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APPLICATIONS OF CESÀRO SUBMETHOD TO APPROXIMATION OF FUNCTIONS IN WEIGHTED ORLICZ SPACES

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Abstract. In this work, the approximation properties of the matrix submethods in weighted Orlicz spaces with Muckenhoupt weights are studied. We obtain some results related to trigonometric approximation using matrix submethods of partial sums of Fourier series of functions in the weighted Orlicz spaces with Muckenhoupt weights.The degree of trigonometric approximations by the matrix methods to the functions have been investigated in weighted Orlicz spaces with Muckenhoupt weights. The error of estimations in this work is obtained in more general terms. In many studies the classical Cesàro method was used to obtain the estimations. In this study to obtain these estimations we use Cesàro submethod instead of classical Cesàro method.

1. Introduction

Let $\mathbb{T}$ denote the interval $[-\pi,\pi]$, $\mathbb{C}$ the complex plane, and $L_p(\mathbb{T})$, $1 \le p \le \infty$, the Lebesgue space of measurable complex-valued functions on $\mathbb{T}$. A convex and continuous function $M : [0,\infty) \to [0,\infty)$ which satisfies the conditions

$$\begin{aligned} M(0) &= 0,\ M(x) > 0 \ \text{ for } x > 0, \\ \lim_{x\to 0} (M(x)/x) &= 0; \quad \lim_{x\to\infty} (M(x)/x) = \infty \end{aligned}$$

is called a *Young function.* We will say that M satisfies the Δ_2-condition if $M(2u) \le cM(u)$ for any $u \ge u_0 \ge 0$ with some constant c, independent of u.

We can consider a right continuous, monotone increasing function $\rho :\ [0,\infty) \to [0,\infty)$ with

$$\rho(0) = 0;\ \lim_{t\to\infty} \rho(t) = \infty \ \text{ and } \rho(t) > 0 \text{ for } t > 0,$$

then the function defined by

$$N(x) = \int_0^{|x|} \rho(t)\, dt$$

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is called $N-$*function*. For a given Young function M, let $\widetilde{L}_M(\mathbb{T})$ denote the set of all Lebesgue measurable functions $f : \mathbb{T} \to \mathbb{C}$ for which

$$\int_{\mathbb{T}} M\left(|f(x)|\right) dx < \infty.$$

The $N-$*function complementary to* M is defined by

$$N\left(y\right) := \max_{x \geq 0} \left(xy - M\left(x\right)\right), \text{ for } y \geq 0.$$

Let N be the complementary Young function of M. It is well-known [27, p. 69], [39, pp. 52-68] that the linear span of $\widetilde{L}_M(\mathbb{T})$ equipped with the *Orlicz norm*

$$\|f\|_{L_M(\mathbb{T})} := \sup\left\{ \int_{\mathbb{T}} |f(x)g(x)|\, dx : g \in \widetilde{L}_N(\mathbb{T}),\ \int_{\mathbb{T}} N\left(|g(x)|\right) dx \leq 1 \right\},$$

or with the *Luxemburg norm*

$$\|f\|^*_{L_M(\mathbb{T})} := \inf\left\{ k > 0 : \int_{\mathbb{T}} M\left(\frac{|f(x)|}{k}\right) dx \leq 1 \right\}$$

becomes a Banach space. This space is denoted by $L_M(\mathbb{T})$ and is called an *Orlicz space* [27, p. 26]. The Orlicz spaces are known as the generalizations of the Lebesgue spaces $L_p(\mathbb{T}),\ 1 < p < \infty$. If $M(x) = M(x,p) := x^p$, $1 < p < \infty$, then Orlicz spaces $L_M(\mathbb{T})$ coindices with the usual Lebesgue spaces $L_p(\mathbb{T}),\ 1 < p < \infty$. Note that the Orlicz spaces play an important role in many areas such as applied mathematics, mechanics, regularity theory, fluid dynamics and statistical physics (e.g., [2], [8], [28] and [41]). Therefore, investigation of approximation of functions by means of Fourier trigonometric series in Orlicz spaces is also important in these areas of research.

The Luxemburg norm is equivalent to the Orlicz norm. The inequalities

$$\|f\|^*_{L_M(\mathbb{T})} \leq \|f\|_{L_M(\mathbb{T})} \leq 2\, \|f\|^*_{L_M(\mathbb{T})}, \qquad f \in L_M(\mathbb{T})$$

hold [27, p. 80].

If we choose $M(u) = u^p/p,\ 1 < p < \infty$ then the complementary function is $N(u) = u^q/q$ with $1/p + 1/q = 1$ and we have the relation

$$p^{-1/p}\, \|u\|_{L_p(\mathbb{T})} = \|u\|^*_{L_M(\mathbb{T})} \leq \|u\|_{L_M(\mathbb{T})} \leq q^{1/q}\, \|u\|_{L_p(\mathbb{T})},$$

where $\|u\|_{L_p(\mathbb{T})} = \left(\int_{\mathbb{T}} |u(x)|^p\, dx \right)^{1/p}$ stands for the usual norm of the $L_p(\mathbb{T})$ space.

