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T-FLATNESS AND BOCHNER FLATNESS OF THE TANGENT BUNDLES OF LIE GROUPS

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Abstract. Let (G, g) be a bi-invariant Lie group and $(TG, \tilde{g})$ be its tangent bundle. In this paper, we compute the $\tilde{T}-$curvature tensor and a Bochner tensor $\tilde{B}$ on $(TG, \tilde{g})$ and show that their flatnesses are related with flatness of the base manifold (G, g).

1. Introduction

In [2], Asgari and Moghaddam introduced a left invariant metric $\tilde{g}$ on the tangent bundle TG of a Lie group (G, g) by using complete and vertical lifts of left invariant vector fields from G. They also presented the Levi-Civita connection, sectional curvature and Ricci tensor formulas of $(TG, \tilde{g})$. In [10], Seifipour and Peyghan studied Cotton, Schouten, Weyl and Bach tensors, and they computed projective, concircular and $M-$projective curvatures on TG when the Lie group G is bi-invariant.

In this paper, we obtain two theorems. In the first theorem, we investigate $\tilde{T}-$flatness of the tangent Lie group TG. $T-$curvature tensor was introduced in [11] by Tripati and Gupta. A lot of well-known curvature tensors including projective, concircular and $M-$projective curvature tensors are special cases of this tensor. In the second theorem, we construct an almost Hermitian structure $\tilde{J}$ on $(TG, \tilde{g})$, compute a Bochner tensor with respect to this structure and study its flatness. More precisely, we will prove the following theorems:

Theorem 1.1. *Let (G, g) be an $m-$dimensional bi-invariant Lie group $(m > 3)$. $(TG, \tilde{g})$ is $\tilde{T}-$flat if and only if (G, g) is flat.*

Theorem 1.2. *Let (G, g) be an $m-$dimensional bi-invariant Lie group $(m > 3)$. The almost Hermitian manifold $(TG, \tilde{g}, \tilde{J})$ is Bochner flat if and only if (G, g) is flat.*

2. Preliminaries

2.1. Tangent bundle of a Lie group. Let (M, g) be an $m-$dimensional Riemannian manifold and TM be its tangent bundle. If (x^i) and (x^i, y^i) are local

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charts on M and TM, respectively, then the complete lift and vertical lift of a vector field $X = X^i\partial_i$ on M are expressed as

$$X^C = X^i\partial_i + y^a(\partial_a X^i)\dot{\partial}_i,\ X^V = X^i\dot{\partial}_i,$$

where $\partial_i = \frac{\partial}{\partial x^i},\ \dot{\partial}_i = \frac{\partial}{\partial y^i}$. Moreover, the complete lift f^C of a smooth function f on M is defined by $f^C = y^i\frac{\partial f}{\partial x^i}$.

Let TG be the tangent bundle of a Lie group G. A Lie group structure on TG obtained by the Lie group structure of G is constructed as follows:

$$(x, v_x).(y, w_y) = (xy, (dl_x)(w_y) + (dr_y)(v_x)),$$

for every $x, y \in G,\ v_x \in T_xG$ and $w_y \in T_yG$, where l_x and r_y are the left and right translations of G by x and y, respectively. Notice that $(TG, .)$ is a Lie group.

For the Lie brackets, we have

$$[X^V, Y^V] = 0,\ [X^C, Y^C] = [X, Y]^C,\ [X^V, Y^C] = [X, Y]^V.$$

Remark that the complete and vertical lifts of any left invariant vector fields of G are left invariant vector fields on the Lie group TG. Moreover, one can decompose a left invariant vector field $\tilde{X}$ into two left invariant vector fields X_1^C and X_2^V as $\tilde{X} = X_1^C + X_2^V$. Also, if $\{X_1, ..., X_m\}$ is a basis for the Lie algebra $\mathfrak{g}$ of G then $\{X_1^V, ..., X_m^V, X_1^C, ..., X_m^C\}$ is a basis for the Lie algebra $\tilde{\mathfrak{g}}$ of TG.

