

EXPONENTIAL STABILITY OF BAM-TYPE NEURAL NETWORKS WITH CONFORMABLE DERIVATIVE

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Abstract. This paper investigates the fractional exponential stability of bidirectional associative memory (BAM) neural networks with conformable derivative. Applying the contraction mapping theorem existence and uniqueness of the equilibrium is studied. Furthermore, an appropriate Lyapunov function is stated to get exponential stability. Numerical examples are given to illustrate the obtained results.

1. Introduction

Recently, artificial neural networks have been created by modeling biological neural networks and have been used in the area of image processing, pattern recognition, associative memory, optimization problems etc. The development of neural networks started with the studies of McCulloch and Pits [26]. The most well-known types of neural networks are Hopfield, Cohen-Grossberg, recurrent and bidirectional associative memory (BAM) neural networks. BAM neural networks were proposed by Kosko [19, 20]. The bidirectional associative memory allows mapping between any two binary patterns. In 1989, it was applied to the spectral density functions derived from process signals by Mathai and Upadhyaya [25]. They obtained that the test patterns were classified successfully. The theory and applications of neural networks are stated in the books [14, 34, 4]. Within the examinations of artificial neural networks systems, one of the aim is to state the conditions for the existence and stability of the unique equilibrium. Thus, numerous researchers have dealt with to get the stability of neural networks. In particular, the stability of BAM-type neural networks has been investigated in the papers [27, 23, 17, 22, 29, 13, 3]. Later, the stability of some neural networks model with piecewise constant arguments has studied in [10, 11, 12].

In 2008, Lundstrom et al. observed that fractional differentiation provides single neurons with a form of adaptation in which no single time scale is preferred [24]. Then in 2009, a fractional Hopfield neural network model was investigated by Boroomand and Menhaj [6]. Later, stability of Hopfield-type systems with Riemann-Liouville and Caputo derivatives has been dealt with in the works

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[9, 31, 36]. In 2014, uniform stability results for a fractional order BAM-type neural networks were obtained [35]. Cao and Bai investigated the stability of BAM neural networks in finite time [7, 8]. In the works, Caputo derivative is considered. In the first work, they used generalized Gronwall inequality to construct the finite-time stability criterion. In the second work, Krasnoselskii's fixed point theorem and the inequality technique were applied to obtain the results. Since Riemann-Liouville and Caputo derivatives do not satisfy some basic properties of integer order derivative, sometimes it is difficult to use them in applications. Recently, Khalil et al. have proposed conformable fractional derivative, which seems an extension of the usual integer-order derivative [18]. Then some properties such as the chain rule, exponential function, and Gronwall's inequality for conformable fractional derivative were established by Abdeljawad [1]. The physical interpretation was investigated by Zhao and Luo [38]. Then diffusion equations with conformable derivative were examined in the papers [39, 37]. Stability analysis of general nonlinear systems with conformable derivative was stated in [32]. Recently, differential equations with conformable derivative have been investigated by the researchers extensively [33, 28, 2, 15, 5]. Ulam-Hyers stability of the integro differential equations with conformable derivative has been dealt with in [33]. In 2021, fractional variational problems were studied in the paper [28]. Ahmed has studied conformable fractional stochastic delay systems with impulses driven by the Rosenblatt process [2]. Hammouch et al. have dealt with an epidemic model with conformable derivatives and the authors explained that fractional order models are useful for describing the mechanism of propagation and controlling epidemics, including Ebola virus disease [15]. A co-infection model with conformable derivative has been investigated in [5]. Moreover, exponential stability and Ulam-type stability of the solutions have been stated in the same paper. When we investigate the studies on neural networks with conformable derivative, we have seen the recent papers [21, 16, 30]. In 2021, the authors examined a Hopfield-type system with conformable derivative, and they stated fractional exponential stability of the equilibrium [21]. Then Huyen et al. obtained LMI conditions for exponential stability for the Hopfield neural networks with conformable derivative [16]. A conformable transfer function for artificial neural networks has been used in the paper [30]. In that paper, the authors obtained that the systems with conformable transfer function is more accurate than the integer-order systems.

When existent works are examined, it is seen that BAM-type neural networks with non-integer order derivatives have been dealt with in the papers [7, 8, 35]. However, the results of the papers [7, 8] can be applied to the systems on a finite interval $[0, T]$. The behavior of the solutions as $t \rightarrow \infty$ can not be explained by the results of the papers [7, 8]. In the paper [35], the uniform stability of a BAM-type fractional order neural network was studied. But, exponential stability is not involved in [35]. On the other hand, it seems that conformable calculus is useful in applications. So, in the present paper, we study the stability of BAM-type neural networks with conformable fractional derivative. Since one of the goals of the study of neural networks is to obtain a convergence rate, in particular, we shall get the conditions for the exponential stability of the equilibrium. We would like to note that there exist some papers on the exponential stability of

BAM-type neural networks [3, 10, 11, 27, 29]. But, in these papers BAM-type neural networks with integer order derivative are investigated. Thus, the main advantage of the present paper is to obtain the results on the exponential stability of BAM-type neural networks with non-integer order derivative. In the present paper, first, we obtain the sufficient conditions for the existence and uniqueness of the equilibrium using the well-known Banach contraction principle. Then Lyapunov function is used to prove the conformable exponential stability of the equilibrium.