If N is complementary to M in Young's sense and $f \in L_M(\mathbb{T})$, $g \in L_N(\mathbb{T})$ then the so-called strong Hölder inequalities [27, p. 80]

$$\int_{\mathbb{T}} |f(x)g(x)|\, dx \leq \|f\|_{L_M(\mathbb{T})}\, \|g\|^*_{L_N(\mathbb{T})},$$

$$\int_{\mathbb{T}} |f(x)g(x)|\, dx \leq \|f\|^*_{L_M(\mathbb{T})}\, \|g\|_{L_N(\mathbb{T})}$$

are satisfied.

The Orlicz space $L_M(\mathbb{T})$ is *reflexive* if and only if the $N-$function M and its complementary function N both satisfy the Δ_2-condition [39, p. 113].

Let $M^{-1}:[0,\infty)\to[0,\infty)$ be the inverse function of the $N-$function M. The *lower* and *upper indices* [3, p. 350]

$$\alpha_M := \lim_{t\to+\infty} -\frac{\log h(t)}{\log t},\ \beta_M := \lim_{t\to o^+} -\frac{\log h(t)}{\log t}$$

of the function

$$h:(0,\infty)\to(0,\infty],\quad h(t):=\lim_{y\to\infty}\sup\frac{M^{-1}(y)}{M^{-1}(ty)},\quad t>0$$

first considered by Matuszewska and Orlicz [30], are called the *Boyd indices* of the Orlicz spaces $L_M(T)$.

It is known that the indices α_M and β_M satisfy $0\le\alpha_M\le\beta_M\le 1,\ \alpha_N+\beta_M=1,\ \alpha_M+\beta_N=1$ and the space $L_M(\mathbb{T})$ is reflexive if and only if $0<\alpha_M\le\beta_M<1$. The detailed information about the Boyd indices can be found in [4]-[7], [29].

A measurable function $\omega:\mathbb{T}\to[0,\infty]$ is called a *weight function* if the set $\omega^{-1}(\{0,\infty\})$ has Lebesgue measure zero. With any given weight ω we associate the ω-*weighted Orlicz space* $L_M(\mathbb{T},\omega)$ consisting of all measurable functions f on $\mathbb{T}$ such that

$$\|f\|_{L_M(\mathbb{T},\omega)} := \|f\omega\|_{L_M(\mathbb{T})}.$$

Let $1<p<\infty,\ 1/p+1/p'=1$ and let ω be a weight function on $\mathbb{T}$. ω is said to satisfy *Muckenhoupt's* A_p*-condition* on $\mathbb{T}$ if

$$\sup_J\left(\frac{1}{|J|}\int_J\omega^p(t)\,dt\right)^{1/p}\left(\frac{1}{|J|}\int_J\omega^{-p'}(t)\,dt\right)^{1/p'}<\infty\ ,$$

where J is any subinterval of $\mathbb{T}$ and $|J|$ denotes its length [36].

Let us indicate by $A_p(\mathbb{T})$ the set of all weight functions satisfying Muckenhoupt's A_p-condition on $\mathbb{T}$.

Let further $t_1,t_2,...,t_n$ be distinet points on $\mathbb{T}$ and let $\lambda_1,\ ...,\lambda_n$ be real numbers. If $1<p<\infty,\ \frac{1}{p}+\frac{1}{q}=1$ and $-\frac{1}{p}<\lambda_j<\frac{1}{q}\ ,\ j=1,...n$ then the weight function

$$\omega(\tau):=\prod_{j=1}^{n}|\tau-t_j|^{\lambda_j},(\tau\in\mathbb{T})$$

belongs to $A_p(\mathbb{T})$ [3, p.30].

According to [24], [25, Lemma 3.3], and [25, Section 2.3] if $L_M(\mathbb{T})$ is reflexive and the weighted function ω satisfies the condition $\omega\in A_{1/\alpha_M}(\mathbb{T})\cap A_{1/\beta_M}(\mathbb{T})$, then the space $L_M(\mathbb{T},\omega)$ is also reflexive.

Let $L_M(\mathbb{T},\omega)$ be a weighted Orlicz space, let $0<\alpha_M\le\beta_M<1$ and let $\omega\in A_{\frac{1}{\alpha_M}}(\mathbb{T})\cap A_{\frac{1}{\beta_M}}(\mathbb{T})$. For $f\in L_M(\mathbb{T},\omega)$ we set

$$(\nu_h f)(x):=\frac{1}{2h}\int_{-h}^{h}f(x+t)\,dt,\ 0<h<\pi,\ x\in T.$$

By reference [18, Lemma 1], the shift operator ν_h is a bounded linear operator on $L_M(\mathbb{T},\omega)$:

$$\|\nu_h(f)\|_{L_M(\mathbb{T},\,\omega)} \le c\,\|f\|_{L_M(\mathbb{T},\,\omega)}\,.$$

The function

$$\Omega_{M,\omega}(\delta,\ f) := \sup_{0<h\le\delta} \|f-(\nu_h f)\|_{L_M(\mathbb{T},\omega)}\,,\ \delta>0$$

is called the *modulus of continuity* of $f\in L_M(\mathbb{T},\omega)$.

