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ON THE GROWTH OF m-TH DERIVATIVES OF ALGEBRAIC POLYNOMIALS IN REGIONS WITH CORNERS IN A WEIGHTED BERGMAN SPACE

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Abstract. In this paper, we study the behavior of growth of the modulus of an arbitrary algebraic polynomial in the weighted Bergman space $A_p(G,h),\ p>0$, in exterior points and in closure of region having exterior nonzero and interior zero angles at a finite number of boundary points. Bernstein-Walsh-type estimates are obtained for algebraic polynomials, in bounded regions with a piecewise-smooth boundary. Using an inequality of the Markov-Nikolskii type, the behavior of polynomials in the whole complex plane is given.

1. Introduction

Let $\mathbb{C}$ be a complex plane and $\overline{\mathbb{C}}:=\mathbb{C}\cup\{\infty\}$; $G\subset\mathbb{C}$ be a finite region, contained origin, bounded by a Jordan curve $L:=\partial G$, $\Omega:=\overline{\mathbb{C}}\backslash\overline{G}$; $\Delta:=\{w:|w|>1\}$ (with respect to $\overline{\mathbb{C}}$). Let function $w=\Phi(z)$ be the univalent conformal mapping of Ω onto Δ normalized by $\Phi(\infty)=\infty$, $\lim_{z\to\infty}\frac{\Phi(z)}{z}>0$, and $\Psi:=\Phi^{-1}$. For $R>1$, let us set $L_R:=\{z:\ |\Phi(z)|=R\}$, $G_R:=intL_R$, $\Omega_R:=extL_R$.

Let $\wp_n$ denote the class of all algebraic polynomials $P_n(z)$ of degree at most $n\in\mathbb{N}$.

Let $\{z_j\}_{j=1}^m\in L$ be a fixed system of distinct points. Consider a so-called generalized Jacobi weight function $h(z)$ being defined as follows:

$$h(z):=h_0(z)\prod_{j=1}^{l}|z-z_j|^{\gamma_j},\quad z\in G_{R_0},\ R_0>1, \tag{1.1}$$

where $\gamma_j>-2$, for all $j=1,2,...,l$, and h_0 is uniformly separated from zero in G_{R_0}, i.e., there exists a constant $c_1(G)>0$ such that for all $z\in G_{R_0}$, $h_0(z)\geq c_1(G)>0$.

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Let $0 < p \leq \infty$ and σ be the two-dimensional Lebesgue measure. For the Jordan region G, we introduce:

$$\|P_n\|_p^p := \|P_n\|_{A_p(h,G)}^p := \iint\limits_G h(z)\,|P_n(z)|^p\,d\sigma,\ 0<p<\infty, \tag{1.2}$$

$$\|P_n\|_\infty := \|P_n\|_{A_\infty(1,G)} := \max_{z\in\overline{G}}|P_n(z)|\,,\ p=\infty;\ \ A_p(1,G)=:A_p(G),$$

and, when L is rectifiable:

$$\|P_n\|_{\mathcal{L}_p(h,L)}^p := \int\limits_L h(z)\,|P_n(z)|^p\,|dz|<\infty,\ 0<p<\infty, \tag{1.3}$$

$$\|P_n\|_{\mathcal{L}_\infty(1,L)} := \max_{z\in L}|P_n(z)|\,;\ \mathcal{L}_p(1,L)=:\mathcal{L}_p(L).$$

The well known classical lemma Bernstein -Walsh [36] shows that for any $P_n(z)\in\wp_n$

$$|P_n(z)| \leq |\Phi(z)|^n\,\|P_n\|_{C(\overline{G})}\,,\quad z\in\Omega. \tag{1.4}$$

holds. If $z\in\overline{G}_R$, then from (1.4) we see that:

$$\|P_n\|_{C(\overline{G}_R)} \leq R^n\,\|P_n\|_{C(\overline{G})}\,. \tag{1.5}$$

Thus, we can estimate the growth uniform norm ($C-$norm) of the polynomial P_n, depending on the extension of the region G up to G_R for some $R>1$, containing G. In particularly, the $C-$norm $\|P_n\|_{C(\overline{G})}$ of polynomials P_n increases no more than a constant, when G expands up to $G_{1+\frac{1}{n}}$. The same effect is observed for the norm (1.3) according to the following estimate [23]:

$$\|P_n\|_{\mathcal{L}_p(L_R)} \leq R^{n+\frac{1}{p}}\,\|P_n\|_{\mathcal{L}_p(L)}\,,\quad p>0. \tag{1.6}$$

The estimate (1.6) has been generalized in [9, Lemma 2.4] for weight function $h(z)\neq 1$, defined as in (1.1) for the $\gamma_j>-1,\ j=1,2,...,l$, as follows:

$$\|P_n\|_{\mathcal{L}_p(h,L_R)} \leq R^{n+\frac{1+\widetilde{\gamma}}{p}}\,\|P_n\|_{\mathcal{L}_p(h,L)}\,,\ \widetilde{\gamma}=\max\{0;\gamma_j:1\leq j\leq\ l\}\,. \tag{1.7}$$

In [2] was given an analog of the estimates (1.4) and (1.7) in the $A_p(h,G)-$norm for the regions with quasiconformal boundary (see: Definition 4.1) with weight function $h(z)$ defined by (1.1), as follows:

$$\|P_n\|_{A_p(h,G_R)} \leq c_1\,{R^*}^{n+\frac{1}{p}}\,\|P_n\|_{A_p(h,G)}\,,\ p>0,$$

where $R^*:=1+c_2(R-1)$ and $c_2>0$, $c_1:=c_1(G,p,c_2)>0$, are constants independent from n and R.

Further, for arbitrary Jordan region G, any $P_n\in\wp_n,\ R_1=1+\frac{1}{n}$, in [5, Theorem1.1] was obtained that

$$\|P_n\|_{A_p(G_R)} \leq cR^{n+\frac{2}{p}}\,\|P_n\|_{A_p(G_{R_1})}\,,\ p>0,$$

is true for arbitrary $R>R_1=1+\frac{1}{n}$, where $c=\left(\frac{2}{e^p-1}\right)^{\frac{1}{p}}\left[1+O(\frac{1}{n})\right],\ n\to\infty$, asymptotically sharp constant.

N. Stylianopoulos in [34] replaced the norm $\|P_n\|_{C(\overline{G})}$ with norm $\|P_n\|_{A_2(G)}$ on the right-hand side of (1.4) and found a new version of the Bernstein-Walsh Lemma: *Assume that L is quasiconformal and rectifiable. Then, there exists a constant $C = C(L) > 0$ depending only on L such that*

$$|P_n(z)| \leq C \frac{\sqrt{n}}{d(z,L)} \|P_n\|_{A_2(G)} |\Phi(z)|^{n+1}, \quad z \in \Omega,$$

where $d(z,L) := \inf \{|\zeta - z| : \ \zeta \in L\}$, *holds for every* $P_n \in \wp_n$.

In this paper, we continue the study of the problem on pointwise estimates of the derivatives $\left|P_n^{(m)}(z)\right|$, $m \geq 1$, in unbounded regions of the complex plane and we obtained estimates as the following type:

$$\left|P_n^{(m)}(z)\right| \leq \eta_n(G,h,p,z) \|P_n\|_p, \ z \in \Omega, \tag{1.8}$$

where $\eta_n(.) \to \infty$ as $n \to \infty$, depending on the properties of the G, h.

Analogous results of (1.8)-type for arbitrary $P_n \in \wp_n$ a different weight function, an unbounded region and some norms were obtained in [6],[7], [8], [10], [11], [12], [22, p.418-428], [29], [34] and the others.

To get an estimate for the $\left|P_n^{(m)}(z)\right|$ on the whole complex plane, we will need an estimate for the $\left|P_n^{(m)}(z)\right|$ in the bounded region $\overline{G}$. To do this, we will use estimates Bernstein - Markov - Nikolsky type for $\left|P_n^{(m)}(z)\right|$, $z \in \overline{G}$, as the following type:

$$\|P_n^{(m)}\|_\infty \leq \nu_n(G,h,p)\|P_n\|_p, \tag{1.9}$$

where $\nu_n := \nu_n(G,h,p) > 0$, $\nu_n \to \infty$ as $n \to \infty$, is a constant depending on the properties of the region G and the weight function h.

Estimates of (1.9)- type for the arbitrary $P_n \in \wp_n$ were studied [2]-[4], [12], [16], [17], [18], [19], [21], [22, pp. 418-428], [24], [26], [27, Sect. 5.3], [28, pp.122-133], [31], [35] and the others.

Therefore, combining the estimates (1.8) and (1.9), we will obtain the estimate for $\left|P_n^{(m)}(z)\right|$ in whole complex plane $\mathbb{C} = \overline{G}_{1+\frac{1}{n}} \cup \Omega_{1+\frac{1}{n}}$:

$$\left|P_n^{(m)}(z)\right| \leq c \|P_n\|_p \begin{cases} \nu_n(G,h,p), & z \in \overline{G}_{1+\frac{1}{n}} \\ \eta_n(G,h,p,d(z,L)\,|\Phi(z)|^{n+1}), & z \in \Omega_{1+\frac{1}{n}}, \end{cases} \tag{1.10}$$

where $c = c(G,p) > 0$ is a constant independent of n, h, P_n, and $\nu_n(G,h,p) \to \infty$, $\eta_n(.) \to \infty$ as $n \to \infty$, depending on the G, h.

In this work, we study similar to (1.10) problem for regions with piecewise Dini-smooth boundary, having exterior nonzero and interior zero angles and weight function $h(z)$ defined in (1.1).

2. Definitions and Notations

Let us give some definitions and notations that will be used later in the text.

Let S be a rectifiable Jordan curve or arc and $z = z(s)$, $s \in [0, \ |S|]$, $|S| :=$ *mes* S (linear measure of S), denote the natural representation of S.

Definition 2.1. [32, p.48](see also [15, p.32])We say that a Jordan curve or arc S called Dini-smooth, if it has a parametrization $z = z(s),\ 0 \leq s \leq |S|$, such that $z^{'}(s) \neq 0,\ 0 \leq s \leq |S|$ and $\left|z^{'}(s_2) - z^{'}(s_1)\right| < g(s_2 - s_1)$, $s_1 < s_2$, where g is an increasing function for which

$$\int_0^1 \frac{g(x)}{x} dx < \infty.$$

Now, we introduce a new class of regions with piecewise Dini-smooth boundary, which have at the boundary points exterior (with respect to $\overline{G}$) nonzero corners and interior cusps simultaneously.

Throughout we denote by $c, c_0, c_1, c_2, ...$ are positive and $\varepsilon_0, \varepsilon_1, \varepsilon_2, ...$ are sufficiently small positive constants (in general, different in different relations), which depend on G in general. Here and in further, for any $k \geq 0$ and $m > k$, notations $j = \overline{k, m}$ denotes $j = k, k+1, ..., m$.

Definition 2.2. [9]We say that a Jordan region $G \in PDS(1; \lambda_1, ..., \lambda_l),\ 0 < \lambda_j \leq 2,\ j = \overline{1, l}$, if $L = \partial G$ consists of the union of finite Dini-smooth arcs $\{L_j\}_{j=0}^{l}$, connecting at the points $\{z_j\}_{j=0}^{l} \in L$, such that L is locally Dini-smooth at z_0 and they have exterior (with respect to $\overline{G}$) angles $\lambda_j \pi,\ 0 < \lambda_j \leq 2$, at the corner points $\{z_j\}_{j=1}^{l} \in L$, where two arcs meet.

Assume without loss of generality, that the Jordan region $G \in PDS(1; \lambda_1, ..., \lambda_l)$, $0 < \lambda_i \leq 2$, with Definition 2.2 is given so that for every point $z_j \in L,\ j = \overline{1, l_1},\ l_1 \leq l$, the region G has exterior nonzero angle (with respect to $\overline{G}$) with $\lambda_j \pi,\ 0 < \lambda_j < 2$, expansion and for every point $z_j \in L,\ j = \overline{l_1 + 1, l}$, has interior zero angles (with respect to $\overline{G}$). If $l_1 = l = 0$, then the region G does not have any angles, and in this case we will write: $G \in DS(1) \equiv DS$; if $l_1 = l \geq 1$, then G has only $\lambda_j \pi,\ 0 < \lambda_j < 2,\ j = \overline{1, l_1}$, exterior nonzero angles and in this case we will write: $G \in PDS(1; \lambda_j)$; if $l_1 = 0$ and $l \geq 1$, then G has only interior zero angles and in this case we will write: $G \in \equiv PDS(1; 2)$.

Throughout this work, we will assume that the points $\{z_j\}_{j=1}^{l} \in L$ defined in (1.1) and in Definition 2.2 are identical and $w_j := \Phi(z_j)$.

For the simplicity of exposition and in order to avoid complex calculations, without loss of generality, we will take $l_1 = 1,\ l = 2$. Then, after this assumption, in the future we will have region $G \in PDS(1; \lambda_1, 2),\ 0 < \lambda_1 < 2$, such that at the point $z_1 \in L$ the region G have exterior nonzero angle $\lambda_1 \pi,\ \ 0 < \lambda_1 < 2$, and at the point $z_2 \in L$ - interior zero angle. Also, we will write $G \in PDS(1; \lambda_1, \lambda_2),\ 0 < \lambda_1, \lambda_2 < 2$, if a region G have only exterior nonzero angles $\lambda_j \pi,\ \ 0 < \lambda_j < 2$, at the points $z_1, z_2 \in L$ and $G \in PDS(1; 2, 2)$, if a region G have only interior zero angles at the point $z_1, z_2 \in L$. The case $l_1 \geq 2,\ l \geq 3$ can be given similarly.

