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DIFFERENCE APPROXIMATION OF THE INVERSE PROBLEM OF DETERMINING THE HIGHEST COEFFICIENT IN A PARABOLIC EQUATION WITH INTEGRAL CONDITIONS

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Abstract. We consider the variational statement of the inverse problem of determining the higher coefficient of a parabolic equation with an integral boundary condition and an integral overdetermination condition. We establish estimates for the rate of convergence of difference approximations of the problem with respect to the state and functional.

1. Introduction

When we study variational statements of coefficient inverse problems for equations of mathematical physics, a number of difficulties arise due to their incorrectness, nonconvexity and describing the state of systems [1, 3]. These circumstances make it difficult to substantiate numerical methods, in particular. Variational statements of coefficient inverse problems for parabolic equations were studied in [1, 3, 4, 6, 7, 12] and others. Convergence of difference approximations of optimal control problems for the coefficients of parabolic equations and variational statements of coefficient inverse problems for these equations were studied in [8, 11] and others. However, these issues have been studied much less for parabolic equations with an integral overdetermination condition.

In this paper, we consider a variational statement of the inverse problem of determining the higher coefficient of a parabolic equation with an integral boundary condition and an integral overdetermination condition. We establish estimates for the rate of convergence of difference approximations of the problem with respect to the state and functional.

2. Variational statement of the inverse problem and its well-posedness

Suppose that it is required to find a pair of functions $\{\upsilon(x),\ u(x,t) = u(x,t;\upsilon)\}$ minimizing the functional

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$$J(\upsilon) = \int_0^l \left| \int_0^T \alpha(t)\, u(x,t;\upsilon)\, dt - \beta(x) \right|^2 dx \tag{2.1}$$

subject to the following boundary value problem

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\upsilon(x)\frac{\partial u}{\partial x}\right) + a(x,t)\, u = f(x,t), \quad (x,t) \in Q_T, \tag{2.2}$$

$$u(x,0) = \varphi(x),\ 0 \le x \le l, \tag{2.3}$$

$$u(0,t) = 0,\ \upsilon(l)\frac{\partial u(l,t)}{\partial x} = \int_0^l H(x,t)\, u(x,t)\, dx,\ 0 < t \le T, \tag{2.4}$$

and the function $\upsilon(x)$ belongs to the set

$$V = \{\upsilon = \upsilon(x) \in W_2^1(0,l) : 0 < \nu \le \upsilon(x) \le \mu,$$
$$|\upsilon'(x)| \le \mu \quad \text{a.e. } (0,l)\}. \tag{2.5}$$

Here $l, T, \nu, \mu > 0$ are given numbers, $Q_T = \{(x,t) :\ 0 < x < l,\ 0 < t \le T\}$, $a(x,t),\ f(x,t),\ \varphi(x),\ H(x,t)$ are given measurable functions that satisfy the following conditions:

$$|a(x,t)| \le \mu \ \text{ a. e. } Q_T, f(x,t) \in L_2(Q_T), \varphi(x) \in W_{2,0}^1(0,l), |H(x,t)| \le \mu,$$
$$\left|\frac{\partial H(x,t)}{\partial t}\right| \le \mu \text{ a.e. } Q_T,$$
$$\alpha(t) \in W_2^1(0,T),\ \beta(x) \in W_2^1(0,l). \tag{2.6}$$

In the work, the function spaces used and the norms correspond to [10, p.23]. In addition, positive constant which is depend on the estimated values and the steps of introduced grids, is denoted by M.

Direct and inverse problems with integral conditions have been studied by various authors in [2, 5, 9] and others.

By the solution of the boundary value problem (2.2)-(2.4) for each control $\upsilon = \upsilon(x) \in V$, we mean a generalized solution from $V_2^{1,0}(Q_T)$. Here $V_2^{1,0}(Q_T)$ is a subspace of $V_2(Q_T)$, whose elements have traces from $L_2(0,l)$ for all $t \in [0,T]$, continuously varying with $t \in [0,T]$ in the norm $L_2(0,l)$. It follows from the results of [14] that, under the above-mentioned assumptions (2.6), the boundary value problem (2.2)-(2.4) has a unique generalized solution from $V_2^{1,0}(Q_T)$ and this solution also belongs to the space $W_2^{1,1}(Q_T)$ for each fixed $\upsilon \in V$, and the following a priori estimate is valid

$$\|u\|_{V_2^{1,0}(Q_T)} + \max_{0 \le t \le T}\left\|\frac{\partial u}{\partial x}(x,t)\right\|_{2,(0,l)} + \left\|\frac{\partial u}{\partial t}\right\|_{2,Q_T} \le$$
$$\le M\left[\|\varphi\|_{2,(0,l)}^{(1)} + \|f\|_{2,Q_T}\right]. \tag{2.7}$$

In addition, it was proved in [14] that problem (2.1)-(2.5) is correctly posed in the weak topology of the space $W_2^1(0,l)$, i.e. the set of optimal control problems (2.1)-(2.5) $V_* = \{\upsilon_* = \upsilon_*(x) \in V : J(\upsilon_*) = J_* \equiv \inf\{J(\upsilon) : \upsilon \in V\}\}$ is nonempty and any minimizing sequence $\{\upsilon_n = \upsilon_n(x)\} \subset V$ of the functional (2.1) converges to the set V_* weakly in $W_2^1(0,l)$.