It can easily be shown that $\Omega_{M,\,\omega}(\cdot,\ f)$ is a continuous, nonnegative and non-decreasing function satisfying the conditions

$$\lim_{\delta\to 0}\Omega_{M,\omega}(\delta,\ f)=0,\ \Omega_{M,\omega}(\delta,\ f+g)\le\Omega_{M,\omega}(\delta,\ f)+\Omega_{M,\omega}(\delta,g)$$

for $f,\ g\in L_M(\mathbb{T},\,\omega)$.

Let $0<\alpha\le 1$. The set of functions $f\in L_M(\mathbb{T},\omega)$ such that

$$\Omega_{M,\omega}(f,\delta)=O(\delta^{\alpha}),\quad \delta>0$$

is called the *Lipschitz class* $Lip(\alpha,M,\omega)$. Let

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}\left(a_k(f)\cos kx+b_k(f)\sin kx\right) \tag{1.1}$$

be the Fourier series of the function $f\in L^1(\mathbb{T})$, where $a_k(f)$ and $b_k(f)$ the Fourier coefficients of the function f. The *n-th partial sum* of series (1.1) is defined, as

$$\begin{aligned} S_n(x,f) &= \frac{a_0}{2}+\sum_{k=1}^{n}\left(a_k(f)\cos kx+b_k(f)\sin kx\right),\\ &= \frac{a_0}{2}+\sum_{k=1}^{n}B_k(x,f)=\sum_{k=0}^{n}B_k(x,f)\ ,\ B_0(x,f):=\frac{a_0}{2}\ ,\\ B_k(x,f) &:= \left(a_k(f)\cos kx+b_k(f)\sin kx\right). \end{aligned}$$

As in [34] we suppose that $\mathbb{F}$ is an infinite subset of $\mathbb{N}$ and consider $\mathbb{F}$ as the range of strictly increasing sequence of positive integers, say $\mathbb{F}=\{\lambda(n)\}_1^{\infty}$. Following [1], [34] the Cesàro submethod C_λ is defined as

$$(C_\lambda x)_n=\frac{1}{\lambda(n)}\sum_{k=1}^{\lambda(n)}x_k,\ n=1,2,...,$$

where $\{x_k\}$ is a sequence of a real or complex numbers. Therefore, the $C_\lambda-$ method yields a subsequence of the Cesàro method C_1, and hence it is regular for any λ. C_λ is obtained by deleting a set of rows from Cesàro matrix. Note that the information about properties of the method C_λ can be found in [1] and [37].

We suppose that $\{p_n\}_0^{\infty}$ is a sequence of positive real numbers. We define the Nörlund means of the series (1.1), as

$$N_n(x,f)=\frac{1}{P_n}\sum_{k=0}^{n}p_{n-k}S_k(f;x)$$

where $P_n := \sum_{m=0}^{n} p_m$. It is clear that if $p_n = 1$ for all $n = 0, 1, 2..$, then $N_n(\cdot; f)$ coincides with Cesàro mean $\sigma_n(\cdot; f)$, defined as

$$\sigma_n(x; f) = \frac{1}{n+1} \sum_{k=0}^{n} p_{n-k} S_k(f; x).$$

We define Nörlund submethod $N_n^{(\lambda)}(\cdot, f)$ by

$$N_n^{(\lambda)}(x, f) = \frac{1}{P_{\lambda(n)}} \sum_{k=0}^{\lambda(n)} p_{\lambda(n)-k} S_k(f; x),$$

where $P_{\lambda(n)} = p_0 + p_1 + p_2 + ... + p_{\lambda(n)} \neq 0$, $(n \geq 0)$, $p_{-1} = P_{-1}$.If $\lambda(n) = n$ then submethod $N_n^{(\lambda)}(\cdot, f)$ gives us classical Nörlund means.

We suppose that $A = (a_{n,k})$ is the infinite lower triangular matrix with non-negative entries. Let

$$S_{\lambda(n)}^{(A)} = \sum_{k=0}^{\lambda(n)} a_{\lambda(n),k}\ ,\ n = 0, 1, ...$$

We define the method $T_n^{(\lambda)}(\cdot, f)$ by

$$T_n^{(\lambda)}(x, f) = \sum_{k=0}^{\lambda(n)} a_{\lambda(n),k} S_k(x, f).$$

Note that if $a_{\lambda(n),k} = \frac{p_{\lambda(n)-k}}{P_{\lambda(n)}}$, then the method $T_n^{(\lambda)}(\cdot, f)$ turns into Nörlund submethod $N_n^{(\lambda)}(\cdot, f)$.

We say that matrix $A = (a_{n,k})$ almost monotone increasing (decreasing) rows if there is a constant $C_1\,(C_2)$, depending only on A, such that $a_{n,k} \leq C_1 a_{n,m}$ $(a_{n,m} \leq C_2 a_{n,k})$, where $0 \leq k \leq m \leq n$.

