function $u^k(x) = u^k(x; \mathrm{v}) \in \mathrm{R}^m$ characterizes the phase state of the k-th subsystem at the point x of the segment with length l^k, $x \in \left[0, l^k\right]$; v is a vector of optimized parameters of external sources, $\mathrm{v} = \left(\mathrm{v}^i \in \Omega_i \subset \mathrm{R}^{M_i}, i \in I\right)$, $\mathrm{v}^i = \left(\mathrm{v}_1^i, ..., \mathrm{v}_{M_i}^i\right)^{\mathrm{T}}$, v_j^i is the j -th component of the external source acting on the i-th node. Let $M = \sum_{i=1}^N M_i$. In the problem given are: square matrix functions $A^k(x) \neq const$ and vector functions $f^k(x)-$ respectively of dimension m and continuous at $x \in [0, l^k]$; row vectors $\underline{g}_j^k = \left(\underline{g}_{j,1}^k, ..., \underline{g}_{j,m}^k\right) \in R^{\underline{n}_i \cdot m}$, $k \in \underline{I} = \{\underline{k}_1, ..., \underline{k}_{\underline{n}_i}\}$, $\bar{g}_j^k = \left(\bar{g}_{j1}^k, ..., \bar{g}_{jm}^k\right) \in R^{\bar{n}_i \cdot m}$, $k \in \bar{I} = \{\bar{k}_1, ..., \bar{k}_{\bar{n}_i}\}$, $j = \overline{1, M_i}$, $i \in I$.

Note that, the total number of differential equations in system (2.1) is equal to Km and the number of unseparated boundary conditions in (2.2) is equal to M and they must be equal to each other: $M = Km$.

Remark. The condition $A^k(x) \neq const$, $x \in \left[0, l^k\right]$ is important for the proposed approach to solving the considered problem. If $A^k(x) = const$, $k \in K$, then we can propose a much simpler approach to solve the problem compared to the one proposed below.

As solution to system (2.1) we mean vector-functions $u^k(x)$, $k = 1, 2, ..., K$, which are continuously differentiable for $x \in [0, l^k]$. We will assume that the boundary value problem (2.1), (2.2) has a unique solution. This, as is known from [18], depends only on the matrices $A^k(x)$, $k = 1, 2, ..., K$, vectors $\underline{g}_j^k$, $k = \overline{1, \underline{n}_i}$, $\bar{g}_j^k$, $k = \overline{1, \bar{n}_i}$, $j = \overline{1, M_i}$, $i \in I$, and does not depend on the other data involved in the problem, in particular, on the unknown vectors v.

Based on practical considerations, the following constraints are imposed on the values of the optimized parameters v^i, $i \in I$:

$$\mathrm{v}^i \in \Omega_i, i \in I. \tag{2.3}$$

We will assume that, the sets of admissible values Ω_i, $i \in I$, are convex and compact.

It is required to find such values for components of the vector v at which the functional

$$\Im(v) = \sum_{k=1}^{K} \int_0^{l^k} f_0^k(u^k(x), x)dx + \Phi(\underline{u}, \bar{u}, \mathrm{v}) \tag{2.4}$$

takes its minimum value. Here, the given functions $f_0^k(u^k, x)$, $\Phi(\underline{u}, \bar{u}, \mathrm{v})$ are continuously differentiable with respect to their arguments and the following designations are used:

$$u = u(x) = (u^1(x), u^2(x), ..., u^K(x)), \underline{u} = \left(\underline{u}^i : i \in I\right) \in \mathrm{R}^{\underline{n}},$$

$$\underline{u}^i = \left(u^{\underline{k}_1}(0), u^{\underline{k}_2}(0), ..., u^{\underline{k}_{\underline{n}_i}}(0)\right)^{\mathrm{T}} \in \mathrm{R}^{\underline{n}_i \cdot m}, \bar{u} = \left(\bar{u}^i : i \in I\right) \in \mathrm{R}^{\bar{n}},$$

$$\bar{u}^i = \left(u^{\bar{k}_1}(l^{\bar{k}_1}), u^{\bar{k}_2}(l^{\bar{k}_2}), ..., u^{\bar{k}_{\bar{n}_i}}(l^{\bar{k}_{\bar{n}_i}})\right)^{\mathrm{T}} \in \mathrm{R}^{\bar{n}_i \cdot m}.$$

The problem (2.1)–(2.4) can be attributed to the class of parametric optimal control problems. The optimized finite-dimensional vector v, which determines the parameters of external sources, has a small dimension in real problems, despite the large dimension of the ODE system (2.1) itself. The solution to the boundary value problem (2.1), (2.2) for given values of the vector v has a certain computational complexity. The complexity is due to non-separated (nonlocal) boundary conditions and, of course, the dimension of the system of differential equations (2.1) itself. Since to solve the optimization problem numerically, it is necessary to determine repeatedly the value of the functional at the current values of the vector v being optimized, and therefore to solve the boundary value problem (2.1), (2.2). Therefore, it is important to use both efficient methods for solving the optimization problem and pay special attention to solving the boundary value problem (2.1), (2.2).

3. Necessary optimality conditions

We'll study the convexity and differentiability of the functional (2.4), obtain formulas for the gradient of the functional and formulate the necessary optimality conditions with respect to the optimized parameters.

Theorem 3.1. *Suppose all the conditions imposed on the functions and parameters involved in problem* (2.1)-(2.4) *are satisfied. If the functions* $f_0^k(u^k(x),x)$,$k \in K$, $\Phi(\underline{u},\bar{u},\mathrm{v})$ *are convex in their arguments, then the functional* $\Im(\mathrm{v})$ *is convex in* v, *and if at least one of these functions is strongly convex, then the functional is strongly convex.*

Due to simplicity, we omit the proof of Theorem 3.1, which can be easily proved using the definition of convexity of a functional.[13]

Next, we study the differentiability of the functional (2.4) and obtain formulas for the components of its gradient with respect to the optimized vector v.