3. Difference approximation of the problem and its well-posedness

We introduce the following grids on the segments $[0,l]$ and $[0,T]$:

$$\omega_h = \{x_i = ih \in [0,l] : i = 1,2,...,N-1,\ h = l/N\},$$
$$\omega_h^+ = \{x_i = ih \in [0,l] : i = 1,2,...,N\},$$
$$\overline{\omega}_h = \{x_i = ih \in [0,l] : i = 0,1,...,N\},$$
$$\omega_\tau = \{t_j = j\tau \in [0,T] : j = 1,2,...,L,\ \tau = T/l\},$$
$$\overline{\omega}_{h*} = \{\overline{x}_i = (i-0.5)\,h \in [0,l] : i = 1,2,...,N\},$$
$$\overset{0}{\omega}_{h*} = \overline{\omega}_{h*}/\ \{\overline{x}_N = (N-0.5)\,h\}.$$

We introduce the following grids on the rectangle $\overline{Q}_T$:

$$\omega_T = \omega_h \times \omega_\tau, \omega_T^+ = \omega_h^+ \times \omega_\tau, \overline{\omega}_T = \overline{\omega}_h \times \omega_\tau, \omega_t^+ = \omega_h^+ \times \left\{t' = \tau, 2\tau, \ldots, t\right\}.$$

Let $\hbar = \hbar\,(x) = h$, if $x \in \omega_h$, $\hbar\,(0) = \hbar\,(l) = 0.5h$.

We introduce the following scalar products and norms for grid functions defined on the corresponding grids:

$$(y,z)_{\omega_h^+} = \sum_{\omega_h^+} \hbar yz,\ \|y\|_{2,\omega_h^+} = (y,y)_{\omega_h^+}^{1/2},$$

$$(y,z)_{\omega_\tau} = \sum_{\omega_\tau} \tau yz,\ \|y\|_{2,\omega_\tau} = (y,y)_{\omega_\tau}^{1/2}, (y_{\overline{x}}, z_{\overline{x}})_{\omega_h^+} = \sum_{\omega_h^+} h y_{\overline{x}} z_{\overline{x}},$$

$$\|y_{\overline{x}}\|_{2,\omega_h^+} = (y_{\overline{x}}, y_{\overline{x}})_{\omega_h^+}^{1/2}, (y,z)_{\omega_T^+} = \sum_{\omega_\tau} \tau\,(y,z)_{\omega_h^+},$$

$$\|y\|_{2,\omega_T^+} = (y,y)_{\omega_T^+}^{1/2}, (y_{\overline{x}}, z_{\overline{x}})_{\omega_t^+} = \sum_{t=\tau}^{t} \tau\,(y_{\overline{x}}, z_{\overline{x}})_{\omega_h^+},$$

$$\|y_{\overline{x}}\|_{2,\omega_t^+} = (y_{\overline{x}}, y_{\overline{x}})_{\omega_t^+}^{1/2}, t \in \omega_\tau, |y|_{\omega_t^+} \equiv \|y\|_{V_2^{1,0}(\omega_t^+)} =$$

$$= \max_{0 \le t' \le t} \left\|y\left(x,t'\right)\right\|_{2,\omega_h^+} + \|y_{\overline{x}}\|_{2,\omega_t^+}, \quad t \in \omega_\tau, (y_{\overline{t}}, z_{\overline{t}})_{\omega_t^+} = \sum_{t=\tau}^{t} \tau\,(y_{\overline{t}}, z_{\overline{t}})_{\omega_h^+},$$

$$\|y_{\overline{t}}\|_{2,\omega_t^+} = (y_{\overline{t}}, y_{\overline{t}})_{\omega_t^+}^{1/2}, t \in \omega_\tau,$$

$$\theta\,(t,y) \equiv |y|_{\omega_t^+} + \sqrt{\tau}\,\|y_{\overline{t}}\,(x,t)\|_{2,\omega_t^+}, \quad t \in \omega_\tau,$$

$$\|\upsilon_h\|_{2,\overline{\omega}_{h*}} = \left(\sum_{\overline{\omega}_{h*}} h\upsilon_h^2\,(x)\right)^{1/2}, \|\upsilon_h\|_{2,\overline{\omega}_{h*}}^{(1)} = \left[\sum_{\overline{\omega}_{h*}} h\upsilon_h^2\,(x) + \sum_{\overset{0}{\omega}_{h*}} h\upsilon_{hx}^2\,(x)\right]^{1/2},$$

$$\|\upsilon_h\|_{\infty,\overline{\omega}_{h*}} = \max_{x \in \overline{\omega}_{h*}} |\upsilon_h\,(x)|\,.$$

We also introduce elementary cells:

$$e_1\,(x) = \{\xi : x - 0.5h \le \xi < x + 0.5h\}\,, x \in \omega_h, e_1\,(0) = \{\xi : 0 \le \xi < 0.5h\}\,,$$
$$e_1\,(l) = \{\xi : l - 0.5h \le \xi \le l\}\,; e_2\,(t) = \{\theta : t - \tau \le \xi < t\}\,,\ t \in \omega_\tau,$$
$$e_1^*\,(x) = \{\xi : x - h \le \xi < x\}\,, x \in \omega_h^+;\ \ e\,(x,t) = e_1\,(x) \times e_2\,(t)\,,\ (x,t) \in \overline{\omega}_T,$$

$$e^*(x,t) = e_1^*(x) \times e_2(t), \ (x,t) \in \omega_T^+.$$

Let S^x, S_-^x, S_-^t be Steklov's one-dimensional averaging operators. In addition, let $S^{tx} = S_-^t S^x$ be the product of the averaging operators S^x and S_-^t.