We will use the relation $f = O(g)$ which means that $f \leq cg$ for a constant c independent of f and g. We are using sums up to $\lambda(n)$ in the nth partial sums S_n and σ_n and writing these sums $S_{\lambda(n)}$ and $\sigma_{\lambda(n)}$, respectively.

In the present paper we study the degree of approximation by the matrix submethods $T_n^{(\lambda)}(\cdot, f)$ of the partial sums of their Fourier of functions in weighted Orlicz spaces. Similar problems about approximation properties of the different sums, constructed according to the Fourier series of given functions in the different spaces have been investigated by several authors (see, for example, [9-23], [26], [31-35], [38] and [40]).

Our main results are the followings:

Theorem 1.1. *Let* $f \in Lip(\alpha, M, \omega)$, $0 < \alpha < 1$, $\omega \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}$ *, let* $A = (a_{n,k})$ *be a lower triangular matrix with* $\left| s_{\lambda(n)}^{(A)} - 1 \right| = O\left(\lambda(n)^{-\alpha}\right)$ *and one of the following conditions holds:*

(i) *A has almost monotone decreasing rows and* $(\lambda(n) + 1)\, a_{\lambda(n),0} = O(1)$,

(ii) *A has almost monotone increasing rows and* $(\lambda(n) + 1)\, a_{\lambda(n),r} = O(1)$,*where* r *is the integer part of* $\frac{n}{2}$. *Then the estimate*

$$\left\| f - T_n^{(\lambda)}(\cdot, f) \right\|_{L_M(\mathbb{T},\omega)} = O(\lambda(n)^{-\alpha})$$

holds.

From Theorem 1.1 we have the following Corollary:

Corollary 1.1. *Let* $f \in Lip(\alpha, M, \omega)$, $0 < \alpha < 1$, $\omega \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}$, *let* $\{p_n\}$ *be a real sequence of positive numbers and one of the following conditions holds:*

(i) $\{p_n\}$ *is almost monotone increasing and* $(\lambda(n)+1)\, p_{\lambda(n)} = O(P_{\lambda(n)})$,

(ii) $\{p_n\}$ *is almost monotone decreasing. Then the relation*

$$\left\| f - N_n^{(\lambda)}(\cdot, f) \right\|_{L_M(\mathbb{T},\omega)} = O(\lambda(n)^{-\alpha})$$

holds.

Theorem 1.2. *Let* $f \in Lip(1, M, \omega)$, $\omega \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}$, *let* $A = (a_{n,k})$ *be a lower triangular matrix with* $\left| s_{\lambda(n)}^{(A)} - 1 \right| = O\left(\lambda(n)^{-1}\right)$ *and one of the following conditions holds:*

(i) $\sum\limits_{k=1}^{\lambda(n)-1} \left| a_{\lambda(n),k-1} - a_{\lambda(n),k} \right| = O(\lambda(n)^{-1})$,

(ii) $\sum\limits_{k=1}^{\lambda(n)-1} (\lambda(n) - k) \left| a_{\lambda(n),k-1} - a_{\lambda(n),k} \right| = O(1)$.

estimate

$$\left\| f - T_n^{(\lambda)}(\cdot, f) \right\|_{L_M(\mathbb{T},\omega)} = O(\lambda(n)^{-1}) \tag{1.2}$$

holds.

From Theorem 1.2 we get the following Corollary:

Corollary 1.2. *Let* $f \in Lip(1, M, \omega)$, $\omega \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}$, *let* $\{p_n\}$ *be a real sequence of positive numbers and the inequality*

$$\sum_{k=1}^{\lambda(n)-1} |p_k - p_{k+1}| = O\left(\frac{P_{\lambda(n)}}{\lambda(n)}\right),$$

holds. Then the following relation holds:

$$\left\| f - N_n^{(\lambda)}(\cdot, f) \right\|_{L_M(\mathbb{T},\omega)} = O(\lambda(n)^{-1}).$$

2. Auxiliary Results

In the proof of the main result we need the following Lemmas:

Lemma 2.1. [21] *Let* $f \in Lip(\alpha, M, \omega)$, $0 < \alpha \le 1$ *and* $\omega \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}$. *Then the relation*

$$\| f - S_n(\cdot, f) \|_{L_M(\mathbb{T},\omega)} = O(n^{-\alpha}), \ n = 1, 2, 3, ...$$

holds.

Lemma 2.2. [21] *Let* $f \in Lip(1, M, \omega)$, $\omega \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}$. *Then the estimate*

$$\| S_n(\cdot, f) - \sigma_n(f) \|_{L_M(\mathbb{T},\omega)} = O(n^{-1}), \ n = 1, 2, 3, ...$$

holds.

Using the proof method in [14, Lemma 6], can be prowed the following lemma:

Lemma 2.3. *Let* $A=(a_{n,k})$ *be an infinite lower triangular matrix with* $\left|s_{\lambda(n)}^{(A)}-1\right|=O\left(\lambda(n)^{-\alpha}\right)$, $0<\alpha<1$ *and one of the following conditions holds:*

(i) A *has almost monotone decreasing rows and* $(\lambda(n)+1)\,a_{\lambda(n),0}=O(1)$,

(ii) A *has almost monotone increasing rows and* $(\lambda(n)+1)\,a_{\lambda(n),r}=O(1)$, *where* r *is the integer part of* $\frac{n}{2}$. *Then the estimate*

$$\sum_{k=1}^{\lambda(n)} k^{-\alpha}a_{\lambda(n),k}=O\left(\lambda(n)^{-\alpha}\right)$$

holds.