In this paper, we deal with bidirectional associative memory (BAM) type neural network

$$\begin{cases} T_\alpha(x_i(t)) = -a_i x_i(t) + \sum_{j=1}^n b_{ij} f_j(y_j(t)) + I_i. \\ T_\alpha(y_i(t)) = -c_i y_i(t) + \sum_{j=1}^n d_{ij} g_j(x_j(t)) + J_i, i = 1, 2, \dots, n \end{cases} \quad (1.1)$$

$$\begin{cases} x(t_0) = x_0 \in \mathbb{R}^n \\ y(t_0) = y_0 \in \mathbb{R}^n \end{cases} \quad (1.2)$$

where $x_i(t)$ and $y_i(t)$ represent the state variable of the i -th unit at time t , f_j and g_j indicate the activation functions of the j th neuron, $a_i > 0$ and $c_i > 0$ denote the rates with which reset their potentials to their resting states when disconnected from the network, b_{ij} , d_{ij} , I_i and J_i are real constant, b_{ij} and d_{ij} represent synaptic connection weights, and I_i , J_i indicate external inputs from the outside of the network. For $w = (u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n)^T \in \mathbb{R}^{2n}$ we shall

use the vector norms $\|w\|_1 = \sum_{k=1}^n (|u_k| + |v_k|)$, $\|w\|_2 = \sqrt{\sum_{k=1}^n (u_k^2 + v_k^2)}$.

We organize this paper as follows: In Section 2, a brief description of the conformable fractional derivative is presented. Section 3 involves the new theorems. To show the existence and uniqueness of the equilibrium of the BAM-type conformable system (1.1) two different norms are used. Then a Lyapunov function is used to obtain fractional exponential stability of the equilibrium. Numerical examples are considered in Section 4 to illustrate the results. The conclusion of the paper is given in the last section.

2. Preliminaries

Now, some basic properties of conformable fractional derivative will be introduced.

Definition 2.1. [18] Let a function $f : [0, \infty) \rightarrow \mathbb{R}$ is given. Then the conformable fractional derivative of the function f of order α is defined by

$$(T_\alpha f)(t) = \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}$$

for all $t > 0$, $\alpha \in (0, 1]$.

Lemma 2.1. [18] Let $0 < \alpha \leq 1$ and f, g be α -differentiable at a point $t > 0$. Then

- (1) $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$.
- (2) $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.

Definition 2.1 was generalized to the interval $[t_0, \infty)$ as follows [1]:

$$(T_\alpha^{t_0} f)(t) = \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon(t - t_0)^{1-\alpha}) - f(t)}{\epsilon}.$$

The fractional conformable exponential function is stated in [1]:

$$E_\alpha(\rho, t) = \exp\left(\rho \frac{t^\alpha}{\alpha}\right)$$

where $\alpha \in (0, 1)$, $\rho \in \mathbb{R}$, and $t \geq 0$.

We shall use $(T_\alpha f)(t)$ instead of $(T_\alpha^{t_0} f)(t)$ for simplicity.

The following result is useful for the proof of exponential stability result.

Lemma 2.2. [32] *Suppose that $0 < \beta < 1$, $g \in ([t_0, \infty), \mathbb{R}^+)$ is β -differentiable on (t_0, ∞) , and*

$$(T_\beta g)(s) \leq -\rho g(s), \quad \rho > 0 \text{ is a constant.}$$

Then

$$g(s) \leq E_\beta(-\rho, s - t_0) g(t_0).$$

3. Main results

We shall state the results in this section. The existence and uniqueness of the equilibrium point is proved by the Banach contraction principle. Then a fractional exponential stability result of the equilibrium for the BAM-type neural networks (1.1) is proved. A vector (x^*, y^*) such that for $i = 1, 2, \dots, n$,

$$\begin{cases} x_i^* = \frac{1}{a_i} \left(\sum_{j=1}^n b_{ij} f_j(y_j^*) + I_i \right) \\ y_i^* = \frac{1}{c_i} \left(\sum_{j=1}^n d_{ij} g_j(x_j^*) + J_i \right) \end{cases} \quad (3.1)$$

where $x^* = (x_1^*, x_2^*, \dots, x_n^*)^T$, $y^* = (y_1^*, y_2^*, \dots, y_n^*)^T$ is called equilibrium point of the system (1.1).