Let us introduce some notation, which is used throughout the work. Let $\widetilde{\lambda}_k := \max\{1; \lambda_k\}$, $\widetilde{\gamma}_k = \max\{0; \gamma_k\}$, $\widetilde{\lambda} := \max\left\{\widetilde{\lambda}_1; \widetilde{\lambda}_2\right\}$; $k = 1, 2$; $\widetilde{\gamma} := \max\{\widetilde{\gamma}_1; \widetilde{\gamma}_2\}$ $\gamma := \max\{\gamma_1; \gamma_2\}$. Further, for $0 < \delta_j < \delta_0 := \frac{1}{4} \min\{|z_i - z_j| : \ i, j = 1, 2, ..., l,\ i \neq j\}$, let $\Omega(z_j,\ \delta_j) := \Omega \cap \{z : |z - z_j| \leq \delta_j\}$; $\delta := \min_{1 \leq j \leq l} \delta_j$; For $L = \partial G$, we put:

$U_{\infty}(L,\delta):=\bigcup\limits_{\zeta\in L}U(\zeta,\ \delta)$ - infinite open cover of the curve L; $U_N(L,\delta):=\bigcup\limits_{j=1}^{N}U_j(L,\delta)$ $\subset U_{\infty}(L,\delta)$-finite open cover of the curve L; $\Omega(\delta):=\Omega(L,\delta):=\Omega\cap U_N(L,\delta)$, $\widehat{\Omega}:=\Omega\setminus\Omega(\delta)$; $\Omega_R(\delta):=\Omega(L_R,\delta):=\Omega_R\cap U_N(L_R,\delta)$, $\widehat{\Omega}_R:=\Omega_R\setminus\Omega_R(\delta)$.

For each $j=1,2$ and $m\geq 1$, we put:

$$p_0(m):=\frac{(\gamma_j+2)\,\lambda_j-2}{(1-\lambda_j)\,(m-1)};\ p_1(m):=\frac{(\gamma_j+2)\,\lambda_j+1}{1-(m-1)\lambda_j}; \tag{2.1}$$
$$p_2:=\frac{(\gamma_j+2)\,\lambda_j+1}{1+\lambda_j};\ p_3:=p_2(\lambda_j,0);\ p_4:=\frac{2\lambda_j+1}{1-(m-1)\lambda_j};$$

$$D_{n,1}^{(+)}=\begin{cases} n^{\left(\frac{\gamma_j+2}{p}\right)\lambda_j}, 2\leq p<p_1(1), 0<\lambda_j\leq 2, \gamma_j>\max\left\{0;2\left(\frac{1}{\lambda_j}-1\right)\right\},\\ n^{1-\frac{1}{p}}, p>p_1(1), 0<\lambda_j\leq 2, \gamma_j>\max\left\{0;2\left(\frac{1}{\lambda_j}-1\right)\right\},\\ n^{\frac{2}{p}}, 2\leq p<3, 0<\lambda_j<1, 0<\gamma_j\leq 2\left(\frac{1}{\lambda_j}-1\right),\\ n^{1-\frac{1}{p}}, p\geq 3, 0<\lambda_j<1, 0<\gamma_j\leq 2\left(\frac{1}{\lambda_j}-1\right),\\ (n\ln n)^{1-\frac{1}{p}}, p=p_1(1), 0<\lambda_j\leq 2, \gamma_j>0,\\ \qquad m=1, z\in\Omega(\delta);\end{cases}$$

$$D_{n,2}^{(+)}:=\begin{cases} n^{\left(\frac{\gamma_j+2}{p}+m-1\right)\lambda_j}, p\geq 2, 1\leq\lambda_j\leq 2, \gamma_j>0,\\ n^{\left(\frac{\gamma_j+2}{p}+m-1\right)\lambda_j}, 2\leq p\leq p_0(m), 0<\lambda_j<1, \gamma_j>2m\left(\frac{1}{\lambda_j}-1\right),\\ n^{1-\frac{1}{p}}, p>p_1(m), 0<\lambda_j<1, \gamma_j>0,\\ n^{\frac{2}{p}+m-1}, p_0(m)<p<p_1(m), 0<\lambda_j<1, \gamma_j>0,\\ (n\ln n)^{1-\frac{1}{p}}, p=p_1(m), 0<\lambda_j\leq\frac{1}{m-1}, 0<\gamma_j\leq 2m\left(\frac{1}{\lambda_j}-1\right),\\ \qquad m\geq 2, z\in\Omega(\delta);\end{cases}$$

$$D_{n,3}^{(+)}:=\begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j}, & 2\leq p<p_2, & 0<\lambda_j\leq 2, & m\geq 1, z\in\widehat{\Omega}(\delta),\\ (n\ln n)^{1-\frac{1}{p}}, & p=p_2, & \frac{1}{\gamma_j}<\lambda_j\leq 2, & m\geq 1, z\in\widehat{\Omega}(\delta),\\ n^{1-\frac{1}{p}}, & p>p_2, & 0<\lambda_j\leq 2, & m\geq 1, z\in\widehat{\Omega}(\delta);\end{cases}$$

$$D_{n,1}^{(-)}:=\begin{cases} n^{\left(\frac{2}{p}\right)\lambda_j}, & 2\leq p<2\lambda_j+1, & 1\leq\lambda_j\leq 2, & m=1, z\in\Omega(\delta),\\ (n\ln n)^{1-\frac{1}{p}}, & p=2\lambda_j+1, & 1\leq\lambda_j\leq 2, & m=1, z\in\Omega(\delta),\\ n^{1-\frac{1}{p}}, & p>2\lambda_j+1, & 1\leq\lambda_j\leq 2, & m=1, z\in\Omega(\delta),\\ n^{\frac{2}{p}}, & 2\leq p<3, & 0<\lambda_j<1, & m=1, z\in\Omega(\delta);\end{cases}$$

$$D_{n,2}^{(-)}:=\begin{cases} n^{\left(\frac{2}{p}+m-1\right)\lambda_j}, & p\geq 2, & 1\leq\lambda_j\leq 2, & m\geq 2, z\in\Omega(\delta),\\ n^{\frac{2}{p}+m-1}, & p\geq 2, & 0<\lambda_j<1, & m\geq 2, z\in\Omega(\delta);\end{cases}$$

$$D_{n,3}^{(-)}:=\left\{\ n^{1-\frac{1}{p}},\quad p\geq 2,\quad 0<\lambda_j\leq 2,\quad m\geq 1, z\in\widehat{\Omega}(\delta).\right.$$

$$M_{n,1}^{(+)}:=\begin{cases} n^{\frac{\gamma_j+2}{p}\lambda_j}, 1<p<2, 0<\lambda_j\leq 2, \gamma_j\geq 2\left(\frac{1}{\lambda_j}-1\right), m=1, z\in\Omega(\delta);\\ n^{\frac{2}{p}}, 1<p<2, 0<\lambda_j\leq 2, 0<\gamma_j<2\left(\frac{1}{\lambda_j}-1\right), m=1, z\in\Omega(\delta);\end{cases}$$

$$M_{n,2}^{(+)} := \begin{cases} n^{\left(\frac{\gamma_j+2}{p}+m-1\right)\lambda_j}, 1<p<2, \begin{cases} 0<\lambda_j\leq 2, \gamma_j \geq \left(\frac{1}{\lambda_j}-1\right)[2+p(m-1)], \\ m\geq 2, z\in\Omega(\delta); \end{cases} \\ n^{m-1+\frac{2}{p}}, 1<p<2, \begin{cases} 0<\lambda_j\leq 2, 0<\gamma_j < \left(\frac{1}{\lambda_j}-1\right)[2+p(m-1)], \\ m\geq 2, z\in\Omega(\delta); \end{cases} \end{cases}$$

$$M_{n,3}^{(+)} := \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j}, & 1<p<2, \quad \frac{1}{\gamma_j+1}<\lambda_j\leq 2, \quad m\geq 1, z\in\widehat{\Omega}(\delta), \\ n^{\frac{1}{p}}, & 1<p<2, \quad 0<\lambda_j\leq\frac{1}{\gamma_j+1}, \quad m\geq 1, z\in\widehat{\Omega}(\delta); \end{cases} \tag{2.2}$$

$$M_{n,1}^{(-)} := \begin{cases} n^{\frac{2}{p}}, & 1<p<2, \quad 0<\lambda_j<1, \quad -2<\gamma_j<-2+\frac{2}{\lambda_j}, \\ n^{\left(\frac{\gamma_j+2}{p}\right)\lambda_j}, & 1<p<2, \quad 0<\lambda_j<1, \quad -2+\frac{2}{\lambda_j}\leq\gamma_j\leq 0, \\ n^{\left(\frac{\gamma_j+2}{p}\right)\lambda_j}, & 1<p<2, \quad 1\leq\lambda_j\leq 2, \quad -2+\frac{1}{\lambda_j}\leq\gamma_j\leq 0, \\ n^{\left(\frac{2}{p}\right)\lambda_j}, & 1<p<2, \quad 1\leq\lambda_j\leq 2, \quad -2<\gamma_j<-2+\frac{1}{\lambda_j}, \\ & m=1, z\in\Omega(\delta), \end{cases}$$

$$M_{n,2}^{(-)} := \begin{cases} n^{\left(\frac{\gamma_j+2}{p}+m-1\right)\lambda_j-\frac{1}{p}}, & 1<p<2, \quad 1\leq\lambda_j\leq 2, \quad -2<\gamma_j\leq 0, \\ n^{\frac{1}{p}+m-1}, & 1<p<2, \quad 0<\lambda_j<1, \quad -2<\gamma_j\leq 0, \\ & m\geq 2, z\in\Omega(\delta), \end{cases}$$

$$M_{n,3}^{(-)} := \begin{cases} n^{\left(\frac{2}{p}-1\right)\lambda_j}, & 1<p\leq 2-\frac{1}{\lambda_j}, \quad 1\leq\lambda_j\leq 2, \quad -2<\gamma_j\leq 0, \\ n^{\frac{1}{p}}, & 2-\frac{1}{\lambda_j}<p<2, \quad 1\leq\lambda_j\leq 2, \quad -2<\gamma_j\leq 0, \\ n^{\frac{1}{p}}, & 1<p<2, \quad 0<\lambda_j<1, \quad -2<\gamma_j\leq 0, \\ & m\geq 1, z\in\widehat{\Omega}(\delta). \end{cases}$$

3. Main Results

Now, we can state our new results.

3.1. The General Estimate (Recurrence Formula). First, we present a general estimate for the $\left|P_n^{(m)}(z)\right|$, for which it will be possible to obtain estimates for the derivative for each order $m=1,2,...$

Theorem 3.1. *Let $p\geq 2$; $G\in PDS(1,\lambda_1,\lambda_2)$, for some $0<\lambda_j\leq 2$, $j=1,2$; $h(z)$ be defined by (1.1) for $l=2$. Then, for any $P_n\in\wp_n$, $n\in\mathbb{N}$, $\gamma_j>-2$, $j=1,2$, and $m=1,2,...$,we have:*

$$\left|P_n^{(m)}(z)\right| \leq c_1 \left|\Phi^{n+1}(z)\right| \cdot \left\{ \frac{\|P_n\|_p}{d(z,L)} A_{n,p}^1(z,m) + \sum_{k=1}^{m} C_m^k B_{n,k}^1(z) \left|P_n^{(m-k)}(z)\right| \right\}, \tag{3.1}$$

where $c_1=c_1(G,\gamma_j,\lambda_j,m,p)>0$ is the constant independent of z and n;

$$A_{n,p}^1(z,m) = \begin{cases} D_n^{(+)} & \gamma_j>0, \\ D_n^{(-)} & -2<\gamma_j\leq 0; \end{cases} \qquad D_n^{(\pm)} := \begin{cases} D_{n,1}^{(\pm)} & m=1, \quad z\in\Omega(\delta), \\ D_{n,2}^{(\pm)} & m\geq 2, \quad z\in\Omega(\delta), \\ D_{n,3}^{(\pm)} & m\geq 1, \quad z\in\widehat{\Omega}(\delta); \end{cases}$$

$$\left|B^1_{n,k}(z)\right| \preceq \left\{ \begin{array}{ll} n^{k\widetilde{\lambda}_j}, & \text{if } z \in \Omega(\delta), \\ 1, & \text{if } z \in \widehat{\Omega}(\delta). \end{array} \right. , \; k=\overline{1,m}, \; \widetilde{\lambda}_j := \max\{1;\lambda_j\}, \; j=1,2;$$

and $D^{(\pm)}_{n,k}$, $k=1,2,3$, *are defined as in (2.1).*

Theorem 3.2. *Let* $1<p<2$; $G \in PDS(1,\lambda_1,\lambda_2)$, *for some* $0<\lambda_j \le 2$, $j=1,2$; $h(z)$ *be defined by (1.1) for* $l=2$. *Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}, \gamma_j > -2$, $j=1,2$, *and* $m=1,2,...$, *we have:*

$$\left|P_n^{(m)}(z)\right| \le c_2 \left|\Phi^{n+1}(z)\right| \cdot \left\{ \frac{\|P_n\|_p}{d(z,L)} A^1_{n,p}(z,m) + \sum_{k=1}^{m} C^k_m B^1_{n,k}(z) \left|P_n^{(m-k)}(z)\right| \right\}, \tag{3.2}$$

where $c_2 = c_2(G,\gamma_j,\lambda_j,m,p) > 0$ *is the constant independent of* z *and* n;

$$A^1_{n,p}(z,m) = \left\{ \begin{array}{ll} M^{(+)}_n & \gamma_j > 0, \\ M^{(-)}_n & -2<\gamma_j \le 0; \end{array} \right. \quad M^{(\pm)}_n := \left\{ \begin{array}{lll} M^{(\pm)}_{n,1} & m=1, & z \in \Omega(\delta), \\ M^{(\pm)}_{n,2} & m \ge 2, & z \in \Omega(\delta), \\ M^{(\pm)}_{n,3} & m \ge 1, & z \in \widehat{\Omega}(\delta); \end{array} \right.$$

$$\left|B^1_{n,k}(z)\right| \preceq \left\{ \begin{array}{ll} n^{k\widetilde{\lambda}_j}, & \text{if } z \in \Omega(\delta), \\ 1, & \text{if } z \in \widehat{\Omega}(\delta). \end{array} \right. , \; k=\overline{1,m}, \; \widetilde{\lambda}_j := \max\{1;\lambda_j\}, \; j=1,2;$$

and $M^{(\pm)}_{n,k}$, $k=1,2,3$, *are defined as in (2.2).*

3.2. Estimate for $|P_n(z)|$.

Theorem 3.3. *Let* $p \ge 2$; $G \in PDS(1,\lambda_1,\lambda_2)$, *for some* $0<\lambda_j\le 2$, $j=1,2$; $h(z)$ *be defined by (1.1) for* $l=2$. *Then, for any* $P_n \in \wp_n$, $n\in\mathbb{N}$, $\gamma_j>-2$, $j=1,2$, *and* $z \in \Omega_R$, *we have:*

$$|P_n(z)| \le c_3 \frac{\left|\Phi^{n+1}(z)\right|}{d(z,L)} A^3_{n,p} \|P_n\|_p, \tag{3.3}$$

where $c_3 = c_3(G,\gamma_j,\lambda_j,p)>0$ *is the constant independent of* z *and* n,

$$A^3_{n,p} := \left\{ \begin{array}{llll} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j}, & 2 \le p < p_2, & 0<\lambda_j\le 2 & \gamma_j > \frac{1}{\lambda_j}, \\ n^{1-\frac{1}{p}}(\ln n)^{1-\frac{1}{p}}, & p=p_2, & 0<\lambda_j\le 2 & \gamma_j > \frac{1}{\lambda_j}, \\ n^{1-\frac{1}{p}}, & p>p_2, & 0<\lambda_j\le 2 & \gamma_j > \frac{1}{\lambda_j}, \\ n^{1-\frac{1}{p}}, & p \ge 2, & 0<\lambda_j\le 2 & 0<\gamma_j \le \frac{1}{\lambda_j}, \\ n^{\frac{1}{p}}, & p\ge 2, & 0<\lambda_j\le 2, & -2<\gamma_j\le 0. \end{array} \right.$$