Also, using the proof scheme developed in [19, Lemma 9] we can prove the following lemma:

Lemma 2.4. *The following relation holds:*

$$\left|\sum_{m=0}^{k} a_{\lambda(n),m}-(k+1)\,a_{\lambda(n),k}\right|\le\sum_{m=1}^{k} m\left|a_{\lambda(n),m-1}-a_{\lambda(n),m}\right|.\quad k=1,2,...,n.$$

3. Proofs of the main results

Proof of Theorem 1.1. Let $f\in Lip(\alpha,M,\omega)$, $0<\alpha<1$, $\omega\in A_{1/\alpha_M}(T)\cap A_{1/\beta_M}$, let $A=(a_{n,k})$ be a lower triangular matrix with $\left|s_{\lambda(n)}^{(A)}-1\right|=O\left(\lambda(n)^{-\alpha}\right)$ and one of the conditions (i) and (ii) of theorem be satisfied. By definitions of $T_n^{(\lambda)}(f)(x)$ and $s_{\lambda(n)}^{(A)}$ we obtain:

$$\begin{aligned}
&T_n^{(\lambda)}(x,f)-f(x)\\
=\ &\sum_{k=0}^{\lambda(n)} a_{\lambda(n),k}S_k(x,f)-f(x)\\
=\ &\sum_{k=0}^{\lambda(n)} a_{\lambda(n),k}S_k(x,f)-f(x)+S_{\lambda(n)}^{(A)}(x,f)-S_{\lambda(n)}^{(A)}(x,f)\\
=\ &\sum_{k=0}^{\lambda(n)} a_{\lambda(n),k}\left[S_k(x,f)-f(x]+S_{\lambda(n)}^{(A)}(x,f)-f(x).\right.
\end{aligned}$$

Taking into account that $\left|s_{\lambda(n)}^{(A)}-1\right|=O\left(\lambda(n)^{-\alpha}\right)$, by using Lemma 2.1 and 2.3 and the last equality, we have:

$$\begin{aligned}
&\left\|f-T_n^{(\lambda)}(\cdot,f)\right\|_{L_M(\mathbb{T},\omega)}\\
=\ &a_{\lambda(n),o}\left\|S_0(\cdot,f)-f\right\|_{L_M(\mathbb{T},\omega)}\\
&+\sum_{k=1}^{\lambda(n)} a_{\lambda(n),k}\left\|S_k(\cdot,f)-f\right\|_{L_M(\mathbb{T},\omega)}+\left|S_{\lambda(n)}^{(A)}-1\right|\left\|f\right\|_{L_M(\mathbb{T},\omega)}\\
=\ &O\left(\lambda(n)^{-1}\right)+O(1)\sum_{k=1}^{\lambda(n)} a_{\lambda(n),k}k^{-\alpha}+O\left(\lambda(n)^{-\alpha}\right)=O\left(\lambda(n)^{-\alpha}\right),
\end{aligned}$$

The proof of Theorem 1.1 is completed.

Proof of Theorem 1.2. Let $f\in Lip(1,M,\omega)$, $\omega\in A_{1/\alpha_M}(T)\cap A_{1/\beta_M}$ and let $A=(a_{n,k})$ be a lower triangular matrix with $\left|s_{\lambda(n)}^{(A)}-1\right|=O\left(\lambda\left(n\right)^{-1}\right)$. Taking into account Lemma 2.1 for $\alpha=1$ we get:

$$\begin{aligned}
&\left\|f-T_n^{(\lambda)}(\cdot,f)\right\|_{L_M(\mathbb{T},\omega)}\\
\leq\ &\left\|S_{\lambda(n)}\left(\cdot,f\right)-T_n^{(\lambda)}(\cdot,f)\right\|_{L_M(\mathbb{T},\omega)}+\left\|f-S_{\lambda(n)}\left(\cdot,f\right)\right\|_{L_M(\mathbb{T},\omega)}\\
=\ &\left\|S_{\lambda(n)}\left(\cdot,f\right)-T_n^{(\lambda)}(\cdot,f)\right\|_{L_M(\mathbb{T},\omega)}+O\left(\lambda\left(n\right)^{-1}\right). \qquad (3.1)
\end{aligned}$$

We put $A_{\lambda(n),k}:=\sum\limits_{m=k}^{\lambda(n)}a_{\lambda(n),m}$. By the definition of $A_{\lambda(n),k}$, we obtain the following equality:

$$\begin{aligned}
T_n^{(\lambda)}(x,f)&=\sum_{k=0}^{\lambda(n)}a_{\lambda(n),k}S_k\left(x,f\right)=\sum_{k=0}^{\lambda(n)}a_{\lambda(n),k}\left(\sum_{m=0}^{k}B_m\left(x,f\right)\right)\\
&=\sum_{k=0}^{\lambda(n)}\left(\sum_{m=k}^{\lambda(n)}a_{\lambda(n),m}\right)B_k\left(x,f\right)=\sum_{k=0}^{\lambda(n)}A_{\lambda(n),k}B_k\left(x,f\right). \qquad (3.2)
\end{aligned}$$