A metric g on a Lie group G is said to be left invariant (right invariant) if

$$\text{(l.i.)}\ \ g_b(u, v) = g_{ab}((dl_a)_b u, (dl_a)_b v),$$

$$\text{(r.i.)}\ \ g_b(u, v) = g_{ba}((dr_a)_b u, (dr_a)_b v),$$

for every $a, b \in G$ and all $u, v \in T_bG$. For shortness, (G, g) is called a left invariant (right invariant) Lie group. A Riemannian metric that is both left and right invariant is called a bi-invariant metric. In this case, (G, g) is called a bi-invariant Lie group.

If g is a left invariant Riemannian metric on a Lie group G, then a left invariant Riemannian metric $\tilde{g}$ on TG is defined by

$$\tilde{g}(X^C, Y^C) = g(X, Y),\ \tilde{g}(X^V, Y^V) = g(X, Y),\ g(X^C, Y^V) = 0, \tag{2.1}$$

where X and Y are two left invariant vector fields on G [2].

In the following propositions, we give the Levi-Civita connection and the Riemannian curvature tensor of the metric $\tilde{g}$ which is defined in (2.1).

Proposition 2.1. [2] *Let (G, g) be a left invariant Lie group with the Levi-Civita connection ∇ and $(TG, \tilde{g})$ be its tangent bundle with the Levi-Civita connection $\tilde{\nabla}$. Then the following relations are satisfied:*

$$\begin{cases} \tilde{\nabla}_{X^C} Y^C = (\nabla_X Y)^C, \\ \tilde{\nabla}_{X^C} Y^V = (\nabla_X Y + \frac{1}{2}ad_Y^* X)^V, \\ \tilde{\nabla}_{X^V} Y^C = (\nabla_X Y + \frac{1}{2}ad_X^* Y)^V, \\ \tilde{\nabla}_{X^V} Y^V = (\nabla_X Y - \frac{1}{2}[X, Y])^C, \end{cases}$$

where $ad_X^* Y$ is the transpose of ad_X with respect to the inner product induced by g on $\mathfrak{g}$.

Proposition 2.2. [10] *Let (G,g) be a left invariant Lie group with the Riemannian curvature tensor R and $(TG,\tilde{g})$ be its tangent bundle with the Riemannian curvature tensor $\tilde{R}$. Then the following relations are satisfied:*

$$\begin{aligned}\tilde{R}(X^C,Y^C)Z^V &= (R(X,Y)Z)^V+\{\frac{1}{2}\nabla_X(ad^*_ZY)+\frac{1}{2}ad^*_{\nabla_YZ+\frac{1}{2}ad^*_ZY}X\\&-\frac{1}{2}\nabla_Y(ad^*_ZX)-\frac{1}{2}ad^*_{\nabla_XZ+\frac{1}{2}ad^*_ZX}Y-\frac{1}{2}ad^*_Z[X,Y]\}^V,\end{aligned}$$

$$\tilde{R}(X^C,Y^C)Z^C=(R(X,Y)Z)^C,$$

$$\begin{aligned}\tilde{R}(X^C,Y^V)Z^V &= (R(X,Y)Z)^C+\{-\frac{1}{2}\nabla_X([Y,Z])-\frac{1}{2}\nabla_Y(ad^*_Z)+\frac{1}{2}[Y,\nabla_XZ]\\&+\frac{1}{4}[Y,ad^*_ZX]-\frac{1}{2}[[X,Y],Z]\}^C,\end{aligned}$$

$$\begin{aligned}\tilde{R}(X^V,Y^C)Z^C &= (R(X,Y)Z)^V+\{\frac{1}{2}ad^*_X(\nabla_YZ)-\frac{1}{2}\nabla_Y(ad^*_ZX)\\&-\frac{1}{2}ad^*_{\nabla_XZ+\frac{1}{2}ad^*_ZX}Y-\frac{1}{2}ad^*_{[X,Y]}Z\}^V,\end{aligned}$$

$$\begin{aligned}\tilde{R}(X^V,Y^V)Z^C &= \{\nabla_X(\nabla_YZ)+\frac{1}{2}\nabla_X(ad^*_YZ)-\frac{1}{2}[X,\nabla_YZ]-\frac{1}{4}[X,ad^*_YZ]\\&-\nabla_Y(\nabla_XZ)-\frac{1}{2}\nabla_Y(ad^*_XZ)+\frac{1}{2}[Y,\nabla_XZ]+\frac{1}{4}[Y,ad^*_XZ]\}^C,\end{aligned}$$

$$\begin{aligned}\tilde{R}(X^V,Y^V)Z^V &= \{\nabla_X(\nabla_YZ)-\frac{1}{2}\nabla_X([Y,Z])+\frac{1}{2}(ad^*_X(\nabla_YZ-\frac{1}{2}[Y,Z]\\&-\nabla_Y(\nabla_XZ)+\frac{1}{2}\nabla_Y([X,Z])-\frac{1}{2}(ad^*_Y(\nabla_XZ-\frac{1}{2}[X,Z])\}^V.\end{aligned}$$

Now, we can give the following corollary from [10].