Definition 3.1. The equilibrium (x^*, y^*) of the conformable system (1.1) is said that fractional exponentially stable if there exist positive constants ρ and K such that

$$\|(x(t), y(t)) - (x^*, y^*)\| \leq K E_\alpha(-\rho, t - t_0) \|(x_0, y_0) - (x^*, y^*)\|, \quad t \geq t_0.$$

The activation functions f_j and g_j are assumed Lipschitz continuous, that is there exist positive constants L_j and M_j , $j = 1, \dots, n$, such that

$$\begin{cases} |f_j(w_1) - f_j(w_2)| \leq L_j |w_1 - w_2|, \forall w_1, w_2 \in \mathbb{R} \\ |g_j(w_1) - g_j(w_2)| \leq M_j |w_1 - w_2|, \forall w_1, w_2 \in \mathbb{R} \end{cases} \quad (3.2)$$

In the following theorem, we shall prove existence of unique equilibrium point using l_1 norm.

Theorem 3.1. *Let (3.2) holds. Suppose that*

$$a_i > M_i \sum_{j=1}^n |d_{ji}|, i = 1, 2, \dots, n, \quad (3.3)$$

and

$$c_i > L_i \sum_{j=1}^n |b_{ji}|, i = 1, 2, \dots, n. \quad (3.4)$$

Then the conformable BAM-type neural networks system (1.1) has a unique equilibrium.

Proof. Substituting $a_i x_i^* = u_i$, $c_i y_i^* = v_i$, $i = 1, 2, \dots, n$, in (3.1) we get

$$\begin{aligned} u_i &= \sum_{j=1}^n b_{ij} f_j \left(\frac{v_j}{c_j} \right) + I_i, \\ v_i &= \sum_{j=1}^n d_{ij} g_j \left(\frac{u_j}{a_j} \right) + J_i. \end{aligned}$$

Consider the mapping $R : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$, $R(\theta) = (N(\theta), P(\theta))^T$ defined by $N(\theta) = (N_1(\theta), N_2(\theta), \dots, N_n(\theta))^T$, $P(\theta) = (P_1(\theta), P_2(\theta), \dots, P_n(\theta))^T$

$$N_i(\theta) = \sum_{j=1}^n b_{ij} f_j \left(\frac{v_j}{c_j} \right) + I_i, \quad i = 1, 2, \dots, n, \quad (3.5)$$

$$P_i(\theta) = \sum_{j=1}^n d_{ij} g_j \left(\frac{u_j}{a_j} \right) + J_i, \quad i = 1, 2, \dots, n, \quad (3.6)$$

where $\theta = (u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n) \in \mathbb{R}^{2n}$.

For $\theta_1 = (u_1^{(1)}, u_2^{(1)}, \dots, u_n^{(1)}, v_1^{(1)}, v_2^{(1)}, \dots, v_n^{(1)})$ and

$\theta_2 = (u_1^{(2)}, u_2^{(2)}, \dots, u_n^{(2)}, v_1^{(2)}, v_2^{(2)}, \dots, v_n^{(2)})$, from the condition (3.2) it is obtained

that

$$\begin{aligned}
 \|R(\theta_1) - R(\theta_2)\|_1 &= \sum_{i=1}^n [|N_i(\theta_1) - N_i(\theta_2)| + |P_i(\theta_1) - P_i(\theta_2)|] \\
 &= \sum_{i=1}^n \left| \sum_{j=1}^n b_{ij} \left(f_j \left(\frac{v_j^{(1)}}{c_j} \right) - f_j \left(\frac{v_j^{(2)}}{c_j} \right) \right) \right| \\
 &\quad + \sum_{i=1}^n \left| \sum_{j=1}^n d_{ij} \left(g_j \left(\frac{u_j^{(1)}}{a_j} \right) - g_j \left(\frac{u_j^{(2)}}{a_j} \right) \right) \right| \\
 &\leq \sum_{i=1}^n \sum_{j=1}^n \left[|b_{ij}| L_j \left| \frac{v_j^{(1)} - v_j^{(2)}}{c_j} \right| \right. \\
 &\quad \left. + |d_{ij}| M_j \left| \frac{u_j^{(1)} - u_j^{(2)}}{a_j} \right| \right] \\
 &= \sum_{i=1}^n \sum_{j=1}^n \left(\frac{|b_{ji}| L_i}{c_i} |v_i^{(1)} - v_i^{(2)}| \right. \\
 &\quad \left. + \frac{|d_{ji}| M_i}{a_i} |u_i^{(1)} - u_i^{(2)}| \right). \tag{3.7}
 \end{aligned}$$

It is clear that the conditions (3.3) – (3.4) imply

$$\sum_{j=1}^n \frac{|d_{ji}| M_i}{a_i} < 1 \text{ and } \sum_{j=1}^n \frac{|b_{ji}| L_i}{c_i} < 1, \quad i = 1, 2, \dots, n.$$

Denote

$$A = \max_{1 \leq i \leq n} \left\{ \frac{L_i}{c_i} \sum_{j=1}^n |b_{ji}|, \frac{M_i}{a_i} \sum_{j=1}^n |d_{ji}| \right\}.$$

So, from the inequality (3.7) we get

$$\begin{aligned}
 \|R(\theta_1) - R(\theta_2)\|_1 &\leq A \sum_{i=1}^n \left(|v_i^{(1)} - v_i^{(2)}| + |u_i^{(1)} - u_i^{(2)}| \right) \\
 &= A \|\theta_1 - \theta_2\|_1.
 \end{aligned}$$

Since $A < 1$, $R : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ is a contraction. So, by the contraction mapping principle there exists a unique fixed point $\theta \in \mathbb{R}^{2n}$ of the map $R : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ such that $R(\theta) = \theta$ and $(x_i^*, y_i^*) \in \mathbb{R}^{2n}$ is the unique equilibrium of the system (1.1).