Since $s_{\lambda(n)}^{(A)}=\sum_{k=0}^{\lambda(n)}a_{\lambda(n),k}$ we obtain:

$$\begin{aligned}
&S_{\lambda(n)}\left(x,f\right)\\
=\ &\sum_{m=0}^{\lambda(n)}B_m\left(x,f\right)=A_{\lambda(n),0}\sum_{k=0}^{\lambda(n)}B_k\left(x,f\right)+\left(1-A_{\lambda(n),0}\right)\sum_{k=0}^{\lambda(n)}B_k\left(x,f\right)\\
=\ &\sum_{k=0}^{\lambda(n)}A_{\lambda(n),0}B_k\left(x,f\right)+\left(1-s_{\lambda(n)}^{(A)}\right)S_{\lambda(n)}\left(x,f\right). \qquad (3.3)
\end{aligned}$$

By using (3.2) and (3.3) we find that

$$T_n^{(\lambda)}(x,f)-S_{\lambda(n)}\left(x,f\right)=\sum_{k=1}^{\lambda(n)}\left(A_{\lambda(n),k}-A_{\lambda(n),0}\right)B_k\left(x,f\right)+\left(s_{\lambda(n)}^{(A)}-1\right)S_{\lambda(n)}\left(x,f\right).$$

The last equality gives us

$$\begin{aligned}
\left\|S_n^{(\lambda)}(\cdot,f)-T_n^{(\lambda)}(\cdot,f)\right\|_{L_M(\mathbb{T},\omega)}&\leq\left\|\sum_{k=1}^{\lambda(n)}\left(A_{\lambda(n),k}-A_{\lambda(n),0}\right)B_k\left(\cdot,f\right)\right\|_{L_M(\mathbb{T},\omega)}\\
&\quad+c_7\left|s_{\lambda(n)}^{(A)}-1\right|\left\|f\right\|_{L_M(\mathbb{T},\omega)}\\
&=\left\|\sum_{k=1}^{\lambda(n)}\left(A_{\lambda(n),k}-A_{\lambda(n),0}\right)B_k\left(\cdot,f\right)\right\|_{L_M(\mathbb{T},\omega)}\\
&\quad+O\left(\lambda\left(n\right)^{-1}\right). \qquad (3.4)
\end{aligned}$$

We denote

$$a^*_{\lambda(n),k} = \frac{A_{\lambda(n),k} - A_{\lambda(n),0}}{k}, \;\; k = 1, 2, ..., n.$$

By using Abel's transformation (see, for example: [21]), we obtain:

$$\begin{aligned} &\sum_{k=1}^{\lambda(n)} \left(A_{\lambda(n),k} - A_{\lambda(n),0}\right) B_k(x, f) \\ = \;\; &\sum_{k=1}^{\lambda(n)} B_k(x, f)\, a^*_{\lambda(n),k} k = a^*_{\lambda(n),\lambda(n)} \sum_{m=1}^{\lambda(n)} m B_m(x, f) \\ &+ \sum_{k=1}^{\lambda(n)-1} \left(a^*_{\lambda(n),k} - a^*_{\lambda(n),k+1}\right) \left(\sum_{m=1}^{k} m B_m(x, f)\right). \end{aligned} \tag{3.5}$$

Taking into account of last equality, we obtain:

$$\begin{aligned} &\left\| \sum_{k=1}^{\lambda(n)} \left(A_{\lambda(n),k} - A_{\lambda(n),0}\right) B_k(\cdot, f) \right\|_{L_M(\mathbb{T},\omega)} \\ \leq \;\; &\left|a^*_{\lambda(n),\lambda(n)}\right| \left\| \sum_{m=1}^{n} m B_m(\cdot, f) - \right\|_{L_M(\mathbb{T},\omega)} \\ &+ \sum_{k=1}^{\lambda(n)-1} \left|a^*_{\lambda(n),k} - a^*_{\lambda(n),k+1}\right| \\ &\times \left(\left\| \sum_{m=1}^{k} m B_m(\cdot, f) \right\|_{L_M(\mathbb{T},\omega)} \right). \end{aligned} \tag{3.6}$$

Now we estimate

$$\left\| \sum_{m=1}^{n} m B_m(\cdot, f) \right\|_{L_M(\mathbb{T},\omega)}.$$

We can write the following equality:

$$S_{\lambda(n)}(x, f) - \sigma_{\lambda(n)}(x, f) = \sum_{k=1}^{\lambda(n)} \frac{k}{\lambda(n) + 1} B_k(x, f). \tag{3.7}$$