Corollary 2.1. *If (G,g) is a bi-invariant Lie group and $\tilde{g}$ is the left invariant Riemannian metric on TG given in (2.1), then for all left invariant vector fields X,Y,Z on G, we have*

$$\text{(i) } \tilde{\nabla}_{X^C}Y^C=\frac{1}{2}[X,Y]^C,\ \tilde{\nabla}_{X^C}Y^V=[X,Y]^V,\ \tilde{\nabla}_{X^V}Y^C=\tilde{\nabla}_{X^V}Y^V=0,$$

$$\begin{aligned}\text{(ii) } \tilde{R}(X^C,Y^C)Z^C &= -\frac{1}{4}[[X,Y],Z]^C,\\ \tilde{R}(X^C,Y^C)Z^V &= \tilde{R}(X^C,Y^V)Z^V=\tilde{R}(X^V,Y^C)Z^C=\tilde{R}(X^V,Y^V)Z^C\\&= \tilde{R}(X^V,Y^V)Z^V=0,\end{aligned}$$

$$\text{(iii) } \tilde{S}(X^C,Y^C)=S(X,Y),\ \tilde{S}(X^C,Y^V)=\tilde{S}(X^V,Y^V)=0,$$

$$\text{(iv) } \tilde{r}=r,$$

where $\tilde{S}$, $\tilde{r}$ and S,r denote the Ricci tensor and the scalar curvature of the metrics $\tilde{g}$ and g, respectively.

2.2. T-Curvature and Bochner Curvature. To prove the theorems in the first section, we shall recall the following definitions.

Definition 2.1. [11] Let (M, g) be an $m-$dimensional (semi-) Riemannian manifold. A $(0, 4) - type$ curvature tensor T on M is defined by

$$\begin{aligned} T(X,Y,Z,W) &= a_0 R(X,Y,Z,W) \\ &\quad + a_1 S(Y,Z)g(X,W) + a_2 S(X,Z)g(Y,W) + a_3 S(X,Y)g(Z,W) \\ &\quad + a_4 g(Y,Z)S(X,W) + a_5 g(X,Z)S(Y,W) + a_6 g(X,Y)S(Z,W) \\ &\quad + a_7 r(g(Y,Z)X - g(X,Z)Y), \end{aligned}$$

where $a_0, ..., a_7$ are some smooth functions on M, and R, S and r are the curvature tensor, the Ricci tensor and the scalar curvature, respectively.

Particular cases of the $T-$curvature tensor are
(1) the quasi-conformal curvature tensor [14] if

$$a_1 = -a_2 = a_4 = -a_5, \ a_3 = a_6 = 0, \ a_7 = -\frac{1}{m}(\frac{a_0}{m-1} + 2a_1),$$

(2) the conformal curvature tensor [4] if

$$a_0 = 1, \ a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{m-2}, \ a_3 = a_6 = 0, \ a_7 = \frac{1}{(m-1)(m-2)},$$

(3) the conharmonic curvature tensor [5] if

$$a_0 = 1, \ a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{m-2}, \ a_3 = a_6 = 0, \ a_7 = 0,$$

(4) the concircular curvature tensor ([12],[13]) if

$$a_0 = 1, \ a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 0, \ a_7 = -\frac{1}{m(m-1)},$$

(5) the pseudo-projective curvature tensor [9] if

$$a_0 = 1, \ a_1 = -a_2, \ a_3 = a_4 = a_5 = a_6 = 0, \ a_7 = -\frac{1}{m}(\frac{a_0}{m-1} + a_1),$$