In the following theorem we shall prove existence of unique equilibrium point using l_2 norm.

Theorem 3.2. *Let assume that (3.2) holds, and that*

$$\left(\sum_{i=1}^n \sum_{j=1}^n \left(\frac{M_i}{a_i} d_{ji} \right)^2 \right)^{1/2} < 1, \tag{3.8}$$

and

$$\left(\sum_{i=1}^n \sum_{j=1}^n \left(\frac{L_i b_{ji}}{c_i} \right)^2 \right)^{1/2} < 1. \quad (3.9)$$

Then the conformable BAM-type neural networks system (1.1) has a unique equilibrium.

Proof. Substitute $a_i x_i^* = u_i$, $c_i y_i^* = v_i$, $i = 1, 2, \dots, n$, in (3.1). Then we get

$$\begin{aligned} u_i &= \sum_{j=1}^n b_{ij} f_j \left(\frac{v_j}{c_j} \right) + I_i, \quad i = 1, 2, \dots, n \\ v_i &= \sum_{j=1}^n d_{ij} g_j \left(\frac{u_j}{a_j} \right) + J_i, \quad i = 1, 2, \dots, n. \end{aligned}$$

Similarly to the proof of Theorem 3.1, we will show that the mapping $R : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$, $R(\theta) = (N(\theta), P(\theta))^T$ defined by $N(\theta) = (N_1(\theta), N_2(\theta), \dots, N_n(\theta))^T$, $P(\theta) = (P_1(\theta), P_2(\theta), \dots, P_n(\theta))^T$ is a contraction with the norm $\|\cdot\|_2$ on \mathbb{R}^{2n} . For $\theta_1, \theta_2 \in \mathbb{R}^{2n}$, $\theta_1 = (u_1^{(1)}, u_2^{(1)}, \dots, u_n^{(1)}, v_1^{(1)}, v_2^{(1)}, \dots, v_n^{(1)})$ and $\theta_2 = (u_1^{(2)}, u_2^{(2)}, \dots, u_n^{(2)}, v_1^{(2)}, v_2^{(2)}, \dots, v_n^{(2)})$, it is obtained that

$$\begin{aligned} \|R(\theta_1) - R(\theta_2)\|_2^2 &= \sum_{i=1}^n \left[(N_i(\theta_1) - N_i(\theta_2))^2 + (P_i(\theta_1) - P_i(\theta_2))^2 \right] \\ &= \sum_{i=1}^n \left[\left| \sum_{j=1}^n b_{ij} \left(f_j \left(\frac{v_j^{(1)}}{c_j} \right) - f_j \left(\frac{v_j^{(2)}}{c_j} \right) \right) \right|^2 \right. \\ &\quad \left. + \left| \sum_{j=1}^n d_{ij} \left(g_j \left(\frac{u_j^{(1)}}{a_j} \right) - g_j \left(\frac{u_j^{(2)}}{a_j} \right) \right) \right|^2 \right] \\ &\leq \sum_{i=1}^n \left[\left(\sum_{j=1}^n |b_{ij}| L_j \left| \frac{v_j^{(1)} - v_j^{(2)}}{c_j} \right| \right)^2 \right. \\ &\quad \left. + \left(\sum_{j=1}^n |d_{ij}| M_j \left| \frac{u_j^{(1)} - u_j^{(2)}}{a_j} \right| \right)^2 \right]. \end{aligned}$$

Hence by Cauchy Schwarz inequality we get

$$\begin{aligned}
\|R(\theta_1) - R(\theta_2)\|_2^2 &\leq \sum_{i=1}^n \left[\sum_{j=1}^n \left(\frac{|b_{ij}| L_j}{c_j} \right)^2 \sum_{j=1}^n |v_j^{(1)} - v_j^{(2)}|^2 \right. \\
&\quad \left. + \sum_{j=1}^n \left(\frac{|d_{ij}| M_j}{a_j} \right)^2 \sum_{j=1}^n |u_j^{(1)} - u_j^{(2)}|^2 \right] \\
&\leq \sum_{i=1}^n \left[\sum_{j=1}^n \left(\frac{|b_{ji}| L_i}{c_i} \right)^2 \sum_{j=1}^n |v_j^{(1)} - v_j^{(2)}|^2 \right. \\
&\quad \left. + \sum_{j=1}^n \left(\frac{|d_{ji}| M_i}{a_i} \right)^2 \sum_{j=1}^n |u_j^{(1)} - u_j^{(2)}|^2 \right]. \quad (3.10)
\end{aligned}$$