Theorem 3.4. *Let* $1<p<2$; $G \in PDS(1,\lambda_1,\lambda_2)$, *for some* $0<\lambda_j\le 2$, $j=1,2$; $h(z)$ *be defined by (1.1) for* $l=2$. *Then, for any* $P_n \in \wp_n$, $n\in\mathbb{N}$, $\gamma_j>-2$, $j=1,2$, *and* $z\in\Omega_R$, *we have:*

$$|P_n(z)| \le c_4 \frac{\left|\Phi^{n+1}(z)\right|}{d(z,L)} A^4_{n,p} \|P_n\|_p, \tag{3.4}$$

where $c_4 = c_4(G, \gamma_j, \lambda_j, p) > 0$ *is the constant independent of* z *and* n;

$$A^4_{n,p} := \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j}, & 1 < p < \gamma_j + 2 - \frac{1}{\lambda_j}, 0 < \lambda_j < 1, \frac{1}{\lambda_j} - 1 < \gamma_j < \frac{1}{\lambda_j}, \\ n^{\frac{1}{p}}, & \gamma_j + 2 - \frac{1}{\lambda_j} \le p < 2, 0 < \lambda_j < 1, \frac{1}{\lambda_j} - 1 < \gamma_j < \frac{1}{\lambda_j}, \\ n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j}, & 1 < p < \gamma_j + 2 - \frac{1}{\lambda_j}, 1 \le \lambda_j \le 2, 0 < \gamma_j < \frac{1}{\lambda_j}, \\ n^{\frac{1}{p}}, & 1 < p < 2, 1 \le \lambda_j \le 2, 0 < \gamma_j < \frac{1}{\lambda_j}, \\ n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j}, & 1 < p < 2, 0 < \lambda_j \le 2, \gamma_j \ge \frac{1}{\lambda_j}, \end{cases}$$

if $\gamma_j > 0$, $j = 1, 2$, *and*

$$A^4_{n,p} := \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j}, & 1 < p < \gamma_j + 2 - \frac{1}{\lambda_j}, 1 \le \lambda_j \le 2, \frac{1}{\lambda_j} - 1 < \gamma_j \le 0, \\ n^{\frac{1}{p}}, & \gamma_j + 2 - \frac{1}{\lambda_j} \le p < 2, 1 \le \lambda_j \le 2, \frac{1}{\lambda_j} - 1 < \gamma_j \le 0, \\ n^{\frac{1}{p}}, & 1 < p < 2, 1 \le \lambda_j \le 2, -2 < \gamma_j \le \frac{1}{\lambda_j} - 1, \\ n^{\frac{1}{p}}, & 1 < p < 2, 0 < \lambda_j < 1, -2 < \gamma_j \le 0, \end{cases}$$

if $-2 < \gamma_j \le 0$, $j = 1, 2$,

We note that Theorems 3.3 and 3.4 for $0 < \lambda_j < 2$ was given in [7]. But, here they are given more clearly.

3.3. Estimate for $|P'_n(z)|$.

Theorem 3.5. *Let* $p \ge 2$; $G \in PDS(1, \lambda_1, \lambda_2)$, *for some* $0 < \lambda_j \le 2$, $j = 1, 2$; $h(z)$ *be defined by (1.1) for* $l = 2$. *Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, $\gamma_j > -2$, $j = 1, 2$, *and* $z \in \Omega_R$, *we have:*

$$\left|P'_n(z)\right| \le c_5 \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z, L)} A^5_{n,p}(z) \left\|P_n\right\|_p, \tag{3.5}$$

where $c_5 = c_5(G, \gamma_j, \lambda_j, p) > 0$ *is the constant independent of* z *and* n;

$$A^5_{n,p}(z) := \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j+\widetilde{\lambda}_j}, 2 \le p < p_2, \begin{cases} 0<\lambda_j\le 2, \\ \gamma_j > \max\left\{\frac{1}{\lambda_j}, 2\left(\frac{1}{\lambda_j}-1\right)\right\}, \end{cases} \\ n^{\frac{2}{p}}, 2\le p<p_2, \begin{cases} 0<\lambda_j<\frac{1}{2}, \\ \frac{1}{\lambda_j}<\gamma_j\le 2\left(\frac{1}{\lambda_j}-1\right), \end{cases} \\ n^{1-\frac{1}{p}+\lambda_j}(\ln n)^{1-\frac{1}{p}}, p=p_2, \begin{cases} 1\le\lambda_j\le 2, \\ \gamma_j>\frac{1}{\lambda_j}, \end{cases} \\ n^{2-\frac{1}{p}}(\ln n)^{1-\frac{1}{p}}, p=p_2, \begin{cases} 0<\lambda_j<\frac{1}{2}, \\ \frac{1}{\lambda_j}<\gamma_j\le 2\left(\frac{1}{\lambda_j}-1\right), \end{cases} \\ n^{2-\frac{1}{p}}, p_2<p<3, \begin{cases} 0<\lambda_j<1, \\ \gamma_j>\frac{1}{\lambda_j}, \end{cases} \\ n^{\left(\frac{\gamma_j+2}{p}\right)\lambda_j}, 2\le p<\frac{\lambda_j(\gamma_j+2)+1}{2}, \begin{cases} \frac{1}{2}\le\lambda_j\le 2, \\ 0<\gamma_j\le\frac{1}{\lambda_j}, \end{cases} \\ n^{2-\frac{1}{p}}, p\ge p_1(1), \begin{cases} 0<\lambda_j<1, \\ 0<\gamma_j\le\frac{1}{\lambda_j}, \end{cases} \\ n^{1-\frac{1}{p}+\lambda_j}, p\ge\frac{\lambda_j(\gamma_j+2)+1}{2}, \begin{cases} 1\le\lambda_j\le 2, \\ \gamma_j>0, \end{cases} \\ n^{\frac{1}{p}+\lambda_j}, 2\le p, \begin{cases} 1\le\lambda_j\le 2, \\ -2<\gamma_j\le 0, \end{cases} \\ n^{1+\frac{1}{p}}, 2\le p<3, \begin{cases} 0<\lambda_j<1, \\ -2<\gamma_j\le 0, \end{cases} \end{cases}$$

if $z \in \Omega(\delta)$ *and*

$$A^5_{n,p}(z) := \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j}, & 2\le p<p_2, & 0<\lambda_j\le 2, & \gamma_j>\frac{1}{\lambda_j}, \\ n^{1-\frac{1}{p}}(\ln n)^{1-\frac{1}{p}}, & p=p_2, & 0<\lambda_j\le 2, & \gamma_j>\frac{1}{\lambda_j}, \\ n^{1-\frac{1}{p}}, & p>p_2, & 0<\lambda_j\le 2, & \gamma_j>0, \\ n^{1-\frac{1}{p}}, & p\ge 2, & 0<\lambda_j\le 2, & -2<\gamma_j\le 0, \end{cases}$$

if $z \in \widehat{\Omega}(\delta)$.

Theorem 3.6. *Let* $1<p<2$; $G \in PDS(1,\lambda_1,\lambda_2)$, *for some* $0<\lambda_j\le 2$, $j=1,2$; $h(z)$ *be defined by (1.1) for* $l=2$. *Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, $\gamma_j > -2$, $j=1,2$, *and* $z \in \Omega_R$, *we have:*

$$\left|P_n'(z)\right| \le c_6 \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,L)} A^6_{n,p}(z) \left\|P_n\right\|_p, \tag{3.6}$$

where $c_6 = c_6(G,\gamma_j,\lambda_j,p) > 0$ *is the constant independent of* z *and* n;

$$A^6_{n,p}(z) := \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j+1}, 1<p<\gamma_j+2-\frac{1}{\lambda_j}, \begin{cases} 0<\lambda_j<1, \\ \frac{1}{\lambda_j}-1<\gamma_j\le 2\left(\frac{1}{\lambda_j}-1\right), \end{cases} \\ n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j+1}, 1<p<2, \begin{cases} \frac{1}{2}<\lambda_j<1, \\ \gamma_j\ge \frac{1}{\lambda_j}, \end{cases} \\ n^{\frac{1}{p}+1}, \gamma_j+2-\frac{1}{\lambda_j}\le p<2, \begin{cases} \frac{1}{2}<\lambda_j<1, \\ \frac{1}{\lambda_j}-1<\gamma_j<\frac{1}{\lambda_j}, \end{cases} \\ n^{\left(\frac{\gamma_j+2}{p}\right)\lambda_j}, 1<p<2, \begin{cases} 1\le\lambda_j\le 2, \\ \gamma_j\ge\frac{1}{\lambda_j}, \end{cases} \\ n^{\frac{1}{p}+\lambda_j}, 1<\gamma_j+2-\frac{1}{\lambda_j}\le p, \begin{cases} 1\le\lambda_j\le 2, \\ 0<\gamma_j<\frac{1}{\lambda_j} \end{cases} \\ n^{\frac{1}{p}+1}, 1<p<2, \begin{cases} 0<\lambda_j<1, \\ -2<\gamma_j<-2+\frac{2}{\lambda_j}, \end{cases} \\ n^{\left(\frac{\gamma_j+2}{p}\right)\lambda_j} 1<p<2, \begin{cases} 0<\lambda_j<1, \\ -2+\frac{2}{\lambda_j}\le\gamma_j\le 0, \end{cases} \\ n^{\frac{1}{p}+\lambda_j}, 2-\frac{1}{\lambda_j}<p<2, \begin{cases} 1\le\lambda_j\le 2, \\ -2<\gamma_j<0 \end{cases} \\ n^{\left(\frac{2}{p}\right)\lambda_j}, 1<p\le 2-\frac{1}{\lambda_j}, \begin{cases} 1\le\lambda_j\le 2, \\ -2<\gamma_j<-2+\frac{1}{\lambda_j}, \end{cases} \end{cases}$$

if $z\in\Omega(\delta)$ *and*

$$A^6_{n,p}(z) := \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j}, & 1<p<\gamma_j+2-\frac{1}{\lambda_j}, 0<\lambda_j<1, \frac{1}{\lambda_j}-1<\gamma_j<\frac{1}{\lambda_j}, \\ n^{\frac{1}{p}}, & \gamma_j+2-\frac{1}{\lambda_j}\le p<2, 0<\lambda_j<1, \frac{1}{\lambda_j}-1<\gamma_j<\frac{1}{\lambda_j}, \\ n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j}, & 1<p<\gamma_j+2-\frac{1}{\lambda_j}, 1\le\lambda_j\le 2, 0<\gamma_j<\frac{1}{\lambda_j}, \\ n^{\frac{1}{p}}, & \gamma_j+2-\frac{1}{\lambda_j}\le p<2, 1\le\lambda_j\le 2, 0<\gamma_j<\frac{1}{\lambda_j}, \\ n^{\left(\frac{2}{p}-1\right)\lambda_j}, & 1<p\le 2-\frac{1}{\lambda_j}, 1\le\lambda_j\le 2, -2<\gamma_j\le 0, \\ n^{\frac{1}{p}}, & 2-\frac{1}{\lambda_j}<p<2, 1\le\lambda_j\le 2, -2<\gamma_j\le 0, \\ n^{\frac{1}{p}}, & 1<p<2, 0<\lambda_j<1, -2<\gamma_j\le 0, \end{cases}$$

if $z\in\widehat{\Omega}(\delta)$.

3.4. Estimate for $\left|P''_n(z)\right|$.

Theorem 3.7. *Let* $p\ge 2$; $G\in PDS(1,\lambda_1,\lambda_2)$, *for some* $0<\lambda_j\le 2$, $j=1,2$; $h(z)$ *be defined by (1.1) for* $l=2$. *Then, for any* $P_n\in\wp_n$, $n\in\mathbb{N}$, $\gamma_j>-2$, $j=1,2$, *and* $z\in\Omega_R$, *we have:*

$$\left|P''_n(z)\right|\le c_7\frac{\left|\Phi^{3(n+1)}(z)\right|}{d(z,L)}\left\|P_n\right\|_p A^7_{n,p}(z), \tag{3.7}$$

where $c_7=c_7(G,\gamma,p)>0$ constant independent of n and z;

$$A^7_{n,p}(z) := A^1_{n,p}(z,2)+A^5_{n,p}(z)\begin{cases} n^{\widetilde{\lambda}_j}, & \text{if } z\in\Omega(\delta), \\ 1, & \text{if } z\in\widehat{\Omega}(\delta), \end{cases}$$

and $A^1_{n,p}(z,2)$ and $A^5_{n,p}(z)$ is defined as in Theorem 3.1 and Theorem 3.5, respectively.