(6) the projective curvature tensor [13] if

$$a_0 = 1, \ a_1 = -a_2 = -\frac{1}{(m-1)}, \ a_3 = a_4 = a_5 = a_6 = a_7 = 0,$$

(7) the $M-$projective curvature tensor [7] if

$$a_0 = 1, \ a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{2(m-1)}, \ a_3 = a_6 = a_7 = 0,$$

(8) the W_0-curvature tensor [7] if

$$a_0 = 1, \ a_1 = -a_5 = -\frac{1}{(m-1)}, \ a_2 = a_3 = \ a_4 = a_6 = a_7 = 0,$$

(9) the W_0^*-curvature tensor [7] if

$$a_0 = 1, \ a_1 = -a_5 = \frac{1}{(m-1)}, \ a_2 = a_3 = \ a_4 = a_6 = a_7 = 0,$$

(10) the W_1-curvature tensor [7] if

$$a_0 = 1,\ a_1 = -a_2 = \frac{1}{(m-1)},\ a_3 = a_4 = a_5 = a_6 = a_7 = 0,$$

(11) the W_1^*-curvature tensor [7] if

$$a_0 = 1,\ a_1 = -a_2 = -\frac{1}{(m-1)},\ a_3 = a_4 = a_5 = a_6 = a_7 = 0,$$

(12) the W_2-curvature tensor [6] if

$$a_0 = 1,\ a_4 = -a_5 = -\frac{1}{(m-1)},\ a_1 = a_2 = a_3 = a_6 = a_7 = 0,$$

(13) the W_3-curvature tensor [7] if

$$a_0 = 1,\ a_2 = -a_4 = -\frac{1}{(m-1)},\ a_1 = a_3 = a_5 = a_6 = a_7 = 0,$$

(14) the W_4-curvature tensor [7] if

$$a_0 = 1,\ a_5 = -a_6 = \frac{1}{(m-1)},\ a_1 = a_2 = a_3 = a_4 = a_7 = 0,$$

(15) the W_5-curvature tensor [8] if

$$a_0 = 1,\ a_2 = -a_5 = -\frac{1}{(m-1)},\ a_1 = a_3 = a_4 = a_6 = a_7 = 0,$$

(16) the W_6-curvature tensor [8] if

$$a_0 = 1,\ a_1 = -a_6 = -\frac{1}{(m-1)},\ a_2 = a_3 = a_4 = a_5 = a_7 = 0,$$

(17) the W_7-curvature tensor [8] if

$$a_0 = 1,\ a_1 = -a_4 = -\frac{1}{(m-1)},\ a_2 = a_3 = a_5 = a_6 = a_7 = 0,$$

(18) the W_8- curvature tensor [8] if

$$a_0 = 1,\ a_1 = -a_3 = -\frac{1}{(m-1)},\ a_2 = a_4 = a_5 = a_6 = a_7 = 0,$$

(19) the W_9-curvature tensor [8] if

$$a_0 = 1,\ a_3 = -a_4 = \frac{1}{(m-1)},\ a_1 = a_2 = a_5 = a_6 = a_7 = 0,$$

(20) the generalized $P-$curvature tensor [3] if

$$a_0, ..., a_6 \text{ are arbitrary constants and } a_7 = 0.$$

A (semi-)Riemannian manifold is called $T-$flat if $T = 0$ [11].

Definition 2.2. [1] Let (M_m, J, g) be an even dimensional almost Hermitian manifold (i.e., g is a Riemannian metric and J is a $(1,1)-$ tensor field which satisfies $J^2 = -I,\ g(JX,Y) = -g(X,JY)$, where $X, Y \in \chi(M)$). Then a Bocher tensor B of $(0,4)-$type on M is defined by

$$\begin{aligned}B(X,Y,Z,W) &= R(X,Y,Z,W) + L(X,W)g(Y,Z) - L(X,Z)g(Y,W) \\ &+L(Y,Z)g(X,W) - L(Y,W)g(X,Z) \\ &+L(JX,W)g(JY,Z) - L(JX,Z)g(JY,W) \\ &+L(JY,Z)g(JX,W) - L(JY,W)g(JX,Z) \\ &-2L(JX,Y)g(JZ,W) - 2L(JZ,W)g(JX,Y),\end{aligned}$$

where

$$\begin{aligned}L(X,Y) &= Eg(SX,Y) + Fg(X,Y), \\ E &= -\frac{1}{m+4},\ F = \frac{r}{2(m+2)(m+4)},\end{aligned}$$

and, S and r denote the Ricci tensor and the scalar curvature of g, respectively.