Let us introduce the following notation. Denote by $u^i = \left(\underline{u}^i, \bar{u}^i\right)^{\mathrm{T}}$ the extended vector of dimension $n_i \cdot m$. Let $G_i = \left(g_{js}^i\right)_{j=1,s=1}^{M_i,n_i \cdot m}$, $i \in I$ be an augmented matrix, each row of which is an extended row vector $g_j^i = (\underline{g}_j^i, \bar{g}_j^i)$ of dimension $n_i \cdot m$. According to the assumption of linear independence of conditions (2.2), we have the following equality

$$\mathrm{rank} G_i = M_i. \tag{3.1}$$

Since the matrix G_i has dimension $M_i \times (n_i \cdot m)$, $M_i \le n_i \cdot m$, $i \in I$, it is possible to extract an invertible submatrix (minor) $\widehat{G}_i$ from the matrix G_i with rank M_i. By changing the order of the columns, we'll again denote the augmented matrix by $G_i = [\widehat{G}_i, \breve{G}_i]$. Here $\breve{G}_i$ is a matrix made up of the columns of the augmented matrix not included in the matrix $\widehat{G}_i$. Similarly, the vector u^i is divided into M_i -dimensional vector $\widehat{u}^i = (\widehat{u}_1^i, ..., \widehat{u}_{M_i}^i)^{\mathrm{T}}$, corresponding to the matrix $\widehat{G}_i$, and $(n_i \cdot m - M_i)$-dimensional vector $\breve{u}^i = (\breve{u}_1^i, ...\breve{u}_{(n_i \cdot m)-M_i}^i)^{\mathrm{T}}$. Let $\widehat{\mu}_j$, $j = 1, ..., M_i$ be the numbers of the columns of the matrix G_i, included in the matrix $\widehat{G}_i$, and $\breve{\mu}_j$, $j = 1, ..., n_i \cdot m - M_i$ be the numbers of the columns of the matrix G_i, included in the matrix $\breve{G}_i$. So $\widehat{u}^i = (u_{\widehat{\mu}_1}^i, ..., u_{\widehat{\mu}_{M_i i}}^i)^{\mathrm{T}}$, $\breve{u}^i = (u_{\breve{\mu}_1}^i, ...u_{\breve{\mu}_{(n_i \cdot m)-M_i}}^i)^{\mathrm{T}}$.

Here and below, we'll mean the derivatives $\frac{\partial f_0^k}{\partial u^k}, \frac{\partial \Phi}{\partial \mathrm{v}}, \frac{\partial \Phi}{\partial \widehat{u}^i} = \left(\frac{\partial \Phi}{\partial u_{\widehat{\mu}_1}^i}, ..., \frac{\partial \Phi}{\partial u_{\widehat{\mu}_{M_i}}^i}\right)$, $\frac{\partial \Phi}{\partial \breve{u}^i} = \left(\frac{\partial \Phi}{\partial u_{\breve{\mu}_1}^i}, ..., \frac{\partial \Phi}{\partial u_{\breve{\mu}_{(n_i \cdot m)-M_i}}^i}\right)$ as row vectors of the corresponding dimension.

The following theorem holds.

Theorem 3.2. *Suppose the conditions imposed on the functions and parameters involved in problem* (2.1)-(2.4) *are satisfied. Then the functional* (2.4) *is differentiable, and the components of its gradient with respect to the optimized parameters* v^i, $i \in I$, *are determined by the formulas:*

$$\nabla_{\mathrm{v}^i} \Im(\mathrm{v}) = \left(\widehat{G}_i^{-1}\right)^{\mathrm{T}} \left(\left(\frac{\partial \Phi}{\partial \widehat{u}^i}\right)^{\mathrm{T}} + \widehat{\psi}^i\right) + \frac{\partial \Phi}{\partial \mathrm{v}}, \quad i \in I, \tag{3.2}$$

where continuously differentiable vector-functions $\psi^k(x) \in R^m, x \in [0, l^k]$, $k = 1, 2, ..., K$, *are the solutions to the following adjoint system of differential equations*

$$\frac{d\psi^k(x)}{dx} = \left(\frac{\partial f_0^k(u^k(x), x)}{\partial u^k}\right)^{\mathrm{T}} - (A^k(x))^{\mathrm{T}}\psi^k(x), x \in (0, l^k), k = 1, 2, ..., K, \quad (3.3)$$

with nonseparated boundary conditions

$$\left(\frac{\partial \Phi}{\partial \breve{u}^i}\right)^{\mathrm{T}} + \breve{\psi}^i - \left(\breve{G}_i\right)^{\mathrm{T}} \left(\overset{\frown}{G}{}_i^{-1}\right)^{\mathrm{T}} \left(\left(\frac{\partial \Phi}{\partial \overset{\frown}{u}{}^i}\right)^{\mathrm{T}} + \overset{\frown}{\psi}{}^i\right) = 0, \; i \in I, \quad (3.4)$$

where $\overset{\frown}{\psi}{}^i = (\psi^i_{\overset{\frown}{\mu}_1}, ..., \psi^i_{\overset{\frown}{\mu}_{M_i i}})^{\mathrm{T}}$, $\breve{\psi}^i = (\psi^i_{\breve{\mu}_1}, ...\psi^i_{\breve{\mu}_{(n_i \cdot m) - M_i}})^{\mathrm{T}}$.

It is interesting to note the following. First, the adjoint problem (3.3), (3.4) has the same specifics as the direct problem. Namely, the boundary conditions (3.4) are also nonseparated. Secondly, in the formulas for the components of the gradient of the functional with respect to sources acting on the $i-$ th point, as can be seen from (3.2), are involved the boundary values of the direct and adjoint variables, which are defined precisely at this point.

Proof. Using the increment of the vector being optimized, we prove differentiability of the functional (2.4) and determine the linear parts of the functional increment [13, 15].