Taking into account of (3.7) and Lemma 2.2, we have:

$$\begin{aligned} &\left\| \sum_{m=1}^{\lambda(n)} m B_m(\cdot, f) \right\|_{L_M(\mathbb{T},\omega)} \\ = \;\; &(\lambda(n) + 1) \left\| S_{\lambda(n)}(\cdot, f) - \sigma_{\lambda(n)}(\cdot, f) - \right\|_{L_M(\mathbb{T},\omega)} \\ = \;\; &(\lambda(n) + 1)\, O\left(\lambda(n)^{-1}\right) = O(1). \end{aligned} \tag{3.8}$$

On the other hand $\left|s_{\lambda(n)}^{(A)}-1\right|=O\left(\lambda(n)^{-1}\right)$ which implies that

$$\begin{aligned} \left|a_{\lambda(n),\lambda(n)}^{*}\right| &= \frac{\left|A_{\lambda(n),\lambda(n)}-A_{\lambda(n),0}\right|}{n}=\frac{\left|a_{\lambda(n),\lambda(n)}-s_{\lambda(n)}^{(A)}\right|}{\lambda(n)} \\ &= \frac{1}{\lambda(n)}\left(s_{\lambda(n)}^{(A)}-a_{\lambda(n),\lambda(n)}\right) \\ &\leq \frac{1}{\lambda(n)}s_{\lambda(n)}^{(A)}=\frac{1}{\lambda(n)}O(1)=O\left(\lambda(n)^{-1}\right). \end{aligned} \tag{3.9}$$

From (3.4), (3.6), (3.8) and (3.9), we have:

$$\begin{aligned} &\left\|S_{\lambda(n)}(\cdot,f)-T_{n}^{(A)}(\cdot,f)-\right\|_{L_{M}(\mathbb{T},\omega)} \\ &= O(1)\,O\left(\lambda(n)^{-1}\right)+O(1)\sum_{k=1}^{\lambda(n)-1}\left|a_{\lambda(n),k}^{*}-a_{\lambda(n),k+1}^{*}\right| \\ &= O\left(\lambda(n)^{-1}\right)+\sum_{k=1}^{\lambda(n)-1}\left|a_{\lambda(n),k}^{*}-a_{\lambda(n),k+1}^{*}\right|. \end{aligned} \tag{3.10}$$

Now we estimate

$$\sum_{k=1}^{\lambda(n)-1}\left|a_{\lambda(n),k}^{*}-a_{\lambda(n),k+1}^{*}\right|.$$

We can show that the following equality holds:

$$a_{\lambda(n),k}^{*}-a_{\lambda(n),k+1}^{*}=\frac{1}{k(k+1)}\left\{(k+1)\,a_{\lambda(n),k}-\sum_{m=0}^{k}a_{\lambda(n),m}\right\}.$$

We suppose that condition (i) of Theorem 1.2 is satisfied. By Lemma 2.4

$$\begin{aligned} &\sum_{k=1}^{\lambda(n)-1}\left|a_{\lambda(n),k}^{*}-a_{\lambda(n),k+1}^{*}\right| \\ &= \sum_{k=1}^{\lambda(n)-1}\frac{1}{k(k+1)}\left|\sum_{m=0}^{k}a_{\lambda(n),m}-(k+1)\,a_{\lambda(n),k}\right| \\ &\leq \sum_{k=1}^{\lambda(n)-1}\frac{1}{k(k+1)}\sum_{m=1}^{k}m\left|a_{\lambda(n),m-1}-a_{\lambda(n),m}\right| \\ &= \sum_{m=1}^{\lambda(n)-1}m\left|a_{\lambda(n),m-1}-a_{\lambda(n),m}\right|\sum_{k=m}^{n-1}\frac{1}{k(k+1)} \\ &\leq \sum_{m=1}^{\lambda(n)-1}m\left|a_{\lambda(n),m-1}-a_{\lambda(n),m}\right|\sum_{k=m}^{\infty}\frac{1}{k(k+1)} \\ &= \sum_{m=1}^{\lambda(n)-1}\left|a_{\lambda(n),m-1}-a_{\lambda(n),m}\right|=O\left(\lambda(n)^{-1}\right). \end{aligned} \tag{3.11}$$

Taking into account of (3.1), (3.10), and (3.11) we have the relation (1.2) of Theorem 1.2. Now under the condition (ii) we prove relation (1.2). Let $r := \left[\frac{n}{2}\right]$. Using Lemma 2.4 we have:

$$\begin{aligned}
&\sum_{k=1}^{\lambda(n)-1} \left|a^*_{\lambda(n),k} - a^*_{\lambda(n),k+1}\right| \\
\leq &\sum_{k=1}^{\lambda(n)-1} \frac{1}{k\,(k+1)} \sum_{m=1}^{k} m \left|a_{\lambda(n),m-1} - a_{\lambda(n),m}\right| \\
\leq &\sum_{k=1}^{r} \frac{1}{k\,(k+1)} \sum_{m=1}^{k} m \left|a_{\lambda(n),m-1} - a_{\lambda(n),m}\right| \\
&+ \sum_{k=r}^{\lambda(n)-1} \frac{1}{k\,(k+1)} \sum_{m=1}^{k} m \left|a_{\lambda(n),m-1} - a_{\lambda(n),m}\right|. \qquad (3.12)
\end{aligned}$$