A (semi-)Riemannian manifold is called Bochner flat if $B = 0$.

3. Proof of Theorems

3.1. Proof of Theorem 1.1. From Corollary 2.1, Definition 2.1 and using (2.1), we obtain the following relations by direct calculations:

$$\tilde{T}(X^V, Y^V, Z^V, W^V) = a_7 r((g(Y,Z)g(X,W) - g(X,Z)g(Y,W)), \tag{3.1}$$

$$\tilde{T}(X^V, Y^C, Z^C, W^V) = a_1 S(Y,Z)g(X,W) + a_7 rg(Y,Z)g(X,W)), \tag{3.2}$$

$$\begin{aligned}\tilde{T}(X^C, Y^C, Z^C, W^C) &= a_0 R(X,Y,Z,W) \\ &+a_1 S(Y,Z)g(X,W) + a_2 S(X,Z)g(Y,W) \\ &+a_3 S(X,Y)g(Z,W) + a_4 g(Y,Z)S(X,W) \\ &+a_5 g(X,Z)S(Y,W) + a_6 + g(X,Y)S(Z,W) \\ &+a_7 r(g(Y,Z)X - g(X,Z)Y),\end{aligned} \tag{3.3}$$

$$\tilde{T}(X^C, Y^V, Z^V, W^C) = a_4 g(Y,Z)S(X,W) + a_7 rg(Y,Z)g(X,W), \tag{3.4}$$

$$\tilde{T}(X^C, Y^C, Z^V, W^C) = \tilde{T}(X^V, Y^V, Z^V, W^C) = 0, \tag{3.5}$$

$$\tilde{T}(X^V, Y^V, Z^C, W^C) = a_6 g(X,Y)S(Z,W), \tag{3.6}$$

$$\tilde{T}(X^V, Y^C, Z^C, W^C) = \tilde{T}(X^V, Y^V, Z^C, W^V) = 0, \tag{3.7}$$

$$\tilde{T}(X^C, Y^C, Z^C, W^V) = \tilde{T}(X^C, Y^V, Z^V, W^V) = 0, \tag{3.8}$$

$$\tilde{T}(X^C, Y^C, Z^V, W^V) = a_3 S(X,Y)g(Z,W), \tag{3.9}$$

where r, S and R denote the scalar curvature, Ricci tensor and curvature tensor of (G,g), respectively. It is obvious that if $R = 0$, then $\tilde{T} = 0$. Conversely, if $\tilde{T} = 0$, then from (3.1) we see that $r = 0$. From (3.2), we occur that $S = 0$. And finally, using (3.3), we get $R = 0$.

Corollary 3.1. *Let (G,g) be an $m-$dimensional bi-invariant Lie group $(m > 3)$ and $(TG, \tilde{g})$ be its tangent bundle. Then we have*

$(TG, \tilde{g})$ is quasi-conformally flat $\Leftrightarrow$ (G, g) is flat.
$(TG, \tilde{g})$ is conformally flat $\Leftrightarrow$ (G, g) is flat.
$(TG, \tilde{g})$ is conharmonically flat $\Leftrightarrow$ (G, g) is flat.
$(TG, \tilde{g})$ is concircularly flat $\Leftrightarrow$ (G, g) is flat.
$(TG, \tilde{g})$ is pseudo-projectively flat $\Leftrightarrow$ (G, g) is flat.
$(TG, \tilde{g})$ is projectively flat $\Leftrightarrow$ (G, g) is flat.
$(TG, \tilde{g})$ is $M-$projectively flat $\Leftrightarrow$ (G, g) is flat.
$(TG, \tilde{g})$ is W_i^* $(i = 0, 1)$ flat $\Leftrightarrow$ (G, g) is flat.
$(TG, \tilde{g})$ is W_i $(i = 0, ..., 9)$ flat $\Leftrightarrow$ (G, g) is flat.
$(TG, \tilde{g})$ is generalized P flat $\Leftrightarrow$ (G, g) is flat.