Let the parameter vector v being optimized gets an increment. Denote: $\tilde{\mathrm{v}} = \mathrm{v} + \Delta\mathrm{v}$. Then the solutions to the boundary value problem (2.1), (2.2) will get increments:

$$\Delta u^k(x; \mathrm{v}) = u^k(x; \mathrm{v} + \Delta\mathrm{v}) - u^k(x; \mathrm{v}) = \tilde{u}^k(x; \tilde{\mathrm{v}}) - u^k(x; \mathrm{v}), \qquad k = 1, 2, ..., K,$$

Here $u^k(x; \mathrm{v})$ and $\tilde{u}^k(x; \tilde{\mathrm{v}})$, $k = 1, 2, ..., K$ are the solutions to the boundary value problems (2.1), (2.2) for optimized parameters v and $\tilde{\mathrm{v}} = \mathrm{v} + \Delta\mathrm{v}$, respectively.

It is easy to show that, $\Delta u^k(x; \mathrm{v})$, $k = 1, 2, ..., K$ are the solutions to the following system of boundary value problems:

$$\frac{d\Delta u^k(x)}{dx} = A^k(x)\Delta u^k(x), k = 1, 2, ..., K, \quad (3.5)$$

$$\sum_{k \in \underline{I}} \underline{g}_j^k \Delta u^k(0) + \sum_{k \in \bar{I}} \bar{g}_j^k \Delta u^k(l^k) = \Delta v_j^i, \quad j = \overline{1, M_i}, \quad i \in I. \quad (3.6)$$

Then for the increment of the functional (2.4) we have

$$\Delta\Im(\mathrm{v}) = \Im(\tilde{\mathrm{v}}) - \Im(\mathrm{v}) = \sum_{k=1}^{K} \int_0^{l^k} \frac{\partial f_0^k}{\partial u^k} \Delta u^k(x) dx +$$

$$+ \frac{\partial \Phi}{\partial \underline{u}} \Delta\underline{u} + \frac{\partial \Phi}{\partial \bar{u}} \Delta\bar{u} + \frac{\partial \Phi}{\partial \mathrm{v}} \Delta\mathrm{v} + \eta, \quad (3.7)$$

$$\eta = o\left(\|\Delta u(x)\|_{L_2^M[0,l]}, \|\Delta\underline{u}\|_{\mathrm{R}^{\underline{n}}}, \|\Delta\bar{u}\|_{\mathrm{R}^{\bar{n}}}, \|\Delta\mathrm{v}\|_{\mathrm{R}^M}\right). \quad (3.8)$$

Here we denoted: $f_0^k = f_0^k(u^k(x), x)$, $\Phi = \Phi(\underline{u}, \bar{u}, \mathrm{v})$, η is the remainder term in the corresponding spaces of functions and finite-dimensional vectors.

As is known from the theory of differential equations [17, 18], under the assumptions made on the data involved in the problem, the following estimation takes place:

$$\|\Delta u(x)\|_{L_2^M[0,l]} \leq O\left(\|\Delta \mathrm{v}\|_{\mathrm{R}^M}\right),$$

and, therefore, we have the following estimations:

$$\|\Delta \underline{u}\|_{\mathrm{R}^{\underline{n}}} = \|\Delta u(0)\|_{\mathrm{R}^{\underline{n}}} \leq O\left(\|\Delta \mathrm{v}\|_{\mathrm{R}^M}\right),$$
$$\|\Delta \bar{u}\|_{\mathrm{R}^{\bar{n}}} = \|\Delta u(l)\|_{\mathrm{R}^{\bar{n}}} \leq O\left(\|\Delta \mathrm{v}\|_{\mathrm{R}^M}\right).$$

Then from (3.8) we have the main estimation

$$\eta = o\left(\|\Delta \mathrm{v}\|_{\mathrm{R}^M}\right),$$

whence it follows that the functional $\Im(\mathrm{v})$ is differentiable with respect to v.

Now we obtain formulas for the components of the functional gradient with respect to v. To do this, we shift the right-hand sides of Eqs. (3.5) to the left and multiply the equalities by as yet arbitrary $m-$ dimensional vector functions $\psi^k(x) \in R^m$, $x \in (0, l^k)$, $k = 1, 2, ..., K$, which are continuously differentiable with respect to their arguments. Let sum the obtained expressions equal to zero and integrate this sum by parts:

$$0 = \sum_{k=1}^{K} \int_0^{l^k} \left[(\psi^k(x))^{\mathrm{T}} \left(\frac{d\Delta u^k(x)}{dx} - A^k(x) \Delta u^k(x) \right) \right] dx =$$

$$= \left[\sum_{k \in \bar{I}_i} (\psi^k(l^k))^{\mathrm{T}} \Delta u^k(l^k) - \sum_{k \in \underline{I}_i} (\psi^k(0))^{\mathrm{T}} \Delta u^k(0) \right] -$$

$$- \sum_{k=1}^{K} \int_0^{l^k} \left[\left(\left(\frac{d\psi^k(x)}{dx} \right)^{\mathrm{T}} + (\psi^k(x))^{\mathrm{T}} A^k(x) \right) \Delta u^k(x) \right] dx. \tag{3.9}$$

Adding the right side of (3.9) to (3.7) and grouping, we obtain:

$$\Delta \Im(\mathrm{v}) = \sum_{k=1}^{K} \int_0^{l^k} \left[\frac{\partial f_0^k}{\partial u^k} - \left(\frac{d\psi^k(x)}{dx} \right)^{\mathrm{T}} - (\psi^k(x))^{\mathrm{T}} A^k(x) \right] \Delta u^k(x) dx +$$

$$+ \frac{\partial \Phi}{\partial \mathrm{v}} \Delta \mathrm{v} + \left(\frac{\partial \Phi}{\partial \underline{u}} - (\underline{\psi})^{\mathrm{T}} \right) \Delta \underline{u} + \left(\frac{\partial \Phi}{\partial \bar{u}} + (\bar{\psi})^{\mathrm{T}} \right) \Delta \bar{u} + \eta, \tag{3.10}$$

where $\underline{\psi} = \left(\underline{\psi}^i : i \in I\right) \in \mathrm{R}^{\underline{n}}$, $\bar{\psi} = \left(\bar{\psi}^i : i \in I\right) \in \mathrm{R}^{\bar{n}}$, $\underline{\psi}^i = (\psi^{\underline{k}_1}(0), ..., \psi^{\underline{k}_{n_i}}(0))^{\mathrm{T}}$, $\bar{\psi}^i = (\psi^{\bar{k}_1}(l^{\bar{k}_1}), ..., \psi^{\bar{k}_{\bar{n}_i}}(l^{\bar{k}_{\bar{n}_i}}))^{\mathrm{T}}$.