We state the problem (2.1)-(2.5) in the following difference approximation form: it is required to find a pair of grid functions $\{\upsilon_h(x), \ y(x,t) = y(x,t;\upsilon_h)\}$ minimizing the grid functional

$$J_{h\tau}(\upsilon_h) = \sum_{x\in\omega_h^+} \hbar \left| \tau \sum_{t\in\omega_\tau} \alpha_\tau(t)\, y(x,t;\upsilon_h) - \beta_h(x) \right|^2 \tag{3.1}$$

subject to the difference boundary value problem

$$y_{\overline{t}} - (\upsilon_h(x-0.5h)\, y_{\overline{x}})_x + a_{h\tau}(x,t)\, y = f_{h\tau}(x,t), \quad (x,t) \in \omega_T, \tag{3.2}$$

$$y(x,0) = \varphi_h(x), \ \ x \in \overline{\omega}_h, \tag{3.3}$$

$$y(0,t) = 0, \upsilon_h(l-0.5h)\, y_{\overline{x}}(l,t) = \sum_{x\in\omega_h^+} hH_{h\tau}(x,t)\, y(x,t) -$$

$$-0.5h\left[y_{\overline{t}}(l,t) + a_{h\tau}(l,t)\, y(l,t) - f_{h\tau}(l,t)\right], t \in \omega_\tau \tag{3.4}$$

and grid controls $\upsilon_h(x)$ are such that

$$\upsilon_h(x) \in V_h = \Big\{\upsilon_h(x) \in W_2^1(\overline{\omega}_{h*}) :$$

$$0 < \nu \le \upsilon_h(x) \le \mu, \ x \in \overline{\omega}_{h*}, |\upsilon_{hx}(x)| \le \mu, \ x \in \overset{0}{\omega}_{h*} \Big\}. \tag{3.5}$$

Here $a_{h\tau}(x,t) = S^{tx}a$, $f_{h\tau}(x,t) = S^{tx}f$, $H_{h\tau}(x,t) = S_-^x H$, $(x,t) \in \omega_T^+$, $\varphi_h(x) = S^x\varphi$, $\varphi_h(0) = 0$, $\beta_h(x) = S^x\beta$, $\quad x \in \omega_h^+$; $\alpha_\tau(t) = S_-^t\alpha$, $t \in \omega_\tau$.

We represent the difference boundary value problem (3.2)-(3.4) in the following form:

$$y_{\overline{t}} = Ay + f_{h\tau}(x,t), \quad (x,t) \in \omega_T^+, \tag{3.6}$$

$$y(x,0) = \varphi_h(x), \ \ x \in \overline{\omega}_h, \tag{3.7}$$

$$y(0,t) = 0, \quad t \in \omega_\tau, \tag{3.8}$$

where

$$Ay = \begin{cases} (\upsilon_h(x-0.5h)\, y_{\overline{x}})_x - a_{h\tau}(x,t)\, y, \ (x,t) \in \omega_T, \\ -\frac{2}{h}\upsilon_h(l-0.5h)\, y_{\overline{x}}(l,t) - a_{h\tau}(l,t)\, y(l,t) + \frac{2}{h}\sum\limits_{\omega_h^+} hH_{h\tau}(x,t)\, y(x,t), \ t \in \omega_\tau \end{cases}.$$

Theorem 3.1. *Let conditions* (2.6) *be satisfied and the step* τ *in the variable* t *satisfies the inequality:*

$$\tau < \tau_0 = \frac{1}{2}\left[2\mu + \mu^2 l + \frac{1}{\nu} + \frac{1}{l}\right]^{-1}\left(2 + \nu^{-1/2}\right)^{-2}. \tag{3.9}$$

Then problem (3.2)-(3.4) *is uniquely solvable for each* $\upsilon_h \in V_h$ *and a priori estimate is valid*

$$\|y(x,t;\upsilon_h)\|_{V_2^{1,0}(\omega_T^+)} + \sqrt{\tau}\, \|y_{\overline{t}}(x,t;\upsilon_h)\|_{2,\omega_T^+} \le$$

$$\le M\left[\|\varphi_h\|_{2,\omega_h^+} + 2\,\|f_{h\tau}\|_{2,1,\omega_T^+}\right]. \tag{3.10}$$

Proof. Let us carry out the main points of the proof of estimate (3.10). We multiply equation (3.6) and $\tau y(x,t')$ as a scalar product $(,)_{\omega_h^+}$ and sum the resulting equality over t' from $t'=\tau$ to $t'=t$, where $t\in\omega_\tau$ is some point. Then using the summation by parts formulas and identity $y_{\bar t}y = 0.5\left(y^2\right)_{\bar t} + 0.5\tau y_{\bar t}^2$, $(x,t)\in\omega_T^+$ to equality

$$\frac{1}{2}\left\|y(x,t)\right\|_{2,\omega_h^+}^2 + \tau\sum_{t'=\tau}^{t} h\sum_{x\in\omega_h^+} \upsilon_h(x-0.5h)\,y_{\bar x}^2(x,t') +$$

$$+\frac{1}{2}\tau^2\sum_{t'=\tau}^{t}\left\|y_{\bar t}(x,t')\right\|_{2,\omega_h^+}^2 = \frac{1}{2}\left\|y(x,0)\right\|_{2,\omega_h^+}^2 - \tau\sum_{t'=\tau}^{t}\big(a_{h\tau}(x,t'),$$

$$y^2(x,t')\big)_{2,\omega_h^+} + \tau\sum_{t'=\tau}^{t}\left(H_{h\tau}(x,t'),y(x,t')\right)_{2,\omega_h^+} y(l,t') +$$

$$+\tau\sum_{t'=\tau}^{t}\left(f_{h\tau}(x,t'),y(x,t')\right)_{2,\omega_h^+}. \tag{3.11}$$