Denote

$$B = \max \left\{ \sum_{i=1}^n \sum_{j=1}^n \left(\frac{L_i}{c_i} b_{ji} \right)^2, \sum_{i=1}^n \sum_{j=1}^n \left(\frac{M_i}{a_i} d_{ji} \right)^2 \right\}.$$

It is clear that the conditions (3.8) – (3.9) imply $B < 1$. So, from the inequality (3.10) we get

$$\|R(\theta_1) - R(\theta_2)\|_2^2 \leq B \sum_{j=1}^n \left[|v_j^{(1)} - v_j^{(2)}|^2 + |u_j^{(1)} - u_j^{(2)}|^2 \right]$$

and so,

$$\|R(\theta_1) - R(\theta_2)\|_2 \leq \sqrt{B} \|\theta_1 - \theta_2\|_2.$$

Since $\sqrt{B} < 1$, $R : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ is a contraction. So, by the contraction mapping principle there exists a unique fixed point $\theta \in \mathbb{R}^{2n}$ of the map $R : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ such that $R(\theta) = \theta$ and $(x_i^*, y_i^*) \in \mathbb{R}^{2n}$ is the unique equilibrium of the system (1.1).

Theorem 3.3. *Let (3.2) holds. Suppose that*

$$a_i > \max \left\{ \sum_{j=1}^n L_j |b_{ij}|, M_i \sum_{j=1}^n |d_{ij}| \right\}, \quad i = 1, 2, \dots, n, \quad (3.11)$$

and

$$c_i > \max \left\{ L_i \sum_{j=1}^n |b_{ji}|, \sum_{j=1}^n M_j |d_{ji}| \right\}, \quad i = 1, 2, \dots, n. \quad (3.12)$$

Then the equilibrium of the system (1.1) is fractional exponentially stable.

Proof. First, it is observed that if (3.11) and (3.12) are satisfied, then the conditions (3.3) and (3.4) are also satisfied. So, under the conditions of Theorem 3.3, there exists unique equilibrium (x^*, y^*) of the system (1.1). Substituting

$z_i(t) = x_i(t) - x_i^*$ and $w_i(t) = y_i(t) - y_i^*$ into (1.1) leads to

$$\begin{cases} T_\alpha(z_i(t)) = -a_i z_i(t) + \sum_{j=1}^n b_{ij} F_j(w_j(t)), i = 1, 2, \dots, n, \\ T_\alpha(w_i(t)) = -c_i w_i(t) + \sum_{j=1}^n d_{ij} G_j(z_j(t)), i = 1, 2, \dots, n, \end{cases} \quad (3.13)$$

where $F_j(w_j(t)) = f_j(w_j(t) + y_j^*) - f_j(y_j^*)$ and $G_j(z_j(t)) = g_j(z_j(t) + x_j^*) - g_j(x_j^*)$. It is easy to see that $F_j(0) = 0$, $G_j(0) = 0$. Moreover, it is observed that F_j , G_j satisfy (3.2) with L_j , M_j , $j = 1, 2, \dots, n$, respectively. Hence, the fractional exponential stability of the zero solution of (3.13) is equivalent to the fractional exponential stability of the equilibrium (x^*, y^*) of (1.1). Therefore, we discuss the stability of the zero solution of (3.13). Now, we shall use following Lyapunov function to discuss the exponential stability of (3.13).

$$V(t) = \frac{1}{2} \sum_{i=1}^n (z_i^2(t) + w_i^2(t)).$$

Calculating the conformable derivative of the function V along the solutions of (3.13), we have

$$\begin{aligned} T_\alpha(V(t)) &= \sum_{i=1}^n (z_i(t) T_\alpha(z_i(t)) + w_i(t) T_\alpha(w_i(t))) \\ &= \sum_{i=1}^n \left[z_i(t) \left(-a_i z_i(t) + \sum_{j=1}^n b_{ij} F_j(w_j(t)) \right) \right. \\ &\quad \left. + w_i(t) \left(-c_i w_i(t) + \sum_{j=1}^n d_{ij} G_j(z_j(t)) \right) \right] \\ &\leq \sum_{i=1}^n (-a_i z_i^2(t) - c_i w_i^2(t) \\ &\quad + \left| z_i(t) \sum_{j=1}^n b_{ij} F_j(w_j(t)) + w_i(t) \sum_{j=1}^n d_{ij} G_j(z_j(t)) \right|). \end{aligned}$$

Since F_j and G_j , $j = 1, 2, \dots, n$ satisfy the condition (3.2) with L_j and M_j respectively, it is obtained that

$$\begin{aligned}
T_\alpha(V(t)) &\leq \sum_{i=1}^n \left(-a_i z_i^2(t) - c_i w_i^2(t) + |z_i(t)| \sum_{j=1}^n |b_{ij}| L_j |w_j(t)| \right. \\
&\quad \left. + |w_i(t)| \sum_{j=1}^n |d_{ij}| M_j |z_j(t)| \right) \\
&= \sum_{i=1}^n \left(-a_i z_i^2(t) - c_i w_i^2(t) + L_i |w_i(t)| \sum_{j=1}^n |b_{ji}| |z_j(t)| \right. \\
&\quad \left. + M_i |z_i(t)| \sum_{j=1}^n |d_{ji}| |w_j(t)| \right) \\
&\leq \sum_{i=1}^n \left[-a_i z_i^2(t) - c_i w_i^2(t) + \sum_{j=1}^n \frac{L_i |b_{ji}|}{2} (|z_j(t)|^2 + |w_i(t)|^2) \right. \\
&\quad \left. + \sum_{j=1}^n \frac{M_i |d_{ji}|}{2} (|z_i(t)|^2 + |w_j(t)|^2) \right].
\end{aligned}$$