Theorem 3.8. *Let* $1 < p < 2$; $G \in PDS(1, \lambda_1, \lambda_2)$, *for some* $0 < \lambda_j \leq 2$, $j = \overline{1, 2}$; $h(z)$ *be defined by (1.1) for* $l = 2$. *Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, $\gamma_j > -2$, $j = 1, 2$, *and* $z \in \Omega_R$, *we have:*

$$\left|P_n''(z)\right| \leq c_8 \frac{\left|\Phi^{3(n+1)}(z)\right|}{d(z, L)} A^8_{n,p}(z) \left\|P_n\right\|_p , \tag{3.8}$$

where $c_8 = c_8(G, \gamma_j, \lambda_j, p) > 0$ constant independent of n and z;

$$A^8_{n,p}(z) := A^1_{n,p}(z, 2) + A^6_{n,p}(z) \begin{cases} n^{\widetilde{\lambda}_j}, & \text{if } z \in \Omega(\delta), \\ 1, & \text{if } z \in \widehat{\Omega}(\delta), \end{cases}$$

and $A^1_{n,p}(z, 2)$ and $A^6_{n,p}(z)$ is defined as in Theorem 3.2 and Theorem 3.6, respectively.

3.5. Estimate $\left|P_n^{(m)}(z)\right|, m \geq 0$, **for bounded regions.** Now, we can state estimates for $\left|P_n^{(m)}(z)\right|$, $m \geq 1$, in the bounded regions $G \in PDS(1, \lambda_1, \lambda_2)$, $0 < \lambda_j \leq 2$, $j = \overline{1, 2}$.

Theorem 3.9. [30, Th.1]*Let* $p > 0$; $G \in PDS(1; \lambda_1, 2)$, *for some* $0 < \lambda_j < 2$; $h(z)$ *be defined by (1.1) for* $l = 2$. *Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, $\gamma_j > -2$, $j = 1, 2$, *and every* $m = 0, 1, 2, ...$, *we have:*

$$\left\|P_n^{(m)}\right\|_\infty \leq c_9 \mu_n \left\|P_n\right\|_p , \tag{3.9}$$

where $c_9 = c_9(G, \gamma_j, \lambda_j, m, p) > 0$ *is a constant independent of* n *and* z,

$$\mu_n := \begin{cases} n^{\frac{(2+\gamma_1)\lambda_1}{p} + 2m} & 0 < \lambda_1 < 2. \quad \gamma_1 \geq \frac{2}{\lambda_1}(2 + \gamma_2) - 2, \quad \gamma_2 \geq 0, \\ n^{2\left(\frac{2+\gamma_2}{p} + m\right)} & 0 < \lambda_1 < 2. \quad 0 < \gamma_1 < \frac{2}{\lambda_1}(2 + \gamma_2) - 2, \quad \gamma_2 \geq 0, \\ n^{2\left(\frac{2}{p} + m\right)}, & 0 < \lambda_1 < 2, \quad -2 < \gamma_1 < 0, \quad -2 < \gamma_2 < 0. \end{cases} \tag{3.10}$$

Corollary 3.1. [30, Th.1]*Let* $p > 0$; $G \in PDS(1; \lambda_1, \lambda_2)$, *for some* $0 < \lambda_j < 2$, $j = 1, 2$; $h(z)$ *be defined by (1.4) for* $l = 2$. *Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, *and every* $m = 0, 1, 2, ...$, *we have:*

$$\left\|P_n^{(m)}\right\|_\infty \leq c_9 \mu_{n,1} \left\|P_n\right\|_p , \tag{3.11}$$

where $c_9 = c_9(G, \gamma_1, \gamma_2, \lambda_1, \lambda_2, p) > 0$ *is the constant, independent of* z *and* n, *and*

$$\mu_{n,1} := \begin{cases} n^{\frac{(2+\gamma_1)\lambda_1}{p}+m\widetilde{\lambda}}, & 0<\lambda_1<2, 0<\lambda_2<2, \begin{cases} \gamma_1 \geq \frac{\lambda_2}{\lambda_1}(2+\gamma_2)-2, \\ \gamma_2>0, \end{cases} \\ n^{\frac{(2+\gamma_2)\lambda_2}{p}+m\widetilde{\lambda}}, & 0<\lambda_1<2, 0<\lambda_2<2, \begin{cases} 0<\gamma_1 < \frac{\lambda_2}{\lambda_1}(2+\gamma_2)-2, \\ \gamma_2 > \frac{1}{\lambda_2}-2, \end{cases} \\ n^{\frac{1}{p}+m\widetilde{\lambda}}(\ln n)^{\frac{1}{p}}, & 0<\lambda_1\leq \frac{1}{2}, 0<\lambda_2\leq\frac{1}{2}, \begin{cases} 0<\gamma_1\leq \frac{1}{\lambda_1}-2, \\ \gamma_2 = \frac{1}{\lambda_2}-2, \end{cases} \\ n^{\frac{1}{p}+m\widetilde{\lambda}}, & 0<\lambda_1\leq \frac{1}{2}, 0<\lambda_2\leq\frac{1}{2}, \begin{cases} -2<\gamma_1< \frac{1}{\lambda_1}-2, \\ -2<\gamma_2 < \frac{1}{\lambda_2}-2, \end{cases} \\ n^{\left(\frac{2}{p}+m\right)\widetilde{\lambda}}, & \frac{1}{2}<\lambda_1<2, \frac{1}{2}<\lambda_2<2, \begin{cases} -2<\gamma_1<0, \\ -2<\gamma_2<0, \end{cases} \end{cases}$$

$$\widetilde{\lambda} := \max\left\{\widetilde{\lambda}_1; \widetilde{\lambda}_2\right\}.$$

Not that, the (3.9) is sharp [30, Th.1].

3.6. Estimate for $|P_n'(z)|$ and $|P_n''(z)|$ in whole plane. According to (1.5) (applied to the polinom $Q_{n-1}(z) := P_n'(z)$), the estimation (3.9) is true also for the points $z \in \overline{G}_R$, $R = 1+\varepsilon_0 n^{-1}$, with a different constant. Therefore, combining estimation (3.9) (for the $z \in \overline{G}_R$) with Theorems 3.5-3.8, we get the growth of $|P_n'(z)|$ and $|P_n''(z)|$ in the whole complex plane, respectively:

Theorem 3.10. *Let $p \geq 2$; $G \in PDS(1,\lambda_1,\lambda_2)$, for some $0<\lambda_j\leq 2$, $j = 1,2$; $h(z)$ be defined by (1.1) for $l=2$. Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, $\gamma_j > -2$, $j=1,2$, and $z \in \mathbb{C}$, we have:*

$$\left|P_n'(z)\right| \leq c_{10}\left\|P_n\right\|_p \begin{cases} \mu_n, & z \in \overline{G}_R, \\ \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,L)} A^5_{n,p}(z), & z \in \Omega_R, \end{cases}$$

where $c_{10} = c_{10}(G,\gamma_j,\lambda_j,p) > 0$ constant independent of n and z; μ_n is defined as in Theorem 3.9, for $m=1$ and $A^5_{n,p}(z)$ is defined as in Theorem 3.5, for all $z \in \Omega_R$.

Theorem 3.11. *Let $1<p<2$; $G \in PDS(1,\lambda_1,\lambda_2)$, for some $0<\lambda_j\leq 2$, $j = 1,2$; $h(z)$ be defined by (1.1) for $l=2$. Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, $\gamma_j > -2$, $j=1,2$,and $z \in \mathbb{C}$, we have:*

$$\left|P_n'(z)\right| \leq c_{11}\left\|P_n\right\|_p \begin{cases} \mu_n, & z \in \overline{G}_R, \\ \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,L)} A^6_{n,p}(z), & z \in \Omega_R, \end{cases}$$

where $c_{11} = c_{11}(G,\gamma_j,\lambda_j,p) > 0$ constant independent of n and z; μ_n is defined as in Theorem 3.9, for $m=1$ and $A^6_{n,p}(z)$ is defined as in Theorem 3.6 for all $z \in \Omega_R$.

Theorem 3.12. *Let $p \geq 2$; $G \in PDS(1,\lambda_1,\lambda_2)$, for some $0<\lambda_j\leq 2$, $j = 1,2$; $h(z)$ be defined by (1.1) for $l=2$. Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, $\gamma_j > -2$, $j=1,2$, and $z \in \mathbb{C}$, we have:*

$$\left|P_n''(z)\right| \leq c_{12}\left\|P_n\right\|_p \begin{cases} \mu_n, & z \in \overline{G}_R, \\ \frac{\left|\Phi^{3(n+1)}(z)\right|}{d(z,L)} A^7_{n,p}(z), & z \in \Omega_R, \end{cases}$$

where $c_{12} = c_{12}(G, \gamma_j, \lambda_j, p) > 0$ *constant independent of* n *and* z; μ_n *is defined as in Theorem 3.9, for* $m = 2$ *and* $A^6_{n,p}(z)$ *is defined as in Theorem 3.7 for all* $z \in \Omega_R$.

Theorem 3.13. *Let* $1 < p < 2$; $G \in PDS(1, \lambda_1, \lambda_2)$, *for some* $0 < \lambda_j \leq 2$, $j = 1, 2$; $h(z)$ *be defined by (1.1) for* $l = 2$. *Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, $\gamma_j > -2$, $j = 1, 2$, *and* $z \in \mathbb{C}$, *we have:*

$$\left|P''_n(z)\right| \leq c_{13} \left\|P_n\right\|_p \begin{cases} \mu_n, & z \in \overline{G}_R, \\ \frac{\left|\Phi^{3(n+1)}(z)\right|}{d(z,L)} A^8_{n,p}(z), & z \in \Omega_R, \end{cases}$$

where $c_{13} = c_{13}(G, \gamma_j, \lambda_j, p) > 0$ *constant independent of* n *and* z; μ_n *is defined as in Theorem 3.9, for* $m = 2$ *and* $A^7_{n,p}(z)$ *is defined as in Theorem 3.8 for all* $z \in \Omega_R$.

Replacing μ_n by $\mu_{n,1}$ from Corollary 3.1, we can write analogs of Theorems 3.10–3.13 for the $G \in PDS(1; \lambda_1, \lambda_2)$, when $0 < \lambda_j < 2$, $j = 1, 2$.

Thus, using Theorems 3.1, 3.2 and estimating the $\left|P^{(m)}_n(z)\right|$ sequentially for each $m \geq 3$, and combining the obtained estimates with Theorem 3.9, we obtain estimates for the $\left|P^{(m)}_n(z)\right|$ for each $m \geq 3$ in whole complex plane.

4. Some auxiliary results

For the nonnegative functions $a > 0$ and $b > 0$, we shall use the notations "$a \preceq b$" (order inequality), if $a \leq cb$ and "$a \asymp b$" are equivalent to $c_1 a \leq b \leq c_2 a$ for some constants c, c_1, c_2 (independent of a and b) respectively.

We can find a well known definition of a K-quasiconformal curve in [13], [25, p.97], [32, p.286] and [33] as follows:

Definition 4.1. The Jordan curve (or arc) L is called $K-$quasiconformal ($K \geq 1$), if there is a $K-$quasiconformal mapping f of the region $D \supset L$ such that $f(L)$ is a circle (or line segment).

Let $F(L)$ denote the set of all sense preserving plane homeomorphisms f of the region $D \supset L$ such that $f(L)$ is a line segment (or circle) and let

$$K_L := \inf \left\{K(f) : f \in F(L)\right\},$$

where $K(f)$ is the maximal dilatation of a such mapping f. Then L is a quasiconformal curve, if $K_L < \infty$, and L is a $K-$quasiconformal curve, if $K_L \leq K$.

Well know that there exist quasiconformal curves which are not rectifiable [20].

According to the "three-point" criterion [13, p.100], every piecewise Dini-smooth curve (without any cusps) is quasiconformal.

Lemma 4.1. [1] *Let* L *be a* $K-$*quasiconformal curve,,* $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z : |z - z_1| \preceq d(z_1, L_{r_0})\}$; $w_j = \Phi(z_j)$, $(z_2, z_3 \in G \cap \{z : |z - z_1| \preceq d(z_1, L_{R_0})\}$; $w_j = \varphi(z_j))$, $j = 1, 2, 3$. *Then,*

a) *The statements* $|z_1 - z_2| \preceq |z_1 - z_3|$ *and* $|w_1 - w_2| \preceq |w_1 - w_3|$ *are equivalent.*

So, statements are $|z_1 - z_2| \asymp |z_1 - z_3|$ *and* $|w_1 - w_2| \asymp |w_1 - w_3|$ *also are equivalent*;

b) *If* $|z_1 - z_2| \preceq |z_1 - z_3|$, *then*

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{K^{-2}} \preceq \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \preceq \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{K^2},$$

where $0 < r_0 < 1$, $R_0 := r_0^{-1}$ *are constants, depending on* G.

Recall that for $0 < \delta_j < \delta_0 := \frac{1}{4}\min\{|z_i - z_j| : i, j = 1, 2, ..., l,\ i \neq j\}$, we put $\Omega(z_j,\ \delta_j) := \Omega \cap \{z : |z - z_j| \leq \delta_j\}$; $\delta := \min\limits_{1\leq j\leq l} \delta_j$, $\Omega(\delta) := \bigcup\limits_{j=1}^{l} \Omega(z_j,\ \delta)$, $\widehat{\Omega} := \Omega \setminus \Omega(\delta)$. Additionally, let $\Delta_j := \Phi(\Omega(z_j,\ \delta))$, $\Delta(\delta) := \bigcup\limits_{j=1}^{l} \Phi(\Omega(z_j,\ \delta))$, $\widehat{\Delta}(\delta) := \Delta \backslash \Delta(\delta)$. Let $w_j := \Phi(z_j)$ and for $\varphi_j := \arg w_j$, $j = 1, 2, ..., l$, we put $\Delta_j^{'} := \left\{t = Re^{i\theta} : R > 1,\ \frac{\varphi_{j-1}+\varphi_j}{2} \leq \theta < \frac{\varphi_j+\varphi_{j+1}}{2}\right\}$, where $\varphi_0 \equiv \varphi_l$, $\varphi_1 \equiv \varphi_{l+1}$; $\Omega_j := \Psi(\Delta_j^{'})$, $L^j := L \cap \overline{\Omega}_j$, $i = 1, 2, ..., l$. Clearly, $\Omega = \bigcup\limits_{j=1}^{l} \Omega_j . L_R^j := L_R \cap \overline{\Omega}^j$. $F^i := \Phi(L^i) = \overline{\Delta}_i^{'} \cap \{\tau : |\tau| = 1\}$, $F_R^i := \Phi(L_R^i) = \overline{\Delta}_i^{'} \cap \{\tau : |\tau| = R\}$, $i = \overline{1, l}$.