Then, using conditions (2.6), the Cauchy inequality with ε from [10, p.33]

$$|ab| \le \frac{\varepsilon}{2}|a|^2 + \frac{1}{2\varepsilon}|b|^2, \quad \varepsilon>0$$

for $\varepsilon=\mu\sqrt{l}$ and the inequality [13, p.290]

$$y^2(l,t) \le \varepsilon\left\|y_{\bar x}(x,t)\right\|_{2,\omega_h^+}^2 + \left(\frac{1}{\varepsilon}+\frac{1}{l}\right)\left\|y(x,t)\right\|_{2,\omega_h^+}^2, \tag{3.12}$$

for $\varepsilon=\nu$, and majorizing the left and right sides of equality (3.11), we obtain the inequality

$$\left\|y(x,t)\right\|_{2,\omega_h^+}^2 + \nu\tau\sum_{t'=\tau}^{t}\left\|y_{\bar x}(x,t')\right\|_{2,\omega_h^+}^2 + \tau^2\sum_{t'=\tau}^{t}\left\|y_{\bar t}(x,t')\right\|_{2,\omega_h^+}^2 \le$$

$$\le \gamma(t)\left[\left\|y(x,0)\right\|_{2,\omega_h^+}^2 + 2\tau\sum_{t'=\tau}^{t}\left\|f_{h\tau}(x,t')\right\|_{2,\omega_h^+}\right] +$$

$$+ct\gamma^2(t) \equiv \delta(t), \quad t\in\omega_\tau, \tag{3.13}$$

where $\gamma(t)=\max\limits_{0\le t'\le t}\left\|y(x,t')\right\|_{2,\omega_h^+}$, $c=2\mu+\mu^2 l+\nu^{-1}+l^{-1}$. From this inequality, we extract three consequences $\forall t\in\omega_\tau$

$$\max_{0\le t'\le t}\left\|y(x,t')\right\|_{2,\omega_h^+}^2 \le \delta(t),\ \tau\sum_{t'=\tau}^{t}\left\|y_{\bar x}(x,t')\right\|_{2,\omega_h^+}^2 \le$$

$$\le \nu^{-1}\delta(t),\ \tau\sum_{t'=\tau}^{t}\tau\left\|y_{\bar t}(x,t')\right\|_{2,\omega_h^+}^2 \le \delta(t).$$

We extract the square root from both parts of these inequalities, and add the resulting inequalities. Then, using condition (3.9), for $t < 2\tau_0$ we obtain the estimate

$$\theta(t,y) \equiv \max_{0\le t'\le t} \left\|y(x,t')\right\|_{2,\omega_h^+} + \left(\tau\sum_{t'=\tau}^{t}\left\|y_{\overline{x}}(x,t')\right\|^2_{2,\omega_h^+}\right)^{1/2} +$$

$$+\sqrt{\tau}\left(\tau\sum_{t'=\tau}^{t}\left\|y_{\overline{t}}(x,t')\right\|^2_{2,\omega_h^+}\right)^{1/2} \le \left[1-\left(2+\nu^{-1/2}\right)\sqrt{ct}\right]^{-2}\left(2+\nu^{-1/2}\right)^2\times$$

$$\times\left[\left\|y(x,0)\right\|_{2,\omega_h^+} + 2\tau\sum_{t'=\tau}^{t}\left\|f_{h\tau}(x,t')\right\|^2_{2,\omega_h^+}\right],\ t\in\omega_\tau,\ t<2\tau_0. \tag{3.14}$$

The interval $[0,T]$ is divided into subintervals $\Delta_1 = [0,\tau_0]$, $\Delta_2 = [\tau_0, 2\tau_0], \ldots, \Delta_n$ of a length not greater than τ_0. Here $\Delta_n = [(n-1)\tau_0, T]$. Each of them satisfies the estimate (3.14). From these estimates, taking into account $\left\|y(x,t)\right\|_{2,\omega_h^+} \le |y|_{\omega_t^+}$, $t\in\omega_\tau$, we derive inequality (3.10) for $t = T$. The unique solvability of problem (3.2)-(3.4) for each $\upsilon_h \in V_h$ is obvious. Theorem 3.1 is proved. □

4. A priori estimate of the error of the difference method by state

Take arbitrary controls $\upsilon(\xi)\in V$, $\upsilon_h(x)\in V_h$. Let $u(\xi,\theta) = u(\xi,\theta;\upsilon)$ be a solution to the boundary value problem (2.2)-(2.4), and $y(x,t) = y(x,t;\upsilon_h)$ be a solution to the difference boundary value problem (3.2)-(3.4) corresponding to the controls $\upsilon\in V$ and $\upsilon_h(x)\in V_h$. Denote $\overline{u}(x,t) = \overline{u}(x,t;\upsilon)$ by the averaging of the function $u(\xi,\theta) = u(\xi,\theta;\upsilon)$, which is determined by the formula

$$\overline{u}(x,t) = \overline{u}(x,t;\upsilon) = \begin{cases} S_-^t u, & (x,t)\in\omega_T^+, \\ S^x\varphi(x), & x\in\overline{\omega}_h,\ t=0, \\ 0, & x=0,\quad t\in\omega_\tau. \end{cases} \tag{4.1}$$