Let us investigate the third and fourth terms in (3.10).

For simplicity of presentation of the calculations given below, instead of matrix and vector operations, we will use their component-wise notation. We'll write the conditions (3.6) in the following form:

$$\begin{pmatrix} \underline{g}_{11}^{\underline{k}_1} \cdots \underline{g}_{1m}^{\underline{k}_1} & \cdots & \underline{g}_{1\underline{n}_i}^{\underline{k}_{\underline{n}_i}} \cdots \underline{g}_{1\underline{n}_i \cdot m}^{\underline{k}_{\underline{n}_i}} \\ \cdots & \cdots & \cdots \\ \underline{g}_{M_i,1}^{\underline{k}_1} \cdots \underline{g}_{M_i,m}^{\underline{k}_1} & \cdots & \underline{g}_{M_i,\underline{n}_i}^{\underline{k}_{\underline{n}_i}} \cdots \underline{g}_{M_i,\underline{n}_i \cdot m}^{\underline{k}_{\underline{n}_i}} \end{pmatrix} \begin{pmatrix} \Delta \underline{u}_1^i \\ \cdots \\ \Delta \underline{u}_{\underline{n}_i \cdot m}^i \end{pmatrix} +$$

$$+\begin{pmatrix} \bar{g}_{11}^{\bar{k}_1}...\bar{g}_{1m}^{\bar{k}_1} \;....\; \bar{g}_{1\bar{n}_i}^{\bar{k}_{\bar{n}_i}}...\bar{g}_{1\bar{n}_i\cdot m}^{\bar{k}_{\bar{n}_i}} \\ ... \; ... \; ... \\ \bar{g}_{M_i,1}^{\bar{k}_1}...\bar{g}_{M_i,m}^{\bar{k}_1} \;...\; \bar{g}_{M_i,\bar{n}_i}^{\bar{k}_{\bar{n}_i}}...\bar{g}_{M_i,\bar{n}_i\cdot m}^{\bar{k}_{\bar{n}_i}} \end{pmatrix} \begin{pmatrix} \Delta\bar{u}_1^i \\ ... \\ \Delta\bar{u}_{\bar{n}_i\cdot m}^i \end{pmatrix} = \begin{pmatrix} \Delta \mathrm{v}_1^i \\ ... \\ \Delta \mathrm{v}_{M_i}^i \end{pmatrix}.$$

Using the notation made above, the relations (3.6) take the following form:

$$\begin{pmatrix} g_{1,1}^i & \cdots & g_{1,(n_i\cdot m)}^i \\ ... & ... & ... \\ g_{M_i,1}^i & \cdots & g_{M_i,(n_i\cdot m)}^i \end{pmatrix} \begin{pmatrix} \Delta u_1^i \\ ... \\ \Delta u_{n_i\cdot m}^i \end{pmatrix} = \begin{pmatrix} \Delta \mathrm{v}_1^i \\ ... \\ \Delta \mathrm{v}_{M_i}^i \end{pmatrix}, i \in I, \tag{3.11}$$

or matrix form as follows:

$$G_i \Delta u^i = \Delta \mathrm{v}^i, \;\; i \in I.$$

Then we can rewrite (3.11) as follows:

$$\begin{pmatrix} \overset{\frown}{g}{}_{1,1}^i & \cdots & \overset{\frown}{g}{}_{1,M_i}^i \\ ... & ... & ... \\ \overset{\frown}{g}{}_{M_i,1}^i & \cdots & \overset{\frown}{g}{}_{M_i,M_i}^i \end{pmatrix} \begin{pmatrix} \Delta \overset{\frown}{u}{}_1^i \\ ... \\ \Delta \overset{\frown}{u}{}_{M_i}^i \end{pmatrix} +$$

$$+\begin{pmatrix} \breve{g}_{1,1}^i & \cdots & \breve{g}_{1,(n_i\cdot m-M_i)}^i \\ ... & ... & ... \\ \breve{g}_{M_i,1}^i & \cdots & \breve{g}_{M_i,(n_i\cdot m-M_i)}^i \end{pmatrix} \begin{pmatrix} \Delta\breve{u}_1^i \\ ... \\ \Delta\breve{u}_{(n_i\cdot m)-M_i}^i \end{pmatrix} = \begin{pmatrix} \Delta \mathrm{v}_1^i \\ ... \\ \Delta \mathrm{v}_{M_i}^i \end{pmatrix}$$

or in the following form:

$$\overset{\frown}{G}_i \,\Delta\, \overset{\frown}{u}{}^i + \breve{G}_i \Delta\breve{u}^i = \Delta v^i, \quad i \in I. \tag{3.12}$$

Taking into account (3.11), we obtain $\overset{\frown}{G}_i$ is an invertible matrix. Then from (3.12) we have

$$\Delta\, \overset{\frown}{u}{}^i = -\left(\overset{\frown}{G}_i\right)^{-1} \breve{G}_i \Delta\breve{u}^i + \overset{\frown}{G}_i{}^{-1} \Delta \mathrm{v}^i, \;\; i \in I. \tag{3.13}$$