The following lemma is a consequence of the results given in [32, pp.41-58], [15, pp.32-36], and estimation for the $\left|\Psi^{'}\right|$ (see, for example, [14, Th.2.8]):

$$\left|\Psi'(\tau)\right| \asymp \frac{d(\Psi(\tau), L)}{|\tau| - 1}. \tag{4.1}$$

Lemma 4.2. *Let a Jordan region* $G \in PDS(1; \lambda_j, 0)$, $0 < \lambda_j \leq 2$, $j = \overline{1, l_1}$. *Then,*

i) for any $w \in \Delta_j$, $|\Psi(w) - \Psi(w_j)| \asymp |w - w_j|^{\lambda_j}$, $|\Psi'(w)| \asymp |w - w_j|^{\lambda_j - 1}$;

ii) for any $w \in \overline{\Delta} \backslash \Delta_j$, $|\Psi(w) - \Psi(w_j)| \asymp |w - w_j|$, $|\Psi'(w)| \asymp 1$.

Lemma 4.3. [3] *Let* L *be a quasiconformal curve*; $h(z)$ *is defined in (1.4). Then, for arbitrary* $P_n(z) \in \wp_n$, *any* $R > 1$ *and* $n = 1, 2, ...,$ *we have:*

$$\|P_n\|_{A_p(h,G_R)} \preceq \widetilde{R}^{n+\frac{1}{p}} \|P_n\|_{A_p(h,G)},\ p > 0, \tag{4.2}$$

where $\widetilde{R} = 1 + c(R - 1)$ *and* c *is independent of* n *and* R.

Lemma 4.4. [17, Lemma 2.4]*Let* $G \in PDS(1; \lambda_1, ..., \lambda_l)$, $0 < \lambda_j \leq 2$, $j = \overline{1, l}$; $h(z)$ *is defined in (1.4). Then, for arbitrary* $P_n(z) \in \wp_n$ *and* $n = 1, 2, ...,$ *we have:*

$$\|P_n\|_{A_p(h,G_{1+c/n})} \preceq \|P_n\|_{A_p(h,G)},\quad p > 0. \tag{4.3}$$

This fact follows from [17, Lemma 2.4], since $PDS(1; \lambda_1, ..., \lambda_l) \subset C_\theta(\lambda_1, ..., \lambda_l)$, $0 < \lambda_j \leq 2$, $j = \overline{1, l}$.

Lemma 4.5. [7, Lemma 2.4]*Let* L *is a* $K-$*quasiconformal curve*; $R = 1 + \frac{c}{n}$. *Then, for any fixed* $\varepsilon \in (0, 1)$ *there exist a level curve* $L_{1+\varepsilon(R-1)}$ *such that the following holds for any polynomial* $P_n(z) \in \wp_n$, $n \in \mathbb{N}$:

$$\|P_n\|_{\mathcal{L}_p\left(\frac{h}{|\Phi'|},\ L_{1+\varepsilon(R-1)}\right)} \prec n^{\frac{1}{p}} \|P_n\|_{A_p(h,\ G)},\ p > 0. \tag{4.4}$$

5. Proofs

Proof of Theorems 3.1 and 3.2. Suppose that $G \in PDS(1, \lambda_1, \lambda_2)$, for some $0 < \lambda_j \leq 2,\ j = \overline{1,2}$; $h(z)$ be defined in (1.1) and let $R = 1 + \frac{1}{n}$, $R_1 := 1 + \frac{R-1}{2}$. For $z \in \Omega$, let us $H_n(z) := \frac{P_n(z)}{\Phi^{n+1}(z)}$. The $m-$th derivative of $H_n(z)$ gives:

$$H_n^{(m)}(z) = \sum_{k=0}^{m} C_m^k \left(\frac{1}{\Phi^{n+1}(z)}\right)^{(k)} P_n^{(m-k)}(z)$$
$$= \frac{P_n^{(m)}(z)}{\Phi^{n+1}(z)} + \sum_{k=1}^{m} C_m^k \left(\frac{1}{\Phi^{n+1}(z)}\right)^{(j)} P_n^{(m-j)}(z),$$

where $C_m^k := \frac{m(m-1)...(m-k+1)}{j!}$. In this we get:

$$P_n^{(m)}(z) = \Phi^{n+1}(z) \left[H_n^{(m)}(z) - \sum_{k=1}^{m} C_m^k \left(\frac{1}{\Phi^{n+1}(z)}\right)^{(k)} P_n^{(m-k)}(z) \right]$$

and

$$\left|P_n^{(m)}(z)\right| \leq \left|\Phi^{n+1}(z)\right| \left\{ \left| \left(\frac{P_n(z)}{\Phi^{n+1}(z)}\right)^{(m)} \right| + \sum_{k=1}^{m} C_m^k \left| \left(\frac{1}{\Phi^{n+1}(z)}\right)^{(k)} \right| \left|P_n^{(m-k)}(z)\right| \right\}. \tag{5.1}$$

Therefore, to estimate $\left|P_n^{(m)}(z)\right|$ for $z \in \Omega$, it suffices to estimate the statements: A) $\left| \left(\frac{P_n(z)}{\Phi^{n+1}(z)}\right)^{(m)} \right|$, $m = 1, 2, ...$; B) $\left| \left(\Phi^{-n-1}(z)\right)^{(k)} \right|$, $k = \overline{1, m}$.

A) Since the function $H_n(z)$ is analytic in Ω, continuous on $\overline{\Omega}$ and $H_n(\infty) = 0$, then Cauchy integral representation for the $m - th$ derivatives gives:

$$H_n^{(m)}(z) = -\frac{1}{2\pi i} \int\limits_{L_{R_1}} H_n(\zeta) \frac{d\zeta}{(\zeta - z)^{m+1}},\ z \in \Omega_R,\ m \geq 1.$$

Then,

$$\left| \left(\frac{P_n(z)}{\Phi^{n+1}(z)}\right)^{(m)} \right| \leq \frac{1}{2\pi} \int\limits_{L_{R_1}} \left| \frac{P_n(\zeta)}{\Phi^{n+1}(\zeta)} \right| \frac{|d\zeta|}{|\zeta - z|^{m+1}} \preceq \frac{1}{d(z, L_{R_1})} \int\limits_{L_{R_1}} |P_n(\zeta)| \frac{|d\zeta|}{|\zeta - z|^m}. \tag{5.2}$$

Denote by

$$A_n(z) := \int\limits_{L_{R_1}} |P_n(\zeta)| \frac{|d\zeta|}{|\zeta - z|^m}, \tag{5.3}$$

and estimate this integral. Let us give some notations.

Let $w_j := \Phi(z_j) = e^{i\varphi_j}$, $j = 1, 2$, and, without loss of generality, we will assume that $0 < \varphi_1 < \varphi_2 < 2\pi$. For $\eta := \min\{\eta_j, j = 1, 2\}$, where $\eta_j = \min\limits_{t \in \partial\Phi(\Omega(z_j,\ \delta_j))} |t - w_j| > 0$, and for any fixed $\rho > 1$,we introduce:

$$\Delta_j(\eta_j) := \{t : |t - w_j| \le \eta_j\} \subset \Phi(\Omega(z_j,\ \delta_j)),$$

$$\Delta(\eta) := \bigcup_{j=1}^{l} \Delta_j(\eta),\ \widehat{\Delta}_j = \Delta\backslash\Delta(\eta_j); \widehat{\Delta}(\eta) := \Delta\backslash\Delta(\eta);\ \Delta_1^{'} := \Delta_1^{'}(1),$$

$$\Delta_1^{'}(\rho) := \left\{t = Re^{i\theta} : R \ge \rho > 1,\ \frac{\varphi_0 + \varphi_1}{2} \le \theta < \frac{\varphi_1 + \varphi_2}{2}\right\},\ \text{and for } j = \overline{2,l},$$

$$\Delta_j^{'} := \Delta_j^{'}(1), \Delta_j^{'}(\rho) := \left\{t = Re^{i\theta} : R \ge \rho > 1,\ \frac{\varphi_{j-1} + \varphi_j}{2} \le \theta < \frac{\varphi_j + \varphi_0}{2}\right\},$$

where $\varphi_0 = 2\pi - \varphi_l;\ \Omega_j := \Psi(\Delta_j^{'}), L_{R_1}^j := L_{R_1} \cap \Omega_j;\ \Omega = \bigcup_{j=1}^{l} \Omega_j.$

For simplicity of calculations, we can limit ourselves to only one point on the boundary, which the weight function has singularities, i.e., let $h(z)$ be defined as in (1.1) for $l = 2$. To estimate $A_n(z)$, first of all replacing the variable $z = \Psi(w)$ and multiplying the numerator and denominator of the integrant by $\prod_{j=1}^{2} |\Psi(\tau) - \Psi(w_j)|^{\frac{\gamma_j}{p}}\ |\Psi'(\tau)|^{\frac{2}{p}}$ and applying the Hölder inequality, we obtain:

$$A_n(z) = \int\limits_{L_{R_1}} |P_n(\zeta)| \frac{|d\zeta|}{|\zeta - z|^m}$$

$$= \sum_{i=1}^{3} \int\limits_{F_{R_1}^i} \frac{\prod_{j=1}^{2} |\Psi(\tau) - \Psi(w_j)|^{\frac{\gamma_j}{p}} \left|P_n(\Psi(\tau))(\Psi'(\tau))^{\frac{2}{p}}\right| |\Psi'(\tau)|^{1-\frac{2}{p}}}{\prod_{j=1}^{2} |\Psi(\tau) - \Psi(w_j)|^{\frac{\gamma_j}{p}} |\Psi(\tau) - \Psi(w)|^m} |d\tau|$$

$$\le \sum_{i=1}^{3} \left(\int\limits_{F_{R_1}^i} \prod_{j=1}^{2} |\Psi(\tau) - \Psi(w_j)|^{\gamma_j} |P_n(\Psi(\tau))|^p \left|\Psi'(\tau)\right|^2 |d\tau| \right)^{\frac{1}{p}}$$

$$\times \left(\int\limits_{F_{R_1}^i} \frac{|\Psi'(\tau)|^{\left(1-\frac{2}{p}\right)q}}{\prod_{j=1}^{2} |\Psi(\tau) - \Psi(w_1)|^{\frac{q\gamma_j}{p}} |\Psi(\tau) - \Psi(w)|^{qm}} |d\tau| \right)^{\frac{1}{q}} =: \sum_{i=1}^{3} A_n^{i,j}(z),$$

where $F_{R_1}^j := \Phi(L_{R_1}^j) = \Delta_j' \cap \{\tau : |\tau| = R_1\}$, $j = 1,2$; $F_{R_1}^3 := \Phi(L_{R_1}^j) \backslash (F_{R_1}^1 \cup F_{R_1}^2)$ and

$$A_n^{i,j}(z) := \left(\int\limits_{F_{R_1}^i} |f_{n,p}(\tau)|^p \, |d\tau| \right)^{\frac{1}{p}}$$

$$\times \left(\int\limits_{F_{R_1}^i} \frac{|\Psi'(\tau)|^{2-q}}{\prod\limits_{j=1}^{2} |\Psi(\tau) - \Psi(w_j)|^{\gamma_j(q-1)} \, |\Psi(\tau) - \Psi(w)|^{qm}} |d\tau| \right)^{\frac{1}{q}}$$

$$=: J_{n,1}^i \cdot J_{n,2}^{i,j}(z),$$

where $f_{n,p}(\tau) := h^{\frac{1}{p}}(\Psi(\tau)) P_n(\Psi(\tau)) (\Psi'(\tau))^{\frac{2}{p}}$, $|\tau| = R_1$.. Applying to Lemma 4.5, we get:

$$J_{n,1}^i \preceq n^{\frac{1}{p}} \|P_n\|_p \,, \; i = 1,2,3.$$

For the estimation of the integral $J_{n,2}^{i,j}(z)$,for $i = 1,2,3$, and $j = 1,2$, we set:

$$E_{R_1}^{11}(w_j) = \left\{ \tau : \tau \in F_{R_1}^j, \; |\tau - w_j| < c_j(R_1 - 1) \right\},$$
$$E_{R_1}^{12}(w_j) := \left\{ \tau : \tau \in F_{R_1}^j, \; c_j(R_1 - 1) \; \leq |\tau - w_j| < \eta \right\},$$
$$E_{R_1}^{13}(w_j) := \left\{ \tau : \tau \in \Phi(L_{R_1}^j), \; |\tau - w_j| \geq \eta \right\},$$

where $0 < c_j < \eta$ is chosen so that $\{\tau : |\tau - w_j| < c_j(R_1 - 1)\} \cap \Delta \neq \varnothing$ and $F_{R_1}^j = \Phi(L_{R_1}^j) = \bigcup\limits_{k=1}^{3} E_{R_1}^{1k}(w_j)$. Taking into consideration these notations, we get:

$$\sum_{i=1}^{3} J_{n,2}^{i,j}(z) =: J_2(z) = \sum_{i=1}^{3} \sum_{j=1}^{2} J_2(E_{R_1}^{1i}(w_j), z) =: \sum_{i=1}^{3} \sum_{j=1}^{2} J_2^{i,j}(z) \tag{5.4}$$

and, consequently,

$$A_n(z) \preceq n^{\frac{1}{p}} \|P_n\|_p \cdot \sum_{j=1}^{2} \sum_{i=1}^{3} J_2^{i,j}(z) =: \sum_{j=1}^{2} \sum_{i=1}^{3} A_{n,i}^j(z), \tag{5.5}$$

where

$$A_{n,i}^j(z) := n^{\frac{1}{p}} \|P_n\|_p \cdot J_2^{i,j}(z), \; i = 1,2,3; \; j = 1,2. \tag{5.6}$$

$$\left(J_2^{i,j}(z)\right)^q := \int\limits_{E_{R_1}^{1i}(w_j)} \frac{|\Psi'(\tau)|^{2-q} \, |d\tau|}{\prod\limits_{j=1}^{2} |\Psi(\tau) - \Psi(w_j)|^{\gamma_j(q-1)} \, |\Psi(\tau) - \Psi(w)|^{qm}}$$

$$\asymp \int\limits_{E_{R_1}^{1i}(w_j)} \frac{|\Psi'(\tau)|^{2-q} \, |d\tau|}{|\Psi(\tau) - \Psi(w_j)|^{\gamma_j(q-1)} \, |\Psi(\tau) - \Psi(w)|^{qm}}, \; i = 1,2,3; \; j = 1,2,$$

since the points w_1 and w_2 are isolated.