We will compare the grid solution $y(x,t;\upsilon_h)$ with the averaging $\overline{u}(x,t;\upsilon)$. Let $z = z(x,t) = z(x,t;\upsilon,\upsilon_h) = y(x,t;\upsilon_h) - \overline{u}(x,t;\upsilon)$, $(x,t)\in\overline{\omega}_T$ be the error of the state difference method. Using conditions (3.6)-(3.8) for the function z, we obtain the problem

$$z_{\overline{t}} = Az + \psi_{h\tau}(x,t),\ \ (x,t)\in\omega_T^+, \tag{4.2}$$

$$z(x,0) = 0,\quad x\in\overline{\omega}_h, \tag{4.3}$$

$$z(0,t) = 0,\quad t\in\omega_\tau, \tag{4.4}$$

where $\psi_{h\tau}(x,t) = f_{h\tau}(x,t) + A\overline{u} - \overline{u}_{\overline{t}}$, $\quad (x,t)\in\omega_T^+$.

We will assume that the generalized solution $u(\xi,\theta) = u(\xi,\theta;\upsilon)$ from $V_2^{1,0}(Q_T)$ also belongs to the space $W_2^{2,1}(Q_T)$. Then, applying the averaging operator S^{tx}

to equation (2.2) at the nodes $(x,t)\in\overline{\omega}_T$, after some transformations we obtain the following representation for $\psi_{h\tau}$:

$$\psi_{h\tau}(x,t)=\begin{cases}\sum\limits_{k=1}^{2}\eta_{1x}^{(k)}(x,t)+\eta_2(x,t)+\eta_{3\bar{t}}(x,t), & (x,t)\in\omega_T,\\ -\frac{2}{h}\sum\limits_{k=1}^{2}\eta_1^{(k)}(x,t)+\eta_2(x,t)+\eta_{3\bar{t}}(x,t)+\frac{2}{h}\eta_4(t), & x=l,\ t\in\omega_\tau,\end{cases}\tag{4.5}$$

where

$$\eta_1^{(1)}(x,t)=\upsilon(x-0.5h,t)\left[\overline{u}_{\overline{x}}(x,t)-S_-^t\frac{\partial u(x-0.5h,t)}{\partial x}\right],\tag{4.6}$$

$$\eta_1^{(2)}(x,t)=[\upsilon_h(x-0,5h)-\upsilon(x-0,5h)]\overline{u}_{\overline{x}}(x,t),\tag{4.7}$$

$$\eta_2(x,t)=S^{tx}(a(x,t)u(x,t))-S^{tx}(a(x,t))\overline{u},\tag{4.8}$$

$$\eta_3(x,t)=S^xu(x,t)-\overline{u}(x,t),\quad (x,t)\in\omega_T^+,\tag{4.9}$$

$$\eta_4(t)=\sum_{x\in\omega_h^+}hS_-^x(H(x,t))\overline{u}(x,t)-S_-^t\left(\int_0^l H(\xi,t)u(\xi,t)d\xi\right),\ t\in\omega_\tau.\tag{4.10}$$

Theorem 4.1. *Let the conditions of theorem 3.1 be satisfied and the solution of problem* (2.2)-(2.4) *belong to the space* $W_2^{2,1}(Q_T)$. *Then, for the solution* z *of problem* (4.2)-(4.4), *the estimate*

$$\|z(x,t)\|_{V_2^{1,0}(\omega_T^+)}+\sqrt{\tau}\,\|z_{\bar{t}}(x,t)\|_{2,\omega_T^+}\le$$

$$\le M\left[\sum_{k=1}^{2}\left\|\eta_1^{(k)}\right\|_{2,\omega_T^+}+\|\eta_2\|_{2,\omega_T^+}+\left(\sum_{t\in\omega_\tau}\|\eta_3(x,t)\|_{2,\omega_h^+}^2\right)^{1/2}+\|\eta_4\|_{2,\omega_\tau}\right]\tag{4.11}$$

holds true.

Proof. Multiplying equations (4.2) scalarly $(,)_{\omega_h^+}$ by $\tau z(x,t')$ and calculating similarly to the obtained equality (3.11), we have

$$\frac{1}{2}\|z(x,t')\|_{2,\omega_h^+}^2+\tau\sum_{t'=\tau}^{t}h\sum_{x\in\omega_h^+}\upsilon_h(x-0.5h)z_{\overline{x}}^2(x,t')+\frac{1}{2}\tau^2\sum_{t'=\tau}^{t}\|z_{\bar{t}}(x,t')\|_{2,\omega_h^+}^2=$$

$$=-\sum_{t'=\tau}^{t}\left(a_{h\tau}(x,t'),z^2(x,t')\right)_{2,\omega_h^+}-\tau\sum_{t'=\tau}^{t}\left(\sum_{k=1}^{2}\eta_1^{(k)}(x,t'),z_{\overline{x}}(x,t')\right)_{2,\omega_h^+}+$$

$$+\tau\sum_{t'=\tau}^{t}\left(\eta_2(x,t'),z(x,t')\right)_{2,\omega_h^+}-$$

$$-\tau\sum_{t'=0}^{t-\tau}\left(\eta_3(x,t'),z_t(x,t')\right)_{2,\omega_h^+}+\left(\eta_3(x,t),z(x,t)\right)_{2,\omega_h^+}+$$

$$+\tau\sum_{t'=\tau}^{t}\left[\sum_{\omega_h^+}hH_{h\tau}(x,t')z(x,t')+\eta_4(t')\right]z(l,t').\tag{4.12}$$