Hence

$$\begin{aligned}
T_\alpha(V(t)) &\leq -\sum_{i=1}^n \left(a_i - \sum_{j=1}^n \frac{L_j |b_{ij}|}{2} - \frac{M_i}{2} \sum_{j=1}^n |d_{ji}| \right) z_i^2(t) \\
&\quad - \sum_{i=1}^n \left(c_i - \frac{L_i}{2} \sum_{j=1}^n |b_{ji}| - \sum_{j=1}^n \frac{M_j |d_{ij}|}{2} \right) w_i^2(t) \\
&\leq -\min_{1 \leq i \leq n} \left\{ \left(a_i - \frac{1}{2} \sum_{j=1}^n L_j |b_{ij}| - \frac{M_i}{2} \sum_{j=1}^n |d_{ji}| \right), \right. \\
&\quad \left. \left(c_i - \frac{L_i}{2} \sum_{j=1}^n |b_{ji}| - \frac{1}{2} \sum_{j=1}^n M_j |d_{ij}| \right) \right\} \sum_{i=1}^n (z_i^2(t) + w_i^2(t))
\end{aligned} \tag{3.14}$$

Denote

$$C = \min_{1 \leq i \leq n} \left\{ \left(a_i - \frac{1}{2} \sum_{j=1}^n (|b_{ij}| L_j + |d_{ji}| M_i) \right), \left(c_i - \frac{1}{2} \sum_{j=1}^n (|b_{ji}| L_i + |d_{ij}| M_j) \right) \right\}$$

By (3.11) and (3.12), $C > 0$. Hence, from (3.14) we get

$$(T_\alpha V(t)) \leq -2CV(t).$$

Applying Lemma 2.2 we obtain that

$$V(t) \leq E_\alpha(-2C, t - t_0) V(t_0).$$

Hence

$$\|(z(t), w(t))\|_2 \leq E_\alpha(-C, t - t_0) \|(z(t_0), w(t_0))\|_2$$

So, we conclude that the zero solution of (3.13) is fractional exponentially stable.

Remark 3.1. In the papers [7, 8], the authors studied the finite time stability of fractional order BAM-type systems. That is, the results are not valid on the interval $[0, \infty)$. In [35], uniform stability results for fractional order BAM-type neural networks were studied. But, exponential stability is not included in that work.

Remark 3.2. In [21, 16], the authors dealt with the fractional exponential stability of neural networks with conformable derivative. But, in the works Hopfield neural networks are studied.

4. Numerical Examples

Example 4.1. Let us consider the system

$$\begin{cases} T_\alpha(x_1(t)) = -0,8x_1(t) - 0,06f(y_1(t)) + 0,05f(y_2(t)) + 0,004 \\ T_\alpha(x_2(t)) = -0,8x_2(t) - 0,03f(y_1(t)) + 0,06f(y_2(t)) + 0,001 \\ T_\alpha(y_1(t)) = -0,6y_1(t) - 0,03f(x_1(t)) + 0,03f(x_2(t)) + 0,001 \\ T_\alpha(y_2(t)) = -0,6y_2(t) - 0,02f(x_1(t)) + 0,04f(x_2(t)) + 0,002 \end{cases} \quad (4.1)$$

where $f(z) = 0.5(|z + 1| - |z - 1|)$ satisfies the Lipschitz condition. Here, it is clear that $a_1 = a_2 = 0.8$, $c_1 = c_2 = 0.6$, $b_{11} = -0.06$, $b_{12} = 0.05$, $I_1 = 0.004$, $b_{21} = -0.03$, $b_{22} = 0.06$, $I_2 = 0.001$, $d_{11} = -0.03$, $d_{12} = 0.03$, $J_1 = 0.001$, $d_{21} = -0.02$, $d_{22} = 0.04$, $J_2 = 0.002$, and the conditions of Theorem 3.3 are satisfied. So, the system (4.1) has a unique equilibrium which is fractional exponentially stable. After computer calculations, it is seen that the equilibrium of the system (4.1) is $(0.00509242, 0.00143881, 0.00148399, 0.00325951)$. In this example, taking $\alpha = 0.1$, and $\alpha = 1$, the fractional exponential stability of the equilibrium is observed in Figure 1, Figure 2, Figure 3 and Figure 4 for the states $x_1(t)$, $x_2(t)$, $y_1(t)$ and $y_2(t)$, respectively.