According to (3.13), we will consider that M_i-dimensional vector of increments $\Delta\, \overset{\frown}{u}{}^i = (\Delta\, \overset{\frown}{u}{}_1^i, ..., \Delta\, \overset{\frown}{u}{}_{M_i}^i)^{\mathrm{T}} = (\Delta u_{\overset{\frown}{\mu}_1}^i, ..., \Delta u_{\overset{\frown}{\mu}_{M_i}}^i)^{\mathrm{T}}$ are independent, and the $(n_i m - M_i)$-dimensional vector $\Delta\breve{u}^i = (\Delta\breve{u}_1^i, ...\Delta\breve{u}_{n_i\cdot m-M_i}^i)^{\mathrm{T}} = (\Delta u_{\breve{\mu}_1}^i, ..., \Delta u_{\breve{\mu}_{M_i}}^i)^{\mathrm{T}}$ are dependent. We take into account (3.13) in (3.10):

$$\sum_{i\in I}\left(\left(\frac{\partial\Phi}{\partial \underline{u}^i} - (\underline{\psi}^i)^{\mathrm{T}}\right)\Delta\underline{u}^i + \left(\frac{\partial\Phi}{\partial\bar{u}^i} + (\bar{\psi}^i)^{\mathrm{T}}\right)\Delta\bar{u}^i\right) =$$

$$= \sum_{i\in I}\left(\frac{\partial\Phi}{\partial u^i} + (\psi^i)^{\mathrm{T}}\right)\Delta u^i = \sum_{i\in I}\left(\frac{\partial\Phi}{\partial\, \overset{\frown}{u}{}^i} + (\overset{\frown}{\psi}{}^i)^{\mathrm{T}}\right)\Delta\, \overset{\frown}{u}{}^i +$$

$$+ \sum_{i\in I}\left(\frac{\partial\Phi}{\partial\breve{u}^i} + (\breve{\psi}^i)^{\mathrm{T}}\right)\Delta\breve{u}^i = \sum_{i\in I}\left(\frac{\partial\Phi}{\partial\breve{u}^i} + (\breve{\psi}^i)^{\mathrm{T}}\right)\Delta\breve{u}^i -$$

$$- \sum_{i\in I}\left(\frac{\partial\Phi}{\partial\, \overset{\frown}{u}{}^i} + (\overset{\frown}{\psi}{}^i)^{\mathrm{T}}\right)\overset{\frown}{G}_i{}^{-1}\breve{G}_i\Delta\breve{u}^i + \sum_{i\in I}\left(\frac{\partial\Phi}{\partial\, \overset{\frown}{u}{}^i} + (\overset{\frown}{\psi}{}^i)^{T}\right)\overset{\frown}{G}_i{}^{-1}\Delta v^i.$$

Taking into account this equality in (3.10), for the increment of the functional (2.4) we finally obtain:

$$\Delta\Im(\mathrm{v}) = \sum_{k=1}^{K}\int_{0}^{l^k}\left[\frac{\partial f_0^k}{\partial u^k} - \left(\frac{d\psi^k(x)}{dx}\right)^{\mathrm{T}} - (\psi^k(x))^T A^k(x)\right]\Delta u^k(x)dx +$$

$$+\sum_{i\in I}\left[\frac{\partial \Phi}{\partial \mathrm{v}^i} + \left(\frac{\partial \Phi}{\partial \widehat{u}^i} + (\widehat{\psi}^i)^{\mathrm{T}}\right)\widehat{G}_i^{\,-1}\right]\Delta \mathrm{v}^i + \tag{3.14}$$

$$+\sum_{i\in I}\left[\frac{\partial \Phi}{\partial \breve{u}^i} + (\breve{\psi}^i)^{\mathrm{T}} - \left(\frac{\partial \Phi}{\partial \widehat{u}^i} + (\widehat{\psi}^i)^{\mathrm{T}}\right)\widehat{G}_i^{\,-1}\breve{G}_i\right]\Delta \breve{u}^i + \eta.$$

Using the arbitrariness of vector functions $\psi^k(x) \in R^m,\ k = 1, 2, ..., K$, we require expressions be equal to zero in square brackets (multipliers $\Delta u^k$and $\Delta \breve{u}^i$). We obtain the boundary value problem (3.3), (3.4) with respect to vector functions $\psi^k(x),\ k = 1, 2, ..., K$, which we will call adjoint with respect to the problem (2.1) - (2.2). The required components of the gradient of the functional $\Im(\mathrm{v})$ will be determined by the linear parts of the functional increment (3.14) according to the increments $\Delta \mathrm{v}^i$.

Thus, Theorem 3.2 is proved.

Let us formulate the optimality conditions [4, 8, 11, 12] for the problem (2.1)–(2.4) in variational form in the following theorem.

Theorem 3.3. *Suppose that the conditions imposed on the functions and parameters involved in problem* (2.1)–(2.4) *is satisfied. For optimality of the parameters* $\mathrm{v}^{i*}\in \Omega_i,\ i \in I$, *it is necessary and sufficient that*

$$\left(\mathrm{grad}_{\mathrm{v}^i}\Im(\mathrm{v}),\ \ \mathrm{v}^i - \mathrm{v}^{i*}\right) \geq 0$$

for all admissible values of the parameters $\mathrm{v}^{i*}\in \Omega_i,\ i \in I$.

The proof of the Theorem 3.3 follows from the convexity and differentiability of the functional of the problem, the convexity and compactness of admissible sets $\Omega_i,\ \ i \in I$ [13, 15].

In many practical applications, external sources are not involved in all nodes of the system, as well as in some nodes, some or all of the values of the source parameters can be given and not optimized. In these cases, the corresponding components of the gradients of the functional are not calculated.

4. The scheme of the numerical solution to the problem

In this section, we propose a numerical scheme for the solution to the problem (2.1) - (2.4). Note that, the solution to the problem (2.1) - (2.4) requires consideration of problems in two levels: the problem of minimizing the functional (upper level) [3, 6, 8, 11, 13, 15] and the boundary value problem (direct and adjoint) with respect to systems of ODEs with nonseparated boundary conditions (lower level) [1, 2, 5, 7, 14, 16].