Using the method of item 2 for obtaining estimates of type (3.10), as well as the formula of summation by parts, inequalities (3.12), (3.14) can be obtained from (4.12) for any $t \in \omega_\tau$ of the following three estimates

$$\max_{0\le t'\le t} \left\| z\left(x,t'\right)\right\|^2_{2,\omega_h^+} \le \delta\left(t\right), \tau \sum_{t'=\tau}^{t} \left\| z_{\overline{x}}\left(x,t'\right)\right\|^2_{2,\omega_h^+} \le \nu^{-1}\delta\left(t\right),$$

$$\tau \sum_{t'=\tau}^{t} \tau \left\| z_{\overline{t}}\left(x,t'\right)\right\|^2_{2,\omega_h^+} \le \delta\left(t\right), \tag{4.13}$$

where

$$\delta\left(t\right) = c_1 t \max_{0\le t'\le t} \left\| z\left(x,t'\right)\right\|_{2,\omega_h^+} + 2c_2 \max_{0\le t'\le t} \left\| z\left(x,t'\right)\right\|_{2,\omega_h^+} \times$$
$$\times \left[\left(\tau \sum_{t'=\tau}^{t} \left\| \eta_2\left(x,t'\right)\right\|^2_{2,\omega_h^+} \right)^{1/2} + \left\| \eta^{(3)}\left(x,t\right)\right\|_{2,\omega_h^+} \right] +$$
$$+2\sum_{k=1}^{2} \left(\tau \sum_{t'=\tau}^{t} \left\| \eta_1^{(k)}\left(x,t'\right)\right\|^2_{2,\omega_h^+} \right)^{1/2} \left(\tau \sum_{t'=\tau}^{t} \left\| z_{\overline{x}}\left(x,t'\right)\right\|^2_{2,\omega_h^+} \right)^{1/2} +$$
$$+2 \left(\tau \sum_{t'=0}^{t-\tau} \left\| \eta_3\left(x,t'\right)\right\|^2_{2,\omega_h^+} \right)^{1/2} \left(\tau \sum_{t'=\tau}^{t} \tau \left\| z_{\overline{t}}\left(x,t'\right)\right\|^2_{2,\omega_h^+} \right)^{1/2} +$$
$$+2c_3 \left(\tau \sum_{t'=\tau}^{t} \eta_4^2\left(t'\right)^2 \right)^{1/2} \left[\left(\tau \sum_{t'=\tau}^{t} \left\| z_{\overline{x}}\left(x,t'\right)\right\|^2_{2,\omega_h^+} \right)^{1/2} + \right.$$
$$\left. + \left(\tau \sum_{t'=\tau}^{t} \left\| z\left(x,t'\right)\right\|^2_{2,\omega_h^+} \right)^{1/2} \right].$$

Here $c_1 = 2\mu + \mu^2 l + \frac{1}{\nu} + \frac{1}{l}$, $c_2 = \max\left(\sqrt{T};1\right)$, $c_3 = \left(\max\left(\nu; \frac{1}{\nu} + \frac{1}{l}\right)\right)^{1/2}$.

Extracting the square root from both parts of inequality (4.13), adding the resulting inequality, after some transformations we arrive at estimate (4.11). Theorem 4.1 is proved. □

Lemma 4.1. *Let the conditions of Theorem 3.1 be satisfied. Then the approximation error components* (4.6)-(4.10) *satisfy the estimates*

$$\left\| \eta_1^{(1)} \right\|_{2,\omega_T^+} \le h \left\| \upsilon \right\|_{\infty,(0,l)} \left\| \frac{\partial^2 u}{\partial \xi^2} \right\|_{2,Q_T}, \tag{4.14}$$

$$\left\| \eta_1^{(2)} \right\|_{2,\omega_T^+} \le \left\| \upsilon_h\left(x\right) - \upsilon\left(x\right) \right\|_{\infty,\overline{\omega}_{h*}} \left\| \frac{\partial u}{\partial \xi} \right\|_{2,Q_T}, \tag{4.15}$$

$$\left\| \eta_2 \right\|_{2,\omega_T^+} \le \sqrt{2} \left\| a \right\|_{\infty,Q_T} \left[3h \left\| \frac{\partial u}{\partial \xi} \right\|_{2,Q_T} + \tau \left\| \frac{\partial u}{\partial \theta} \right\|_{2,Q_T} \right], \tag{4.16}$$

$$\left(\sum_{t\in\omega_\tau} \left\| \eta^{(3)}\left(x,t\right) \right\|^2_{2,\omega_h^+} \right)^{1/2} \le M \left(\frac{h^{3/2}}{\tau^{1/2}} + \tau^{1/2} \right) \left\| u \right\|^{(2,1)}_{2,Q_T}, \tag{4.17}$$

$$\|\eta_4\|_{2,\omega_\tau} \le \sqrt{2l}\,\|H\|^{(0,1)}_{\infty,Q_T}\left[h^{3/2}\left\|\frac{\partial u}{\partial \xi}\right\|_{2,Q_T} + h^{1/2}\tau\,\|u\|_{2,Q_T}\right]. \tag{4.18}$$

Proof. Let us consider only the proof of estimates (4.18).