For any $k,j=1,2$, denote by

$$E^{1k}_{R_1,j}(w_j):=\left\{\tau\in E^{1k}_{R_1}(w_j):|\Psi(\tau)-\Psi(w_j)|\geq|\Psi(\tau)-\Psi(w)|\right\},$$
$$E^{1k}_{R_1,2}(w_j):=E^{1k}_{R_1}(w_j)\backslash E^{1k}_{R_1,1}(w_j),$$

$$\left[I(E^{1k}_{R_1,1}(w_j))\right]^q:=\begin{cases}\int\limits_{E^{1k}_{R_1,1}(w_j)}\frac{|\Psi'(\tau)|^{2-q}|d\tau|}{|\Psi(\tau)-\Psi(w)|^{\gamma_j(q-1)+qm}}, & if\ \gamma_j\geq 0,\\ \int\limits_{E^{1k}_{R_1,1}(w_j)}\frac{|\Psi(\tau)-\Psi(w_j)|^{(-\gamma_j)(q-1)}|\Psi'(\tau)|^{2-q}|d\tau|}{|\Psi(\tau)-\Psi(w)|^{qm}}, & if\ \gamma_j<0,\end{cases}\tag{5.7}$$

$$\left[I(E^{1k}_{R_1,2}(w_j))\right]^q:=\int\limits_{E^{1k}_{R_1,2}(w_j)}\frac{|\Psi'(\tau)|^{2-q}|d\tau|}{|\Psi(\tau)-\Psi(w_j)|^{\gamma_j(q-1)+qm}}\ ,\ k=1,2,$$

and estimate the last integrals.

Given the possible values q ($q>2$ and $q<2$) and γ_j ($-2<\gamma_j<0$ and $\gamma_j\geq 0$), we will consider the cases separately.

Case 1. Let $1<q\leq 2$ ($p\geq 2$).

1.1. Let $\gamma_j>0$. If $z\in\Omega(\delta)$, applying Lemma 4.2, we get:

$$\begin{aligned}\left[I(E^{11}_{R_1,1}(w_j))\right]^q&\preceq\int\limits_{E^{11}_{R_1,1}(w_j)}\frac{|\tau-w_j|^{(2-q)(\lambda_j-1)}|d\tau|}{|\tau-w|^{[\gamma_j(q-1)+qm]\lambda_j}}\\&\preceq n^{[\gamma_j(q-1)+qm]\lambda_j-(2-q)(\lambda_j-1)}mesE^{11}_{R_1,1}(w_j)\\&\preceq n^{[\gamma_j(q-1)+qm]\lambda_j-(2-q)(\lambda_j-1)-1};\\I(E^{11}_{R_1,1}(w_j))&\preceq n^{\left(\frac{\gamma_j+2}{p}+m-1\right)\lambda_j-\frac{1}{p}}.\end{aligned}\tag{5.8}$$

$$\begin{aligned}\left[I(E^{11}_{R_1,2}(w_j))\right]^q&\preceq\int\limits_{E^{11}_{R_1,1}(w_j)}\frac{|\tau-w_j|^{(2-q)(\lambda_j-1)}|d\tau|}{|\tau-w_j|^{[\gamma_j(q-1)+qm]\lambda_j}}\\&\preceq n^{[\gamma_j(q-1)+qm]\lambda_j-(2-q)(\lambda_j-1)}mesE^{11}_{R_1,2}(w_j)\\&\preceq n^{[\gamma_j(q-1)+qm]\lambda_j-(2-q)(\lambda_j-1)-1};\\I(E^{11}_{R_1,2}(w_j))&\preceq n^{\left(\frac{\gamma_j+2}{p}+m-1\right)\lambda_j-\frac{1}{p}}.\end{aligned}\tag{5.9}$$

Then, from (5.8) and (5.9), we have:

$$I(E^{11}_{R_1,1}(w_j))+I(E^{11}_{R_1,2}(w_j))\preceq n^{\left(\frac{\gamma_j+2}{p}+m-1\right)\lambda_j-\frac{1}{p}}.\tag{5.10}$$

Further, analogously, applying (4.1) and Lemmas 4.2, 4.1, for $m \geq 1$, we get:

$$\left[I(E^{12}_{R_1,1}(w_j))\right]^q \preceq n^{2-q} \int\limits_{E^{12}_{R_1,1}(w_j)} \frac{|d\tau|}{|\tau - w|^{[\gamma_j(q-1)+qm-(2-q)]\lambda_j}}$$

$$\preceq N_1 := \begin{cases} n^{[\gamma_j(q-1)+qm-(2-q)]\lambda_j+1-q}, & [\gamma_j(q-1)+qm-(2-q)]\lambda_j > 1, \\ n^{2-q}\ln n, & [\gamma_j(q-1)+qm-(2-q)]\lambda_j = 1, \\ n^{2-q}, & [\gamma_j(q-1)+qm-(2-q)]\lambda_j < 1; \end{cases}$$

$$I(E^{12}_{R_1,1}(w_j))$$

$$\preceq N_2 := \begin{cases} n^{\left(\frac{\gamma_j+2}{p}+m-1\right)\lambda_j-\frac{1}{p}}, & 2 \leq p < p_1(m), \begin{cases} 0 < \lambda_j < \frac{1}{2m}, \\ \gamma_j > \frac{1}{\lambda_j} - 2m, \end{cases} \\ n^{\left(\frac{\gamma_j+2}{p}+m-1\right)\lambda_j-\frac{1}{p}}, & 2 \leq p < p_1(m), \frac{1}{2m} < \lambda_j \leq \frac{1}{m-1}, \gamma_j > 0, \\ n^{\left(\frac{\gamma_j+2}{p}+m-1\right)\lambda_j-\frac{1}{p}}, & p \geq 2, 0 < \lambda_j < \frac{1}{2m}, 0 < \gamma_j \leq \frac{1}{\lambda_j} - 2m, \\ n^{\left(\frac{\gamma_j+2}{p}+m-1\right)\lambda_j-\frac{1}{p}}, & p \geq 2, \frac{1}{m-1} < \lambda_j \leq 2, \gamma_j > 0, \\ n^{1-\frac{2}{p}}(\ln n)^{1-\frac{1}{p}}, & p = p_1(m), \frac{1}{\gamma_j+2m} < \lambda_j \leq \frac{1}{m-1}, \gamma_j > 0, \\ n^{1-\frac{2}{p}}, & p > p_1(m), 0 < \lambda_j < \frac{1}{2m}, \gamma_j > \frac{1}{\lambda_j} - 2m, \\ n^{1-\frac{2}{p}}, & p > p_1(m), \frac{1}{2m} \leq \lambda_j \leq \frac{1}{m-1}, \gamma_j > 0, \\ n^{1-\frac{2}{p}}, & p \geq 2, 0 < \lambda_j < \frac{1}{2m}, 0 < \gamma_j \leq \frac{1}{\lambda_j} - 2m, \\ n^{1-\frac{2}{p}}, & p \geq 2, \frac{1}{m-1} < \lambda_j \leq 2, \gamma_j > 0. \end{cases} \quad (5.11)$$

$$\left[I(E^{12}_{R_1,2}(w_j))\right]^q \preceq n^{2-q} \int\limits_{E^{12}_{R_1,2}(w_j)} \frac{|d\tau|}{|\tau - w_j|^{[\gamma_j(q-1)+qm-(2-q)]\lambda_j}} \preceq N_1$$

$$I(E^{12}_{R_1,2}(w_j)) \preceq N_2 \quad (5.12)$$

Then, from (5.11) and (5.12), we obtain:

$$I(E^{12}_{R_1,1}(w_j)) + I(E^{12}_{R_1,2}(w_j)) \preceq N_2$$

For $\tau \in E^{13}_{R_1}(w_j)$ we see that $\eta < |\tau - w_j| < 2\pi R_1$. Therefore, $|\Psi(\tau) - \Psi(w_j)| \succeq 1$, from Lemma 4.1 and for $|\tau - w_j| \geq \eta$, $|\Psi(\tau) - \Psi(w)| \asymp |\tau - w|$, from Lemma 4.2. Then, for $w \in \Delta(w_j, \eta)$, applying (4.1), we get:

$$\left(J^{3,j}_2(z)\right)^q \preceq \int\limits_{E^{13}_{R_1}(w_j)} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w)|^{qm}} \preceq \int\limits_{E^{13}_{R_1}(w_j)} \frac{|d\tau|}{|\tau - w|^{qm}} \preceq n^{qm-1}; \quad (5.13)$$

$$J^{3,j}_2(z) \preceq n^{m-1+\frac{1}{p}}, z \in \Omega(\delta); \quad J^{3,j}_2(z) \preceq 1, \ z \in \widehat{\Omega}(\delta).$$

Combining (5.8-5.13), for $p \geq 2, \gamma_j > 0$, $j = 1, 2$, $m \geq 1$ and $z \in \Omega(\delta)$, we get:

$$\sum_{k=1}^{3} J^{k,j}_2(z) \preceq n^{\left(\frac{\gamma_j+2}{p}+m-1\right)\lambda_j-\frac{1}{p}} + N_2 + n^{m-1+\frac{1}{p}}.$$

After simple calculations, for $m=1$, we have:

$$\sum_{k=1}^{3} J_2^{k,j}(z) \preceq \begin{cases} n^{\left(\frac{\gamma_j+2}{p}\right)\lambda_j-\frac{1}{p}}, & 2\le p<p_1(1), \begin{cases} 0<\lambda_j\le 2, \\ \gamma_j>\max\left\{0;2\left(\frac{1}{\lambda_j}-1\right)\right\}, \end{cases} \\ n^{1-\frac{2}{p}}, & p>p_1(1), \begin{cases} 0<\lambda_j\le 2, \\ \gamma_j>\max\left\{0;2\left(\frac{1}{\lambda_j}-1\right)\right\}, \end{cases} \\ n^{\frac{1}{p}}, & 2\le p<3, 0<\lambda_j<1, 0<\gamma_j\le 2\left(\frac{1}{\lambda_j}-1\right), \\ n^{1-\frac{2}{p}}, & p\ge 3, 0<\lambda_j<1, 0<\gamma_j\le 2\left(\frac{1}{\lambda_j}-1\right), \\ n^{1-\frac{2}{p}}(\ln n)^{1-\frac{1}{p}}, & p=p_1(1), 0<\lambda_j\le 2, \gamma_j>0, \end{cases} \tag{5.14}$$

and, for $m\ge 2$,

$$\sum_{k=1}^{3} J_2^{k,j}(z) \preceq \begin{cases} n^{\left(\frac{\gamma_j+2}{p}+m-1\right)\lambda_j-\frac{1}{p}}, & p\ge 2, 1\le\lambda_j\le 2, \gamma_j>0, \\ n^{\left(\frac{\gamma_j+2}{p}+m-1\right)\lambda_j-\frac{1}{p}}, & 2\le p\le p_0(m), \begin{cases} \frac{1}{m-1}<\lambda_j<1, \\ \gamma_j>2m\left(\frac{1}{\lambda_j}-1\right), \end{cases} \\ n^{1-\frac{2}{p}}, & p>p_1(m), 0<\lambda_j<1, \gamma_j>0, \\ n^{\frac{1}{p}+m-1}, & p_0(m)<p<p_1(m), 0<\lambda_j<1, \gamma_j>0, \\ n^{1-\frac{2}{p}}(\ln n)^{1-\frac{1}{p}}, & p=p_1(m), \begin{cases} 0<\lambda_j\le\frac{1}{m-1}, \\ 0<\gamma_j\le 2m\left(\frac{1}{\lambda_j}-1\right). \end{cases} \end{cases} \tag{5.15}$$