Upper level. Suppose admissible sets Ω_i have a simple structure (ball, parallelepiped, etc.). To determine the optimal values for v, applying formula (3.2) to compute the gradient components of the functional of problem (2.1) – (2.4), one can use effective first-order optimization methods, for example, the gradient projection method [13, 15]:

$$\mathrm{v}^{t+1} = P_{\Omega_i}\left[\mathrm{v}^t - \alpha_t \nabla_{\mathrm{v}^i} \Im(\mathrm{v}^t)\right], \quad i \in I, \quad t = 1, 2, ... \tag{4.1}$$

Here $P_{\Omega_i}[\bullet]$ is the projection operator of an arbitrary point v^i onto an admissible set Ω_i, $\alpha_t \geq 0$ is a step of one-dimensional minimization.

It is necessary to compute the components of the gradient of the functional $\Im(\mathrm{v})$ for the current values of the vector v at each iteration of the procedure (4.1). For this purpose, the direct boundary value problem (2.1), (2.2) is first solved, and then – the adjoint boundary value problem (3.3), (3.4). The results of the solution are substituted into formula (3.2) to compute the components of the functional gradient.

Lower level. The direct (2.1), (2.2) and adjoint (3.3), (3.4) boundary value problems are two-point problems of large dimension. To solve them, we propose an approach based on the use of the operation proposed in [7] for shifting unseparated boundary conditions of direct and adjoint initial boundary value problems. The approach allows us to shift each variable value from one end to the other in each condition separately. As a result, it will be necessary to solve an algebraic system of equations with a weakly and arbitrarily filled matrix with respect to the values of all the variables of the problem at one of the ends, and then obtained Cauchy problems are solved separately for each subsystem. The scheme for solving these boundary value problems makes it easy to parallelize the computational process with respect to each variable in each condition (2.2), (3.4), and these processes are carried out separately for each subsystem (2.1), (3.3).

5. The results of numerical experiments

We present the results of numerical experiments obtained by the solution to the following problem of optimization the values of source parameters for a system (Fig. 1),in which

$$N = 4,\ \ K = 3,\ \ m = 2,\ \ M = 6, I = \{1,2,3,4\}, \underline{I}_1 = \emptyset,\ \bar{I}_1 = \{2\}, \underline{I}_2 = \{1,3\},$$
$$\bar{I}_2 = \{4\}, \underline{I}_3 = \emptyset,\ \ \bar{I}_3 = \{2\}\ \underline{I}_4 = \{2\},\ \bar{I}_4 = \emptyset,\ \ l^k = 1,\ \ k \in \{[1],[2],[3]\}.$$

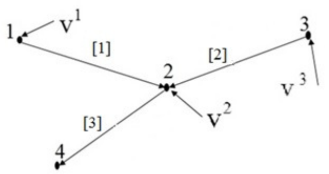

FIGURE 1. The scheme of the system.

The system consists of three subsystems of two-dimensional ODEs:

$$\begin{cases} \frac{du_1^{[1]}}{dx} = u_2^{[1]} + x^2 - 3x - 2, & \frac{du_2^{[1]}}{dx} = xu_1^{[1]} - 2u_2^{[1]} + 5x + 3, \\ \frac{du_1^{[2]}}{dx} = u_2^{[2]} + 4 - 4x^2 + x, & \frac{du_2^{[2]}}{dx} = u_1^{[2]} - xu_2^{[2]} + 5 + 3x^2 - 2x, \\ \frac{du_1^{[3]}}{dx} = u_2^{[3]} - x - 5, & \frac{du_2^{[3]}}{dx} = u_1^{[3]} - u_2^{2,4} + 5. \end{cases} \tag{5.1}$$

The subsystems with numbers [1], [2], [3] are interconnected by initial and / or final (boundary) values of phase states in the form (2.2). Each node of 1,3,4 has one condition, and node 2 has three conditions $M_1 = 1,\ M_2 = 3,\ M_3 = 1,\ M_4 = 1$. In total, 6 conditions are given, of which three are nonseparated:

$$\begin{array}{lll} u_1^{[1]}(0) = \mathrm{v}^1, & u_2^{[2]}(1) - u_2^{[3]}(0) = \mathrm{v}_1^2, & u_2^{[1]}(1) - u_2^{[2]}(1) = \mathrm{v}_2^2 \\ u_1^{[1]}(1) + u_1^{[2]}(1) + u_1^{[3]}(0) = \mathrm{v}_3^2, & u_1^{[2]}(0) = \mathrm{v}^3, & u_2^{[3]}(1) = 0. \end{array} \tag{5.2}$$

Thus, in conditions (5.2) the matrices $G_i,\ i = 1,3,4$, have the dimension $M_1 \times m = 1 \times 2$, and the dimension of G_2 is equal to $M_2 \times n_2 m = 3 \times 6$,

$$G_1 = \begin{pmatrix} \underline{g}_{1,1}^{[1]} & \underline{g}_{1,2}^{[1]} \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

$$G_2 = \begin{pmatrix} \underline{g}_{1,1}^{[3]} & \underline{g}_{1,2}^{[3]} & \bar{g}_{1,1}^{[1]} & \bar{g}_{1,2}^{[1]} & \bar{g}_{1,1}^{[2]} & \bar{g}_{1,2}^{[2]} \\ \underline{g}_{2,1}^{[3]} & \underline{g}_{2,2}^{[3]} & \bar{g}_{2,1}^{[1]} & \bar{g}_{2,2}^{[1]} & \bar{g}_{2,1}^{[2]} & \bar{g}_{2,2}^{[2]} \\ \underline{g}_{3,1}^{[3]} & \underline{g}_{3,2}^{[3]} & \bar{g}_{3,1}^{[1]} & \bar{g}_{3,2}^{[1]} & \bar{g}_{3,1}^{[2]} & \bar{g}_{3,2}^{[2]} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix},$$

$$G_3 = \begin{pmatrix} \underline{g}_{1,1}^{[2]} & \underline{g}_{1,2}^{[2]} \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}, G_4 = \begin{pmatrix} \bar{g}_{1,1}^{[3]} & \bar{g}_{1,2}^{[3]} \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}. \tag{5.3}$$

As can be seen from (5.1), (5.2), the sources acting on the nodes 1 and 3 have one-parameter per each, those on node 2 have three-parameters, and node 4 has no external sources: $\mathrm{v}^4 \equiv 0$, i.e. $\mathrm{v}^1,\ \mathrm{v}^3 \in R^1,\ \ \mathrm{v}^2 \in R^3$.