3.2. Proof of Theorem 1.2. Define a $(1, 1)-$tensor field $\tilde{J}$ on $(TG, \tilde{g})$ as

$$\tilde{J}(X^C) = -X^V, \ \tilde{J}(X^V) = X^C.$$

One can easily show that $\tilde{J}^2 = -I$ and $\tilde{g}(\widetilde{J\tilde{X}}, \tilde{Y}) = -g(\tilde{X}, \widetilde{J\tilde{Y}})$ for all $\tilde{X} = X^V, X^C$ and $\tilde{Y} = Y^V, Y^C$. Therefore, $(TG, \tilde{g}, \tilde{J})$ is an almost Hermitian manifold. Using this structure, Corollary 2.1 and Definition 2.2, we get

$$\tilde{B}(X^V, Y^V, Z^V, W^V) = F(g(X, W)g(Y, Z) - g(X, Z)g(Y, W)), \tag{3.10}$$

$$\begin{aligned} \tilde{B}(X^V, Y^C, Z^C, W^V) &= Fg(X, W)g(Y, Z) + [ES(Y, Z) + Fg(Y, Z)]g(X, W) \\ &\quad +[ES(X, Z) + Fg(X, Z)]g(Y, W) \\ &\quad +[ES(Y, W) + Fg(Y, W)]g(X, Z) \\ &\quad 2[ES(X, Y) + Fg(X, Y)]g(Z, W) \\ &\quad +2Fg(Z, W)g(X, Y), \end{aligned} \tag{3.11}$$

$$\begin{aligned} \tilde{B}(X^C, Y^C, Z^C, W^C) &= R(X, Y, Z, W) + [ES(X, W) + Fg(X, W)]g(Y, Z) \\ &\quad -[ES(X, Z) + Fg(X, Z)]g(Y, W) \\ &\quad +[ES(Y, Z) + Fg(Y, Z)]g(X, W) \\ &\quad -[ES(Y, W) + Fg(Y, W)]g(X, Z), \end{aligned} \tag{3.12}$$

$$\begin{aligned} \tilde{B}(X^C, Y^V, Z^V, W^C) &= [ES(X, W) + Fg(X, W)]g(Y, Z) + Fg(Y, Z)g(X, W) \\ &\quad +Fg(X, Z)g(Y, W) \\ &\quad +[ES(Y, W) + Fg(Y, W)]g(X, Z) \\ &\quad +2Fg(X, Y)g(Z, W) \\ &\quad +2[ES(Z, W) + Fg(Z, W)]g(X, Y), \end{aligned} \tag{3.13}$$

$$\tilde{B}(X^C, Y^C, Z^V, W^C) = \tilde{B}(X^V, Y^V, Z^V, W^C), \tag{3.14}$$

$$\begin{aligned} \tilde{B}(X^V, Y^V, Z^C, W^C) &= -[ES(X, W) + Fg(X, W)]g(Y, Z) \\ &\quad -[ES(X, Z) + Fg(X, Z)]g(Y, W) \\ &\quad +[ES(Y, Z) + Fg(Y, Z)]g(X, W) \\ &\quad -[ES(Y, W) + Fg(Y, W)]g(X, Z), \end{aligned} \tag{3.15}$$

$$\tilde{B}(X^V, Y^C, Z^C, W^C) = 0, \tag{3.16}$$

$$\tilde{B}(X^V, Y^V, Z^C, W^V) = \tilde{B}(X^C, Y^C, Z^C, W^V) = \tilde{B}(X^C, Y^V, Z^V, W^V) = 0, \tag{3.17}$$

$$\tilde{B}(X^C, Y^C, Z^V, W^V) = 2Fg(X, W)g(Y, Z) - 2Fg(X, Z)g(Y, W), \tag{3.18}$$

where $E = -\frac{1}{2m+4}$ and $F = \frac{\tilde{r}}{2(2m+2)(2m+4)} = \frac{r}{2(2m+2)(2m+4)}$, and r, S, R denote the scalar curvature, Ricci tensor and curvature tensor of (G, g), respectively. Clearly, if $R = 0$, then $\tilde{B} = 0$. Conversely, if $\tilde{B} = 0$, then from (3.10) we occur that $r = 0$. From (3.11), we conclude $S = 0$. Using (3.12), we obtain $R = 0$.

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