We estimate

$$\sum_{k=1}^{r} \frac{1}{k\,(k+1)} \sum_{m=1}^{k} m \left|a_{\lambda(n),m-1} - a_{\lambda(n),m}\right|.$$

Since the condition (ii) satisfies, using Abel's transformation, we get:

$$\begin{aligned}
&\sum_{k=1}^{r} \frac{1}{k\,(k+1)} \sum_{m=1}^{k} m \left|a_{\lambda(n),m-1} - a_{\lambda(n),m}\right| \\
\leq &\sum_{k=1}^{r} \left|a_{\lambda(n),k-1} - a_{\lambda(n),k}\right| \\
= &\sum_{k=1}^{r} \frac{1}{(\lambda(n) - k)} (\lambda(n) - k) \left|a_{\lambda(n),k-1} - a_{\lambda(n),k}\right| \\
\leq &\frac{1}{(\lambda(n) - r)} O(1) = O\left(\lambda(n)^{-1}\right). \qquad (3.13)
\end{aligned}$$

Now we estimate the second term on the right side of (3.12). The following inequality holds:

$$\begin{aligned}
&\sum_{k=r}^{\lambda(n)-1} \frac{1}{k\,(k+1)} \sum_{m=1}^{k} m \left|a_{\lambda(n),m-1} - a_{\lambda(n),m}\right|, \\
= &\sum_{k=r}^{\lambda(n)-1} \frac{1}{k\,(k+1)} \sum_{m=1}^{r} m \left|a_{\lambda(n),m-1} - a_{\lambda(n),m}\right| \\
+ &\sum_{k=r}^{\lambda(n)-1} \frac{1}{k\,(k+1)} \sum_{m=r}^{k} m \left|a_{\lambda(n),m-1} - a_{\lambda(n),m}\right| \\
&:= J_{n_1} + J_{n_2}. \qquad (3.14)
\end{aligned}$$

Since $\sum_{k=1}^{r}\left|a_{\lambda(n),k-1}-a_{\lambda(n),k}\right|=O\left(\lambda(n)^{-1}\right)$, from (3.12) we write

$$\begin{aligned} J_{n_1} &\leq \sum_{k=r}^{\lambda(n)-1}\frac{1}{k(k+1)}\sum_{m=1}^{r}\left|a_{\lambda(n),m-1}-a_{\lambda(n),m}\right| \\ &= O\left(\lambda(n)^{-1}\right)\sum_{k=r}^{\lambda(n)-1}\frac{1}{k+1} \\ &= O\left(\lambda(n)^{-1}\right)(\lambda(n)-r)\frac{1}{r+1}=O\left(\lambda(n)^{-1}\right). \end{aligned} \tag{3.15}$$

Now we estimate the expression J_{n_2} :

$$\begin{aligned} J_{n_2} &\leq \sum_{k=r}^{\lambda(n)-1}\frac{1}{(k+1)}\sum_{m=r}^{k}\left|a_{\lambda(n),m-1}-a_{\lambda(n),m}\right| \\ &\leq \frac{1}{r+1}\sum_{k=r}^{\lambda(n)-1}\left(\sum_{m=r}^{k}\left|a_{\lambda(n),m-1}-a_{\lambda(n),m}\right|\right) \\ &\leq \frac{2}{\lambda(n)}\sum_{k=r}^{\lambda(n)-1}\left(\sum_{m=r}^{k}\left|a_{\lambda(n),m-1}-a_{\lambda(n),m}\right|\right) \\ &\leq \frac{2}{\lambda(n)}\sum_{k=r}^{\lambda(n)-1}(\lambda(n)-k)\left|a_{\lambda(n),k-1}-a_{\lambda(n),k}\right| \\ &\leq \frac{2}{\lambda(n)}\sum_{k=1}^{\lambda(n)-1}(\lambda(n)-k)\left|a_{\lambda(n),k-1}-a_{\lambda(n),k}\right| \\ &= \frac{2}{\lambda(n)}O(1)=O\left(\lambda(n)^{-1}\right). \end{aligned} \tag{3.16}$$

Now combining (3.14)- (3.16) we get:

$$\sum_{k=r}^{\lambda(n)-1}\frac{1}{k(k+1)}\sum_{m=1}^{k}m\left|a_{\lambda(n),m-1}-a_{\lambda(n),m}\right|=O\left(\lambda(n)^{-1}\right).$$

The last relation, (3.12) and (3.13) gives us

$$\sum_{k=1}^{\lambda(n)-1}\left|a^{*}_{\lambda(n),k}-a^{*}_{\lambda(n),k+1}\right|=O\left(\lambda(n)^{-1}\right). \tag{3.17}$$

Therefore, by (3.1), (3.10) and (3.17) we obtain that

$$\left\|f-T_n^{(\lambda)}(\cdot,f)-\right\|_{L_M(\mathbb{T},\omega)}=O\left(\lambda(n)^{-1}\right).$$

The proof of Theorem 1.2 is completed.

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