We assume that the parameters of the sources at the predetermined three nodes of the set I are unknown and it is required to determine them by minimizing the functional

$$\Im(\mathrm{v}) = \int_0^1 \left[\left(u_2^{[1]}(x) - 2x - 1\right)^2 + \left(u_2^{[2]}(x) - 3x\right)^2 + \left(u_2^{[3]}(x) - x - 3\right)^2 \right] dx. \tag{5.4}$$

The exact solution to the problem (5.1)-(5.4) is: $\mathrm{v}^{1*} = (1),\ \mathrm{v}^{2*} = (0,0,-1),\ \mathrm{v}^{3*} = (-2),\ u_1^{[1]^*}(x) = -x + 1,\ u_2^{[1]^*}(x) = 2x + 1,\ \ u_1^{[2]^*}(x) = 2x - 2,\ u_2^{[2]^*}(x) = 3x,\ \ u_1^{[3]^*}(x) = x - 1,\ \ u_2^{[3]^*}(x) = x + 3$, and $\Im(\mathrm{v}^*) = 0$.

There are no constraints on controls and source parameters in the problem. Therefore, at the upper level, to solve the optimization problem, one can use unconstrained optimization methods, in particular, the conjugate gradient method [13, 15].

According to formulas (3.3), the adjoint boundary-value problem has the form:

$$\begin{cases} \frac{d\psi_1^{[1]}}{dx} = -x\psi_2^{[1]}, & \frac{d\psi_2^{[1]}}{dx} = 2[u_2^{[1]}(x) - 2x - 1] - \psi_1^{[1]} + 2\psi_2^{[1]}, \\ \frac{d\psi_1^{[2]}}{dx} = -\psi_2^{[2]}, & \frac{d\psi_2^{[2]}}{dx} = 2[u_2^{[2]}(x) - 3x] - \psi_1^{[2]} + x\psi_2^{[2]}, \\ \frac{d\psi_1^{[3]}}{dx} = -\psi_2^{[3]}, & \frac{d\psi_2^{[3]}}{dx} = 2[u_2^{[3]}(x) - x - 3] - \psi_1^{[3]} + \psi_2^{[3]}. \end{cases} \tag{5.5}$$

Taking into account (5.3), the submatrices of the matrix $G_i,\ \ i = 1,3,4$, consist of one element. Then, according to (3.4), the inverse matrices to singleton matrices $\widehat{G}_1 = (1)$, $\widehat{G}_3 = (1)$, $\widehat{G}_4 = (1)$ coincide with the matrices themselves $\widehat{G}_i,\ i = 1,3,4$, and $\breve{G}_1 = (0)$,$\breve{G}_3 = (0)$, $\breve{G}_4 = (0)$. A submatrix $\widehat{G}_2$ of rank $M_2 = 3$ can be extracted from the matrix G_2, for example,

$$\widehat{G}_2=\begin{pmatrix} \underline{g}_{1,1}^{[3]} & \bar{g}_{1,2}^{[1]} & \bar{g}_{1,2}^{[2]} \\ \underline{g}_{2,1}^{[3]} & \bar{g}_{2,2}^{[1]} & \bar{g}_{2,2}^{[2]} \\ \underline{g}_{3,1}^{[3]} & \bar{g}_{3,2}^{[1]} & \bar{g}_{3,2}^{[2]} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\breve{G}_2 = \begin{pmatrix} \underline{g}_{1,2}^{[3]} & \bar{g}_{1,1}^{[1]} & \bar{g}_{1,1}^{[2]} \\ \underline{g}_{2,2}^{[3]} & \bar{g}_{2,1}^{[1]} & \bar{g}_{2,1}^{[2]} \\ \underline{g}_{3,2}^{[3]} & \bar{g}_{3,1}^{[1]} & \bar{g}_{3,1}^{[2]} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Then the components of the vectors $\widehat{u}^i$, $i = 1, 3, 4$ corresponding to the columns of the submatrices $\widehat{G}_i$, $i = 1, 3, 4$, are: $\widehat{u}^1 = \left(u_1^{[1]}(0)\right)$, $\widehat{u}^3 = \left(u_1^{[2]}(0)\right)$, $\widehat{u}^4 = \left(u_1^{[3]}(1)\right)$, and the vectors corresponding to the columns of the submatrices $\breve{G}_i$, $i = 1, 3, 4$ are the vectors: $\breve{u}^1 = \left(u_2^{[1]}(0)\right)$, $\breve{u}^3 = \left(u_2^{[2]}(0)\right)$, $\breve{u}^4 = \left(u_2^{[3]}(1)\right)$. The vector $\widehat{u}^2 = \left(u_1^{[3]}(0),\ u_2^{[1]}(1),\ u_2^{[2]}(1)\right)^{\mathrm{T}}$ corresponds to the columns of the submatrices $\widehat{G}_2$ and the vector corresponding to the columns of the submatrix $\breve{G}_2$ is $\breve{u}^2 = \left(u_2^{[3]}(0),\ u_1^{[1]}(1),\ u_1^{[2]}(1)\right)^{\mathrm{T}}$. Then the boundary conditions at all nodes for the adjoint system (5.5) can be obtained according to (3.4) from the following relations:

$$\left(\breve{G}_1\right)^{\mathrm{T}} \left(\widehat{G}_1^{-1}\right)^{\mathrm{T}} \psi_1^{[1]}(0) = \psi_2^{[1]}(0),$$