Using (4.1) and (4.10) it can be shown that the function $\eta_4\,(t)\,,\quad t\in\omega_\tau$ has the representation

$$\eta_4\,(t) = \frac{1}{\tau}\int_{t-\tau}^{t}\left[\sum_{x\in\omega_h^+}\int_{x-h}^{x}\left(H\,(\xi,t)\int_{\xi}^{x}\frac{\partial u\,(\xi_1,\theta)}{\partial \xi_1}d\xi_1\right)d\xi\right]d\theta +$$

$$+\frac{1}{\tau}\int_{t-\tau}^{t}\left[\sum_{x\in\omega_h^+}\int_{x-h}^{x}\left(u\,(\xi,\theta)\int_{\theta}^{t}\frac{\partial H\,(\xi,\theta_1)}{\partial \theta_1}d\theta_1\right)d\xi\right]d\theta.$$

Evaluating this representation, we get

$$|\eta_4\,(t)| \le \frac{h^{3/2}}{\tau^{1/2}}\sum_{x\in\omega_h^+}\max_{x-h\le\xi\le x}|H\,(\xi,t)|\left\|\frac{\partial u}{\partial \xi}\right\|_{2,e^*(x,t)} +$$

$$+h^{1/2}\tau^{1/2}\sum_{x\in\omega_h^+}\max_{(\xi,\theta)\in e^*(x,t)}\left|\frac{\partial H\,(\xi,\theta)}{\partial \theta}\right|\|u\|_{2,e^*(x,t)}\,,\quad t\in\omega_\tau.$$

Based on the resulting inequality, we establish estimate (4.18).

Estimates (4.14), (4.15), (4.16) are proved in a similar way. Estimate (4.17) was proved in [11]. □

Theorem 4.2. *Let the conditions of Theorem 4.1 be satisfied. Then for any controls $\upsilon\in V$ and $\upsilon_h\in V_h$ the estimate of the error of the difference method with respect to the state is*

$$\|y\,(x,t;\upsilon_h) - \overline{u}\,(x,t;\upsilon)\|_{V_2^{1,0}(\omega_T^+)} + \sqrt{\tau}\,\|y_{\bar t}\,(x,t;\upsilon_h) - \overline{u}_{\bar t}\,(x,t;\upsilon)\|_{2,\omega_T^+} \le$$

$$\le M\left[h + \frac{h^{3/2}}{\tau^{1/2}} + \tau^{1/2} + \|\upsilon_h\,(x) - \upsilon\,(x)\|_{\infty,\overline{\omega}_{h*}}\right] = ME\,(h,\tau,\upsilon,\upsilon_h)\,. \tag{4.19}$$

Corollary 4.1. *We can obtain various estimates of the convergence rate of the difference method with respect to the state from (4.19). Let $\upsilon_h\,(x) = \upsilon\,(x)\,,\quad x\in\overline{\omega}_{h*}$. Then the difference boundary value problem (3.2)-(3.4) has convergence rate estimates $O\left(h^{1/2}\right)$ for $\tau\sim h$ or $\tau\sim h^2$ in the grid norm $V_2^{1,0}\left(\omega_T^+\right)$, convergence rate estimates $O\left(h^{5/8}\right)$ for $\tau\sim h^{5/4}$ or $\tau\sim h^{7/4}$, convergence rate estimates $O\left(h^{3/4}\right)$ for $\tau\sim h^{3/2}$.*

5. Estimates of the error and convergence rate of approximations with respect to the functional

Theorem 5.1. *Let the conditions of Theorem 4.1 be satisfied. Then, for any controls $\upsilon\in V$ and $\upsilon_h\in V_h$ for the error of the grid functional (3.2), we have the estimate*

$$|J\,(\upsilon) - J_{h\tau}\,(\upsilon_h)| \le M\left[h + \tau + E\,(h,\tau,\upsilon,\upsilon_h)\right]. \tag{5.1}$$

To study the relationship between problems (2.1)-(2.5) and (3.1)-(3.5), we introduce two mappings $P_h : W_2^1\left(\omega_h^+\right) \to W_2^1(0,l)$ and $Q_h : W_2^1(0,l) \to W_2^1\left(\omega_h^+\right)$, acting according to the rules:

$$P_h \upsilon_h(\xi) = \begin{cases} \upsilon_h(0.5h), \ 0 \le \xi \le 0.5h, \\ \upsilon_h(x-0.5h) + \upsilon_{hx}(x-0.5h)(\xi - x + 0.5h), \\ x - 0.5h \le \xi \le x + 0.5h, x \in \omega_h, \\ \upsilon_h(l-0.5h), \ l - 0.5h \le \xi \le l, \end{cases}$$

$$Q_h \upsilon(x) = \upsilon(x-0.5h), \ x \in \omega_h^+,$$

It is easy to show that for any controls $\upsilon_h \in V_h$, $\upsilon \in V$, there are inclusion places $P_h \upsilon_h(\xi) \in V$, $Q_h \upsilon(x) \in V_h$.