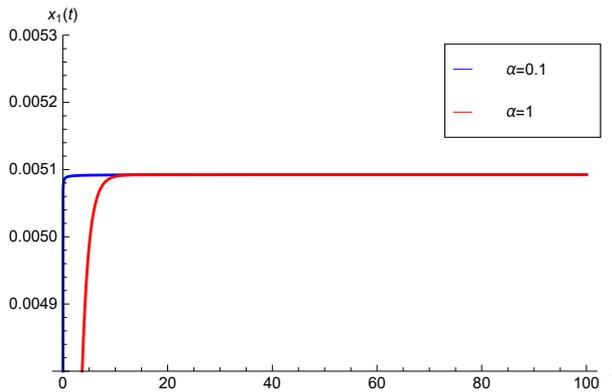


FIGURE 1. Time responses of the state $x_1(t)$ for the system (4.1)

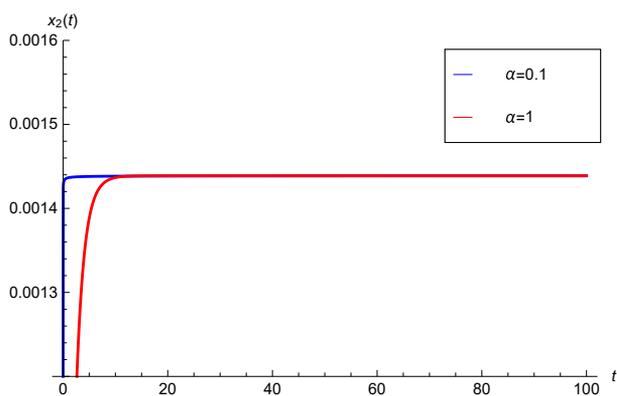


FIGURE 2. Time responses of the state $x_2(t)$ for the system (4.1)

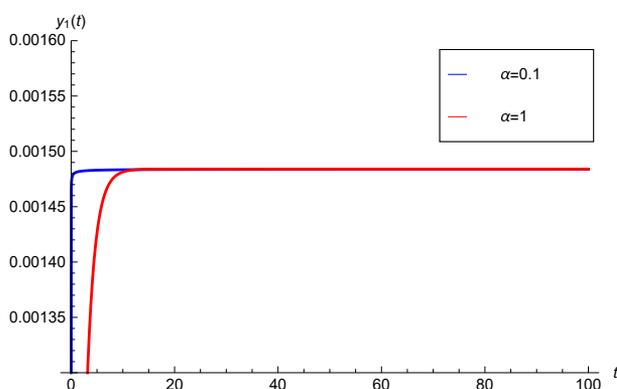


FIGURE 3. Time responses of the state $y_1(t)$ for the system (4.1)

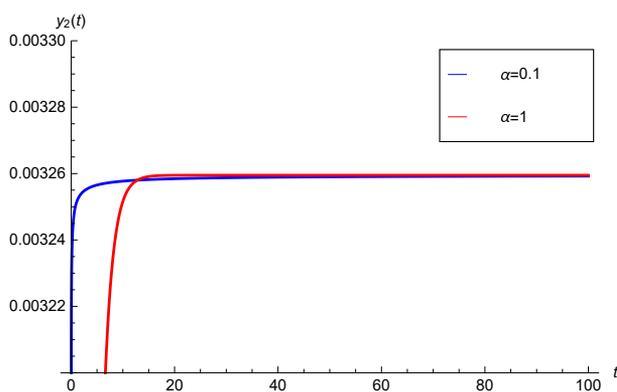


FIGURE 4. Time responses of the state $y_2(t)$ for the system (4.1)

Remark 4.1. From the numeric simulations of the system (4.1), it is observed that the equilibrium is exponentially stable in both cases $\alpha = 0.1$, and $\alpha = 1$. But, when $\alpha = 0.1$, the equilibrium of this system is approached faster.

Example 4.2. Now, consider the following system

$$\begin{cases} T_\alpha(x_1(t)) = -0,8x_1(t) - 0,06f(y_1(t)) + 0,05f(y_2(t)) + 0,02 \\ T_\alpha(x_2(t)) = -0,6x_2(t) - 0,03f(y_1(t)) + 0,06f(y_2(t)) + 0,001 \\ T_\alpha(y_1(t)) = -0,6y_1(t) - 0,03g(x_1(t)) + 0,03g(x_2(t)) + 0,004 \\ T_\alpha(y_2(t)) = -0,6y_2(t) - 0,02g(x_1(t)) + 0,04g(x_2(t)) + 0,002 \end{cases} \quad (4.2)$$

where $f(z) = \frac{1}{z^2+1}$ and $g(z) = 0.5(|z+1| - |z-1|)$ satisfy the Lipschitz condition. It can be seen that the conditions Theorem 3.3 are satisfied. So, there exists fractional exponentially stable unique equilibrium of the system (4.2). The states $x_1(t)$, $x_2(t)$, $y_1(t)$ and $y_2(t)$ of the system (4.2) can be seen in Figure 5, Figure 6, Figure 7 and Figure 8, respectively. Exponential stability of the equilibrium (0.012513, 0.0516663, 0.00862483, 0.00636099) is observed for $\alpha = 0.1$, $\alpha = 0.5$ and $\alpha = 0.9$.