If $z\in\widehat{\Omega}(\delta)$, then for any $m\ge 1$, we have:

$$\begin{aligned} \left(J_2^{1,j}(z)\right)^q &\preceq \int\limits_{E_{R_1}^{11}(w_j)} \frac{|\tau-w_j|^{(\lambda_j-1)(2-q)}|d\tau|}{|\tau-w_j|^{\gamma_j(q-1)\lambda_j}} \\ &\preceq \int\limits_{E_{R_1}^{11}(w_j)} \frac{|d\tau|}{|\tau-w_j|^{\gamma_j(q-1)\lambda_j-(\lambda_j-1)(2-q)}} \\ &\preceq n^{\gamma_j(q-1)\lambda_j-(\lambda_j-1)(2-q)} mes E_{R_1}^{11}(w_j) \preceq n^{\gamma_j(q-1)\lambda_j-(\lambda_j-1)(2-q)-1}; \end{aligned}$$

$$J_2^{1,j}(z) \preceq n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j-\frac{1}{p}}. \tag{5.16}$$

$$\begin{aligned} \left(J_2^{2,j}(z)\right)^q &\preceq n^{2-q} \int\limits_{E_{R_1}^{12}(w_j)} \frac{|\tau-w_j|^{\lambda_j(2-q)}|d\tau|}{|\tau-w_j|^{\gamma_j(q-1)\lambda_j}} \\ &\preceq n^{2-q} \int\limits_{E_{R_1}^{12}(w_j)} \frac{|d\tau|}{|\tau-w_j|^{[\gamma_j(q-1)-(2-q)]\lambda_j}} \\ &\preceq \begin{cases} n^{[\gamma_j(q-1)-(2-q)]\lambda_j+1-q}, & [\gamma_j(q-1)-(2-q)]\lambda_j>1, \\ n^{2-q}\ln n, & [\gamma_j(q-1)-(2-q)]\lambda_j=1, \\ n^{2-q}, & [\gamma_j(q-1)-(2-q)]\lambda_j<1; \end{cases} \end{aligned}$$

$$J_2^{2,j}(z) \preceq \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j-\frac{1}{p}}, & 2 \leq p < p_2, \quad \frac{1}{\gamma_j} < \lambda_j \leq 2, \\ n^{(1-\frac{2}{p})}(\ln n)^{1-\frac{1}{p}}, & p = p_2, \quad \frac{1}{\gamma_j} < \lambda_j \leq 2, \\ n^{1-\frac{2}{p}}, & p > p_2, \quad \frac{1}{\gamma_j} < \lambda_j \leq 2, \\ n^{1-\frac{2}{p}}, & p \geq 2, \quad 0 < \lambda_j \leq \frac{1}{\gamma_j}. \end{cases} \tag{5.17}$$

$$\left(J_2^{3,j}(z)\right)^q \preceq \int\limits_{E_{R_1}^{13}(w_j)} \frac{|d\tau|}{|\tau - w_j|^{\gamma_j(q-1)\lambda_j}} \preceq 1;$$

$$J_2^{3,j}(z) \preceq 1 \tag{5.18}$$

Therefore, from (5.16)-(5.18), we obtain:

$$\sum_{k=1}^{3} J_2^{k,j}(z) \preceq n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j-\frac{1}{p}} + \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j-\frac{1}{p}}, & 2 \leq p < p_2, \frac{1}{\gamma_j} < \lambda_j \leq 2, \\ n^{(1-\frac{2}{p})}(\ln n)^{1-\frac{1}{p}}, & p = p_2, \frac{1}{\gamma_j} < \lambda_j \leq 2, \\ n^{1-\frac{2}{p}}, & p > p_2, \frac{1}{\gamma_j} < \lambda_j \leq 2, \\ n^{1-\frac{2}{p}}, & p \geq 2, 0 < \lambda_j \leq \frac{1}{\gamma_j}, \end{cases}$$

$$\preceq \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j-\frac{1}{p}}, & 2 \leq p < p_2, \quad 0 < \lambda_j \leq 2, \\ n^{(1-\frac{2}{p})}(\ln n)^{1-\frac{1}{p}}, & p = p_2, \quad \frac{1}{\gamma_j} < \lambda_j \leq 2, \\ n^{1-\frac{2}{p}}, & p > p_2, \quad 0 < \lambda_j \leq 2. \end{cases}$$

Combining (5.5), (5.6), (5.16)-(5.18), in case $\gamma_j > 0$, we obtain:

$$A_n(z) \preceq D_n^{(+)} \|P_n\|_p, \tag{5.19}$$

where

$$D_n^{(+)} := \begin{cases} D_{n,1}^{(+)}, & m = 1, \quad z \in \Omega(\delta), \\ D_{n,2}^{(+)}, & m \geq 2, \quad z \in \Omega(\delta), \\ D_{n,3}^{(+)}, & m \geq 1, \quad z \in \widehat{\Omega}(\delta), \end{cases}$$

and $D_{n,i}^{(+)}$, $i = 1, 2, 3$, are defined as in (2.1).

1.2. If $\gamma_j \leq 0$, for $w \in \Delta(w_1,\eta) \cap \Omega_R(\delta) : |\Psi(\tau)-\Psi(w_1)| \preceq |\Psi(\tau)-\Psi(w)|$, according to Lemma 4.1, analogously we have:

$$\left(I(E^{11}_{R_1,1}(w_j))\right)^q \preceq \int\limits_{E^{11}_{R_1,1}(w_j)} \frac{|\tau-w_j|^{(2-q)(\lambda_j-1)-\gamma_j(q-1)\lambda_j}\,|d\tau|}{|\tau-w|^{qm\lambda_j}} \tag{5.20}$$

$$\preceq \left(\frac{1}{n}\right)^{(2-q)(\lambda_j-1)-\gamma_j(q-1)\lambda_j} \int\limits_{E^{11}_{R_1,1}(w_j)} \frac{|d\tau|}{|\tau-w|^{qm\lambda_j}}$$

$$\preceq \left(\frac{1}{n}\right)^{(2-q)(\lambda_j-1)-\gamma_j(q-1)\lambda_j-qm\lambda_j} mesE^{11}_{R_1,1}(w_j)$$

$$\preceq \left(\frac{1}{n}\right)^{(2-q)(\lambda_j-1)-\gamma_j(q-1)\lambda_j-qm\lambda_j+1};$$

$$I(E^{11}_{R_1,1}(w_j)) \preceq n^{\left(\frac{2}{p}+m-1\right)\lambda_j-\frac{1}{p}}.$$

$$\left(I(E^{11}_{R_1,2}(w_j))\right)^q \preceq \int\limits_{E^{11}_{R_1,2}(w_j)} \frac{|\tau-w_j|^{(2-q)(\lambda_j-1)-\gamma_j(q-1)\lambda_j}\,|d\tau|}{|\tau-w_j|^{qm\lambda_j}}$$

$$\preceq \left(\frac{1}{n}\right)^{(2-q)(\lambda_j-1)-\gamma_j(q-1)\lambda_j} \int\limits_{E^{11}_{R_1,2}(w_j)} \frac{|d\tau|}{|\tau-w_j|^{qm\lambda_j}}$$

$$\preceq \left(\frac{1}{n}\right)^{(2-q)(\lambda_j-1)-\gamma_j(q-1)\lambda_j-qm\lambda_j+1};$$

$$I(E^{11}_{R_1,2}(w_j)) \preceq n^{\left(\frac{2}{p}+m-1\right)\lambda_j-\frac{1}{p}}.$$

For $\tau \in E^{12}_{R_1}$ we see that $|\tau-w_1|<\eta$ and from Lemma 4.1, $|\Psi(\tau)-\Psi(w_1)| \preceq 1$. Then, for $w \in \Delta(w_1,\eta)\cap\Omega_R(\delta) : |\Psi(\tau)-\Psi(w_1)| \preceq |\Psi(\tau)-\Psi(w)|$ and $m \geq 1$, applying Lemma 4.2, we get:

$$\left(I(E^{12}_{R_1,1}(w_j))\right)^q \preceq n^{2-q} \int\limits_{E^{12}_{R_1,1}(w_j)} \frac{|d\tau|}{|\tau-w|^{[qm-(2-q)]\lambda_j}}$$

$$\preceq N_3 := \begin{cases} n^{[qm-(2-q)]\lambda_j+1-q}, & [qm-(2-q)]\,\lambda_j > 1, \\ n^{2-q}\ln n, & [qm-(2-q)]\,\lambda_j = 1, \\ n^{2-q}, & [qm-(2-q)]\,\lambda_j < 1; \end{cases}$$

$$I(E^{12}_{R_1,1}(w_j)) \preceq N_4 := \begin{cases} n^{\left(\frac{2}{p}+m-1\right)\lambda_j-\frac{1}{p}}, & 2 \leq p < p_4, & \frac{1}{2m} < \lambda_j \leq \frac{1}{m-1}, \\ n^{1-\frac{2}{p}}(\ln n)^{1-\frac{1}{p}}, & p = p_4, & \frac{1}{2m} < \lambda_j < \frac{1}{m-1} \\ n^{1-\frac{2}{p}}, & p > p_4. & \frac{1}{2m} < \lambda_j \leq \frac{1}{m-1}, \\ n^{1-\frac{2}{p}}, & p \geq 2, & 0 < \lambda_j \leq \frac{1}{2m}, \\ n^{\left(\frac{2}{p}+m-1\right)\lambda_j-\frac{1}{p}}, & p \geq 2 & \frac{1}{m-1} < \lambda_j \leq 2, \end{cases} \tag{5.21}$$

$$\left(I(E^{12}_{R_1,2}(w_j))\right)^q \preceq n^{2-q} \int\limits_{E^{12}_{R_1,2}(w_j)} \frac{|d\tau|}{|\tau - w_j|^{[qm-(2-q)]\lambda_j}} \preceq N_3$$

$$I(E^{12}_{R_1,2}(w_j)) \preceq N_4 \tag{5.22}$$

For $\tau \in E^{13}_{R_1}$ and each $w \in \Delta(w_1,\eta) \cap \Omega_R(\delta)$ we have $\eta < |\tau - w_1| < 2\pi R_1$. Therefore, from Lemma 4.1 and applying (4.4), we get:

$$\begin{aligned}\left(I(E^{13}_{R_1}(w_j))\right)^q &:= \int\limits_{E^{13}_{R_1}(w_j)} \frac{|\Psi(\tau)-\Psi(w_1)|^{(-\gamma_j)(q-1)}\,|\Psi'(\tau)|^{2-q}\,|d\tau|}{|\Psi(\tau)-\Psi(w)|^{qm}} \\ &\preceq \int\limits_{E^{13}_{R_1}(w_j)} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w)|^{qm}} \preceq \int\limits_{E^{13}_{R_1}(w_j)} \frac{|d\tau|}{|\tau - w|^{qm}} \preceq n^{qm-1},\end{aligned}$$

$$I(E^{13}_{R_1}(w_j)) \preceq n^{\frac{1}{p}+m-1}. \tag{5.23}$$

Therefore, combining (5.20)-(5.23) in case of $\gamma_j \le 0$ for $z \in \Omega(\delta)$, we have:

$$\sum_{k=1}^{3} J_2^{k,j}(z) \preceq n^{\left(\frac{2}{p}+m-1\right)\lambda_j-\frac{1}{p}} + n^{\frac{1}{p}+m-1} \tag{5.24}$$

$$+\begin{cases} n^{\left(\frac{2}{p}+m-1\right)\lambda_j-\frac{1}{p}}, & 2\le p<p_4, \quad \frac{1}{2m}<\lambda_j\le\frac{1}{m-1}, \quad m\ge 1,\\ n^{1-\frac{2}{p}}\left(\ln n\right)^{1-\frac{1}{p}}, & p=p_4, \quad \frac{1}{2m}<\lambda_j<\frac{1}{m-1} \quad m\ge 1,\\ n^{1-\frac{2}{p}}, & p>p_4. \quad \frac{1}{2m}<\lambda_j\le\frac{1}{m-1}, \quad m\ge 1,\\ n^{1-\frac{2}{p}}, & p\ge 2, \quad 0<\lambda_j\le\frac{1}{2m}, \quad m\ge 1,\\ n^{\left(\frac{2}{p}+m-1\right)\lambda_j-\frac{1}{p}}, & p\ge 2, \quad \frac{1}{m-1}<\lambda_j\le 2, \quad m\ge 1,\end{cases}$$

$$\preceq\begin{cases} n^{\left(\frac{2}{p}\right)\lambda_j-\frac{1}{p}}, & 2\le p<2\lambda_j+1, \quad 1\le\lambda_j\le 2, \quad m=1,\\ n^{1-\frac{2}{p}}\left(\ln n\right)^{1-\frac{1}{p}}, & p=2\lambda_j+1, \quad 1\le\lambda_j\le 2, \quad m=1,\\ n^{1-\frac{2}{p}}, & p>2\lambda_j+1. \quad 1\le\lambda_j\le 2, \quad m=1,\\ n^{\frac{1}{p}}, & 2\le p<3, \quad 0<\lambda_j<1, \quad m=1,\\ n^{1-\frac{2}{p}}, & p\ge 3, \quad 0<\lambda_j<1, \quad m\ge 2,\\ n^{\left(\frac{2}{p}+m-1\right)\lambda_j-\frac{1}{p}}, & p\ge 2, \quad 1\le\lambda_j\le 2, \quad m\ge 2,\\ n^{\frac{1}{p}+m-1}, & p\ge 2, \quad 0<\lambda_j<1, \quad m\ge 2.\end{cases}$$

If $z\in\widehat{\Omega}(\delta)$, then $|w-w_1|\geq\eta$, from Lemma 4.2 and from (4.4), we get:

$$\left(J_2^{1,j}(z)\right)^q \preceq n^{(2-q)} \int\limits_{E_{R_1}^{11}(w_j)} |\Psi(\tau)-\Psi(w_j)|^{(-\gamma_j)(q-1)+(2-q)}\,|d\tau|$$

$$\preceq n^{(2-q)} \int\limits_{E_{R_1}^{11}(w_j)} |\Psi(\tau)-\Psi(w_j)|^{(-\gamma_j)(q-1)+2-q}\,|d\tau| \preceq 1;$$

$$J_2^{1,j}(z)\preceq 1.$$

$$\left(J_2^{2,j}(z)\right)^q \preceq \int\limits_{E_{R_1}^{12}(w_j)} \left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q} |\Psi(\tau)-\Psi(w_j)|^{(-\gamma_j)(q-1)}\,|d\tau| \preceq n^{(2-q)};$$

$$J_2^{2,j}(z)\preceq n^{1-\frac{2}{p}}.$$

$$\left(J_2^{3,j}(z)\right)^q \preceq \int\limits_{E_{R_1}^{13}(w_j)} \frac{|\Psi'(\tau)|^{2-q}\,|d\tau|}{|\Psi(\tau)-\Psi(w_j)|^{\gamma_j(q-1)}} \preceq n^{(2-q)} \int\limits_{E_{R_1}^{13}(w_j)} |d\tau| \preceq n^{(2-q)};$$

$$J_2^{3,j}(z)\preceq n^{\left(1-\frac{2}{p}\right)}.$$

Combining the last tree estimates, in case of $\gamma_j<0$ for $z\in\widehat{\Omega}(\delta)$, we have:

$$\sum_{k=1}^{3} J_2^{k,j}(z)\preceq n^{\left(1-\frac{2}{p}\right)}. \tag{5.25}$$

Then, for the $\gamma_j\leq 0$, from (5.24-5.25), for any $m\geq 1$, we obtain:

$$\sum_{k=1}^{3} J_2^{k,j}(z)\preceq \begin{cases} n^{\left(\frac{2}{p}\right)\lambda_j-\frac{1}{p}}, & 2\leq p<2\lambda_j+1, \quad 1\leq\lambda_j\leq 2, \quad m=1,\\ n^{1-\frac{2}{p}}(\ln n)^{1-\frac{1}{p}}, & p=2\lambda_j+1, \quad 1\leq\lambda_j\leq 2, \quad m=1,\\ n^{1-\frac{2}{p}}, & p>2\lambda_j+1, \quad 1\leq\lambda_j\leq 2, \quad m=1,\\ n^{\frac{1}{p}}, & 2\leq p<3, \quad 0<\lambda_j<1, \quad m=1,\\ n^{1-\frac{2}{p}}, & p\geq 3, \quad 0<\lambda_j<1, \quad m\geq 2,\\ n^{\left(\frac{2}{p}+m-1\right)\lambda_j-\frac{1}{p}}, & p\geq 2, \quad 1\leq\lambda_j\leq 2, \quad m\geq 2,\\ n^{\frac{1}{p}+m-1}, & p\geq 2, \quad 0<\lambda_j<1, \quad m\geq 2,\\ & \text{for } z\in\Omega(\delta); \end{cases}$$

$$\sum_{k=1}^{3} J_2^{k,j}(z)\preceq n^{\left(1-\frac{2}{p}\right)}, \text{ for } z\in\widehat{\Omega}(\delta).$$

Therefore, for the $\gamma_j\leq 0$, from (5.24), we have:

$$A_n(z)\preceq \|P_n\|_p\cdot D_n^{(-)}, \tag{5.26}$$

where

$$D_n^{(-)}:=\begin{cases} D_{n,1}^{(-)}, & m=1, \quad z\in\Omega(\delta),\\ D_{n,2}^{(-)}, & m\geq 2, \quad z\in\Omega(\delta),\\ D_{n,3}^{(-)}, & m\geq 1, \quad z\in\widehat{\Omega}(\delta), \end{cases}$$

and $D_{n,i}^{(-)}$, $i=1,2,3$, are defined as in (2.1).