Lemma 5.1. *For arbitrary controls $\upsilon \in V$, $\upsilon_h \in V_h$, we have the estimates*

$$\max\left\{\left|J(\upsilon) - J_{h\tau}(Q_h\upsilon)\right|, \left|J(P_h\upsilon_h) - J_{h\tau}(\upsilon_h)\right|\right\} \le M\left[h + \frac{h^{3/2}}{\tau^{1/2}} + \tau^{1/2}\right].$$

These estimates follow from (5.1) and are based on the definitions of mappings P_h, Q_h.

Theorem 5.2. *Let the conditions of Theorem 4.1 be satisfied. Then approximations* (3.1)-(3.5) *satisfy the estimate*

$$\left|J_{h\tau*} - J_*\right| \le M\left[h + \frac{h^{3/2}}{\tau^{1/2}} + \tau^{1/2}\right]. \tag{5.2}$$

Here $J_{h\tau*} \equiv \inf\left\{J_{h\tau}(\upsilon_h) : \upsilon_h \in V_h\right\}$.

If, in addition, the sequence of controls $\{\upsilon_{h\varepsilon}\}$ is determined from the conditions

$$\upsilon_{h\tau} \in V_h, J_{h\tau*} \le J_{h\tau}(\upsilon_{h\varepsilon}) \le J_{h\tau*} + \varepsilon_{h\tau},\ \varepsilon_{h\tau} \ge 0,\ \varepsilon_{h\tau} \to 0 \ for\ h,\tau \to 0 \tag{5.3}$$

then the sequence $\{P_h\ \upsilon_{h\varepsilon}\}$ satisfies the estimate

$$0 \le J(P_h\upsilon_{h\varepsilon}) - J_* \le M\left[h + \frac{h^{3/2}}{\tau^{1/2}} + \tau^{1/2}\right] + \varepsilon_{h\tau}. \tag{5.4}$$

Proof. Let's take some control $\upsilon_* \in V_*$. It's obvious that $Q_h\upsilon_* \in V_h$. From this and Lemma 4.1 it follows

$$J_{h\tau*} \le J_{h\tau}(Q_h\upsilon_*) \le J(\upsilon_*) +$$
$$+M\left[h + \tau^{1/2} + h^{3/2}\tau^{-1/2}\right] = J_{h*} + M\left[h + \tau^{1/2} + h^{3/2}\tau^{-1/2}\right]. \tag{5.5}$$

Let's take some control

$$\upsilon_{h*} \in V_{h*} = \left\{\upsilon_{h*} \in V_h : J_{h\tau}(\upsilon_{h*}) = J_{h\tau*} \equiv \inf\left\{J_{h\tau}(\upsilon_h) : \upsilon_h \in V_h\right\}\right\}.$$

Then it is clear that $P_h\upsilon_{h*} \in V_h$. From this and Lemma 4.1 it follows

$$J_* \le J(P_h\upsilon_{h*}) \le J_{h\tau}(\upsilon_{h*}) + M\left[h + \tau^{1/2} + h^{3/2}\tau^{-1/2}\right] =$$
$$= J_{h\tau*} + M\left[h + \tau^{1/2} + h^{3/2}\tau^{-1/2}\right]. \tag{5.6}$$

Estimates (5.2) follow from (5.5), (5.6).

Consider the sequence $\{ \upsilon_{h\varepsilon}\} \subset V_h$ from (5.3). Then $P_h\, \upsilon_{h\varepsilon} \in V$ and

$$0 \leq J\left(P_h \upsilon_{h\varepsilon}\right) - J_* = \left[J\left(P_h \upsilon_{h\varepsilon}\right) - J_{h\tau}\left(\upsilon_{h\varepsilon}\right)\right] + \left[J_{h\tau}\left(\upsilon_{h\varepsilon}\right) - J_{h\tau *}\right] + \left[J_{h\tau *} - J_*\right].$$

From this, and also form inequalities (5.2), (5.3) and Lemma 4.1 it follows estimate (5.4). Theorem 5.2 is proved. □

Corollary 5.1. *From* (5.2) *and* (5.4) *one can obtain different estimates of the rate of convergence. For example, for* $\tau \sim h$ *or* $\tau \sim h^2$ *the estimates* $|J_{h\tau *} - J_*| \leq Mh^{1/2}$, $0 \leq J_{h\tau}\left(\upsilon_{h\varepsilon}\right) - J_* \leq Mh^{1/2} + \varepsilon_{h\tau}$, *for* $\tau \sim h^{5/4}$ *or* $\tau \sim h^{7/4}$ *the estimates* $|J_{h\tau *} - J_*| \leq Mh^{5/8}$, $0 \leq J_{h\tau}\left(\upsilon_{h\varepsilon}\right) - J_* \leq Mh^{5/8} + \varepsilon_{h\tau}$, *for* $\tau \sim h^{3/2}$ *the estimates* $|J_{h\tau *} - J_*| \leq Mh^{3/4}$, $0 \leq J_{h\tau}\left(\upsilon_{h\varepsilon}\right) - J_* \leq Mh^{3/4} + \varepsilon_{h\tau}$ *are valid.*

Moreover, in these cases the sequence of controls $\{P_h\, \upsilon_{h\varepsilon}\}$ is minimizing for problem (2.1)-(2.5) and converges $W_2^1\left(0, l\right)$ weakly to V_*.

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