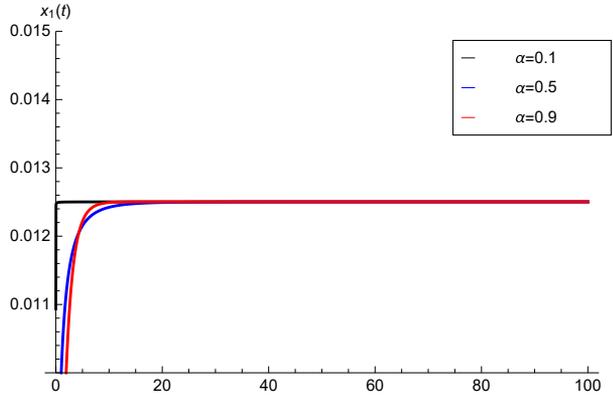


FIGURE 5. Time responses of the state $x_1(t)$ for the system (4.2)

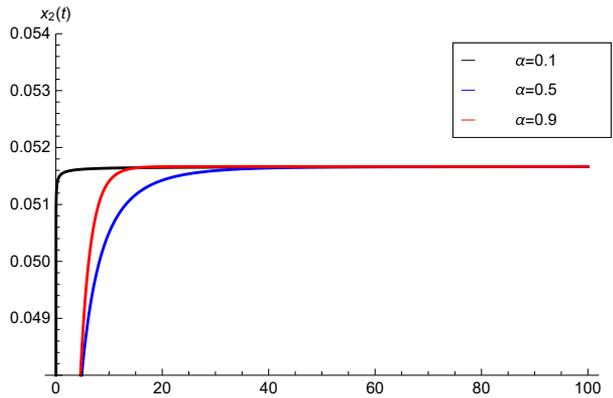


FIGURE 6. Time responses of the state $x_2(t)$ for the system (4.2)

Remark 4.2. In order to check the validity of our results, two examples are given for different α values. Figures 1-8 show the time responses of the state variables. It is seen that the equilibrium point is exponentially stable for $t \geq t_0$, which is the main advantage of the proposed system when we compare it with the existing results on the non-integer order BAM-type neural networks.

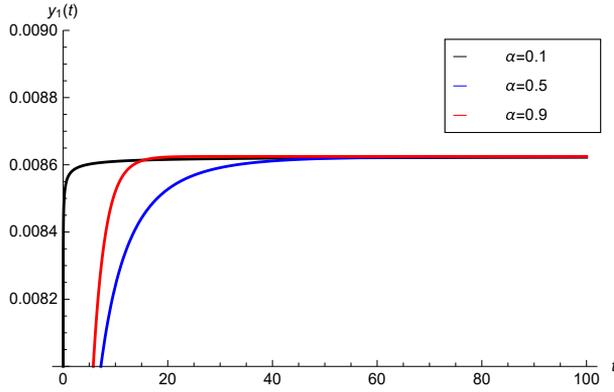


FIGURE 7. Time responses of the state $y_1(t)$ for the system (4.2)

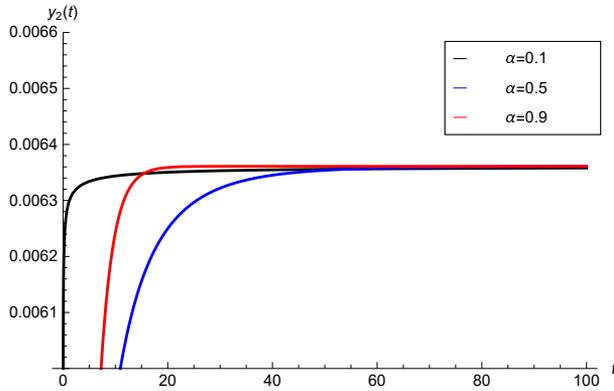


FIGURE 8. Time responses of the state $y_2(t)$ for the system (4.2)

Remark 4.3. Since the conformable derivative has following property [1], we don't need to write a new algorithm for numerical simulations.

If x is differentiable, then

$$(T_\alpha^{t_0} x)(t) = (t - t_0)^{1-\alpha} x'(t).$$

So, we could use Mathematica software for the numerical simulations.

Remark 4.4. In the present paper, we have obtained the results for $0 < \alpha \leq 1$. Although the convergence rate for different values of α is not calculated in the present paper, it is thought that exponential stability, and α synchronization for neural networks can be investigated for $n < \alpha \leq n + 1$, $n \in \mathbb{N}$.

5. Conclusion

In this paper, a BAM-type neural network with conformable derivative is investigated. Mainly, sufficient conditions for the exponential stability of the unique equilibrium are obtained. Numerical examples are examined to observe the validity of the theoretical results. We aim to investigate delay models as well as impulsive systems with conformable derivative in future works.

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