$$\left(\breve{G}_2\right)^{\mathrm{T}} \left(\widehat{G}_2^{-1}\right)^{\mathrm{T}} \begin{pmatrix} -\psi_1^{[3]}(0) \\ \psi_2^{[1]}(1) \\ \psi_2^{[2]}(1) \end{pmatrix} = \begin{pmatrix} -\psi_2^{[3]}(0) \\ \psi_1^{[1]}(1) \\ \psi_1^{[2]}(1) \end{pmatrix}$$

$$\left(\breve{G}_2\right)^{\mathrm{T}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \left(\widehat{G}_2^{-1}\right)^{\mathrm{T}} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\left(\breve{G}_3\right)^{\mathrm{T}} \left(\widehat{G}_3^{-1}\right)^{\mathrm{T}} \psi_1^{[2]}(0) = \psi_2^{[2]}(0), \left(\breve{G}_4\right)^{\mathrm{T}} \left(\widehat{G}_4^{-1}\right)^{\mathrm{T}} \psi_1^{[3]}(1) = \psi_2^{[3]}(1).$$

Hence we have:

$$\psi_2^{[1]}(0) = 0, \qquad \psi_2^{[2]}(0) = 0, \qquad \psi_2^{[3]}(1) = 0,$$

$$-\psi_2^{[1]}(1) - \psi_2^{[2]}(1) = -\psi_2^{[3]}(0), \quad -\psi_1^{[3]}(0) = \psi_1^{[1]}(1), \quad -\psi_1^{[3]}(0) = \psi_1^{[2]}(1). \tag{5.6}$$

The gradient of the functional (5.4) according to (3.2) is determined by the formulas:

$$\nabla_{\mathrm{v}^1} \Im(\mathrm{v}) = -\psi_1^{[1]}(0),$$

$$\nabla_{\mathrm{v}^2} \Im(\mathrm{v}) = \left(\widehat{G}_2^{-1}\right)^{\mathrm{T}} \begin{pmatrix} -\psi_1^{[3]}(0) \\ \psi_2^{[1]}(1) \\ \psi_2^{[2]}(1) \end{pmatrix} = \begin{pmatrix} \psi_2^{[1]}(1) + \psi_2^{[2]}(1) \\ \psi_2^{[1]}(1) \\ -\psi_1^{[3]}(0) \end{pmatrix},$$

$$\nabla_{\mathrm{v}^3} \Im(\mathrm{v}) = -\psi_1^{[2]}(0), \tag{5.7}$$

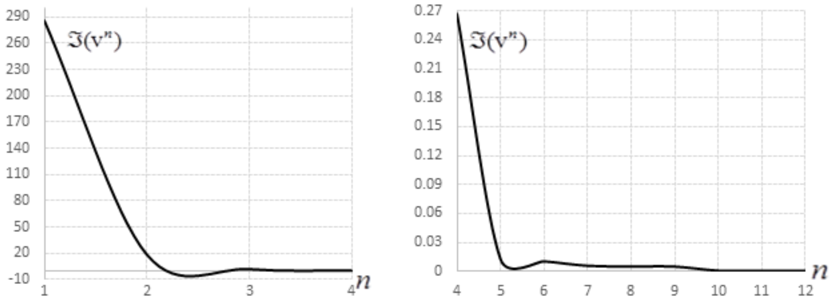

FIGURE 2. Values of the functional on iterations of the conjugate gradient method.

We used the sweep method proposed in [7] to solve the direct and adjoint boundary value problems. Auxiliary Cauchy problems used in the sweep method were solved by the fourth-order Runge–Kutta method with step $h = 0,01$ [16].

The table shows the values of the optimal parameters v^* obtained by the conjugate gradient projection method with a given accuracy 10^{-5} in terms of the functional for four different initial values of the parameters v^0. n number of iterations.

Table. The results of the numerical solution to the problem (5.1)–(5.4) using different initial values of the parameters v^0.

	v^0	v^*	$\Im(\mathrm{v}^0)$	$\Im(\mathrm{v}^*)$	n
1	(1; (0; 0; -1.7); -2.8)	(1; (0; 0; -0.999); -1.999)	0.125	$9.46 \cdot 10^{-9}$	21
2	(1; (2; 2; 1.5); 3)	(0.999; (-0.0001; 0; -1); -2)	26.058	$5.96 \cdot 10^{-9}$	18
3	(3; (3; 3; 3); 3)	(0.999; (0; 0; -1.0002); -2)	42.217	$1.91 \cdot 10^{-9}$	31
4	(3; (-2; 5; -2); -4)	(1; (0; 0; -1.0001); -2)	12.639	$1.04 \cdot 10^{-9}$	16
5	(-5; (-4; 5; 5); -4)	(0.998; (0; 0; -1.002); -2)	285.536	$4.79 \cdot 10^{-8}$	17

As can be seen from the table, the obtained values of the parameters of external sources and of the functional to be minimized are quite close for various initial optimization points used. It can be seen from the fig.2 that already at the third iteration of the conjugate gradient method, the value of the functional is close to zero. This indicates that the problem of optimizing the parameters of nonlocal boundary conditions is well conditioned, we have considered at least within the framework of not only this test problem, but also other problems.

6. Conclusion

We study an optimization problem described by ODE system consisting of large numbers of subsystems. Subsystems are interconnected in an arbitrary order only by nonseparated initial or final values of phase variables. The right sides of the boundary conditions involve the values of parameters of external sources that are optimized with respect to a given objective quality functional. In the paper, we investigate the convexity and differentiability of the objective functional, and obtain the optimality conditions for the values of the source parameters. It is shown that, the adjoint problem has the same specifics as the direct problem, and in the expressions for the components of the functional gradient with respect to the source parameters participate the boundary values of the direct and adjoint variables, defined only at the corresponding nodes. We use an approach to the numerical solution to the corresponding boundary value problems based on the sweep method when solving direct and adjoint boundary value problems, which significantly increases the efficiency of solving the optimization problem as a whole, because of allowing parallelizing the sweep process for each condition of the subsystem.

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