Therefore, combining (5.2), (5.3), (5.26), for any $p\geq 2$, $\gamma_j>-2$, we obtain:

$$\left|\left(\frac{P_n(z)}{\Phi^{n+1}(z)}\right)^{(m)}\right| \preceq \frac{1}{d(z,L_{R_1})} D_n \|P_n\|_p, \tag{5.27}$$

where

$$D_n := \begin{cases} D_n^{(+)} & \gamma_j>0, \\ D_n^{(-)} & -2<\gamma_j\leq 0; \end{cases} \quad D_n^{(\pm)} := \begin{cases} D_{n,1}^{(\pm)} & m=1, \quad z\in\Omega(\delta), \\ D_{n,2}^{(\pm)} & m\geq 2, \quad z\in\Omega(\delta), \\ D_{n,3}^{(\pm)} & m\geq 1, \quad z\in\widehat{\Omega}(\delta), \end{cases}$$

and $D_{n,k}^{(\pm)}$, $k=1,2,3$, defines as in (2.1).

Case 2. Let $q>2$ $(1<p<2)$. Then, $2-q<0$, and so

$$\left[I(E_{R_1,1}^{1k})(w_j)\right]^q := \begin{cases} \int\limits_{E_{R_1,1}^{1k}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}|\Psi(\tau)-\Psi(w)|^{\gamma_j(q-1)+qm}}, & if\ \gamma_j\geq 0, \\ \int\limits_{E_{R_1,1}^{1k}} \frac{|\Psi(\tau)-\Psi(w_j)|^{(-\gamma_j)(q-1)}|d\tau|}{|\Psi'(\tau)|^{q-2}|\Psi(\tau)-\Psi(w)|^{qm}}, & if\ \gamma_j<0, \end{cases} \tag{5.28}$$

$$\left[I(E_{R_1,2}^{1k})(w_j)\right]^q := \int\limits_{E_{R_1,2}^{1k}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}\,|\Psi(\tau)-\Psi(w_j)|^{\gamma_j(q-1)+qm}}, \; k=1,2,$$

$$\left(J_2^{3,j}(z)\right)^q := \int\limits_{E_{R_1}^{13}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}\,|\Psi(\tau)-\Psi(w_j)|^{\gamma_j(q-1)}\,|\Psi(\tau)-\Psi(w)|^{qm}}.$$

2.1. If $\gamma_j>0$ and $z\in\Omega(\delta)$, applying Lemma's 4.1 and 4.2 to (5.28), we obtain:

$$\begin{aligned} \left[I(E_{R_1,1}^{11})(w_j)\right]^q &\preceq \int\limits_{E_{R_1,1}^{11}} \frac{|d\tau|}{|\tau-w|^{(q-2)(\lambda_j-1)+[\gamma_j(q-1)+qm]\lambda_j}} \\ &\preceq n^{(q-2)(\lambda_j-1)+[\gamma_j(q-1)+qm]\lambda_j} mes E_{R_1,1}^{11} \\ &\preceq n^{(q-2)(\lambda_j-1)+[\gamma_j(q-1)+qm]\lambda_j-1}; \end{aligned}$$

$$I(E_{R_1,1}^{11})(w_j) \preceq n^{\left(\frac{\gamma_j+2}{p}+m-1\right)\lambda_j-\frac{1}{p}}.$$

$$\begin{aligned} \left[I(E_{R_1,2}^{11})(w_j)\right]^q &\preceq \int\limits_{E_{R_1,2}^{11}} \frac{|d\tau|}{|\tau-w_j|^{(q-2)(\lambda_j-1)+[\gamma_j(q-1)+qm]\lambda_j}} \\ &\preceq n^{(q-2)(\lambda_j-1)+[\gamma_j(q-1)+qm]\lambda_j} mes E_{R_1,2}^{11} \\ &\preceq n^{(q-2)(\lambda_j-1)+[\gamma_j(q-1)+qm]\lambda_j-1}; \end{aligned}$$

$$I(E_{R_1,2}^{11})(w_j) \preceq n^{\left(\frac{\gamma_j+2}{p}+m-1\right)\lambda_j-\frac{1}{p}}.$$

Then, for $m\geq 1$, we have:

$$I(E_{R_1,1}^{11}(w_j)) + I(E_{R_1,2}^{11}(w_j)) \preceq n^{\left(\frac{\gamma_j+2}{p}+m-1\right)\lambda_j-\frac{1}{p}}. \tag{5.29}$$

$$\left[I(E_{R_1,1}^{12})(w_j)\right]^q$$

$$\preceq \int\limits_{E^{12}_{R_1,1}} \left(\frac{|\tau|-1}{d(\Psi(\tau),L)}\right)^{q-2} \frac{|d\tau|}{|\tau-w|^{[\gamma_j(q-1)+qm]\lambda_j}}$$

$$\preceq N_5 := \begin{cases} n^{[\gamma_j(q-1)+qm+q-2]\lambda_j+1-q}, & [\gamma_j(q-1)+qm+q-2]\lambda_j>1,\\ n^{2-q}\ln n, & [\gamma_j(q-1)+qm+q-2]\lambda_j=1,\\ n^{2-q}, & [\gamma_j(q-1)+qm+q-2]\lambda_j<1; \end{cases}$$

$$I(E^{12}_{R_1,1}(w_j)) \preceq N_6 := \begin{cases} n^{\left(\frac{\gamma_j+2}{p}+m-1\right)\lambda_j-\frac{1}{p}}, 1<p<p_1(m), \begin{cases} 0<\lambda_j<\frac{1}{2m},\\ 0<\gamma_j<\frac{1}{\lambda_j}-2m, \end{cases}\\ n^{1-\frac{2}{p}}(\ln n)^{1-\frac{1}{p}}, p=p_1(m), \begin{cases} 0<\lambda_j<\frac{1}{2m},\\ 0<\gamma_j<\frac{1}{\lambda_j}-2m, \end{cases}\\ n^{1-\frac{2}{p}}, p_1(m)<p<2, \begin{cases} 0<\lambda_j<\frac{1}{2m},\\ 0<\gamma_j<\frac{1}{\lambda_j}-2m, \end{cases}\\ n^{1-\frac{2}{p}}, 1<p<2, \frac{1}{m-1}<\lambda_j\le 2, \gamma_j>0,\\ n^{\left(\frac{\gamma_j+2}{p}+m-1\right)\lambda_j-\frac{1}{p}}, 1<p<2, 2\ge\lambda_j>\frac{1}{m-1}, \gamma_j>0, \end{cases} \tag{5.30}$$

$$\left[I(E^{12}_{R_1,2})(w_j)\right]^q \preceq \int\limits_{E^{12}_{R_1,2}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}|\Psi(\tau)-\Psi(w_j)|^{\gamma_j(q-1)+qm}} \preceq N_5$$

$$I(E^{12}_{R_1,1}(w_j)) \preceq N_6 \tag{5.31}$$

From (5.30) and (5.31), we have:

$$I(E^{12}_{R_1,1}(w_j)) + I(E^{12}_{R_1,2}(w_j)) \preceq N_6$$

For $\tau \in E^{13}_{R_1}$ we see that $\eta < |\tau-w_1| < 2\pi R_1$. Therefore, from Lemma 4.1, we have $|\Psi(\tau)-\Psi(w_1)| \succeq 1$. For $|\tau-w_j| \ge \eta$, $|\Psi(\tau)-\Psi(w)| \asymp |\tau-w|$, from Lemma 4.2. Then, applying Lemma 4.2 and (4.1), we get:

$$J_2^{3,j}(z) \preceq \left(\int\limits_{E^{13}_{R_1}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w)|^{qm}}\right)^{\frac{1}{q}} \preceq n^{m-1+\frac{1}{p}},\ z\in\Omega(\delta); \tag{5.32}$$

$$J_2^{3,j}(z) \preceq 1,\ z\in\widehat{\Omega}(\delta).$$

From (5.29)- (5.32), for $\gamma_j>0$, $m\ge 1$ and $z\in\Omega(\delta)$, we obtain:

$$\sum_{k=1}^{3} J_2^{k,j}(z) \preceq n^{\left(\frac{\gamma_j+2}{p}+m-1\right)\lambda_j-\frac{1}{p}} + n^{m-1+\frac{1}{p}} + N_6.$$

Therefore, for $1<p<2$, $\gamma_j>0, 0<\lambda_j\le 2, m\ge 1$ and $z\in\Omega(\delta)$, we have:

$$\sum_{k=1}^{3} J_2^{k,j}(z) \preceq \begin{cases} n^{\left(\frac{\gamma_j+2}{p}+m-1\right)\lambda_j-\frac{1}{p}}, & \gamma_j \ge \left(\frac{1}{\lambda_j}-1\right)[2+p(m-1)],\\ n^{m-1+\frac{1}{p}}, & 0<\gamma_j\le\left(\frac{1}{\lambda_j}-1\right)[2+p(m-1)]. \end{cases} \tag{5.33}$$

If $z \in \widehat{\Omega}(\delta)$, then $|w-w_1| \geq \eta$, $|\tau - w| \geq \frac{\eta}{2}$, from (4.1) and Lemma 4.2, for $m \geq 1$, we have:

$$\left(J_2^{1,j}(z)\right)^q \preceq \int\limits_{E_{R_1}^{11}(w_j)} \frac{|d\tau|}{|\tau - w_j|^{\gamma_j(q-1)\lambda_j+(q-2)(\lambda_j-1)}} \tag{5.34}$$
$$\preceq n^{\gamma_j(q-1)\lambda_j+(q-2)(\lambda_j-1)} mes E_{R_1}^{11}(w_j) \preceq n^{\gamma_j(q-1)\lambda_j+(q-2)(\lambda_j-1)-1};$$
$$J_2^{1,j}(z) \preceq n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j - \frac{1}{p}}.$$

$$\left(J_2^{2,j}(z)\right)^q \preceq \int\limits_{E_{R_1}^{12}(w_j)} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}\,|\Psi(\tau)-\Psi(w_j)|^{\gamma_j(q-1)}} \tag{5.35}$$
$$\preceq \left(\frac{1}{n}\right)^{q-2} \int\limits_{E_{R_1}^{12}(w_j)} \frac{|d\tau|}{|\tau - w_j|^{[\gamma_j(q-1)+q-2]\lambda_j}}$$
$$\preceq \begin{cases} n^{[\gamma_j(q-1)+q-2]\lambda_j+1-q}, & [\gamma_j(q-1)+q-2]\,\lambda_j > 1, \\ n^{2-q}\ln n, & [\gamma_j(q-1)+q-2]\,\lambda_j = 1, \\ n^{2-q}, & [\gamma_j(q-1)+q-2]\,\lambda_j < 1; \end{cases}$$
$$J_2^{2,j}(z) \preceq \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j-\frac{1}{p}}, & 1<p<p_2, \quad 0<\lambda_j<\frac{1}{\gamma_j}, \\ n^{1-\frac{2}{p}}(\ln n)^{1-\frac{1}{p}}, & p=p_2, \quad 0<\lambda_j<\frac{1}{\gamma_j}, \\ n^{1-\frac{2}{p}}, & p>p_2, \quad 0<\lambda_j<\frac{1}{\gamma_j}, \\ n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j-\frac{1}{p}}, & 1<p<2, \quad \frac{1}{\gamma_j}\leq\lambda_j\leq 2. \end{cases}$$

$$\left(J_2^{3,j}(z)\right)^q \preceq \int\limits_{E_{R_1}^{13}(w_j)} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}} \preceq \int\limits_{E_{R_1}^{13}(w_j)} |d\tau| \preceq 1 \quad J_2^{3,j}(z) \preceq 1. \tag{5.36}$$

Therefore, from (5.34)- (5.36), for $1<p<2$, $\gamma_j>0, 0<\lambda_j\leq 2, m\geq 1$ and $z \in \widehat{\Omega}(\delta)$, we obtain:

$$\sum_{k=1}^{3} J_2^{k,j}(z) \preceq n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j-\frac{1}{p}} + \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j-\frac{1}{p}}, & 1<p<p_2, \quad 0<\lambda_j<\frac{1}{\gamma_j}, \\ n^{1-\frac{2}{p}}(\ln n)^{1-\frac{1}{p}}, & p=p_2, \quad 0<\lambda_j<\frac{1}{\gamma_j}, \\ n^{1-\frac{2}{p}}, & p>p_2. \quad 0<\lambda_j<\frac{1}{\gamma_j}, \\ n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j-\frac{1}{p}}, & 1<p<2, \quad \frac{1}{\gamma_j}\leq\lambda_j\leq 2, \end{cases}$$

$$\sum_{k=1}^{3} J_2^{k,j}(z) \preceq \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j-\frac{1}{p}}, & 1<p<2, \quad \frac{1}{\gamma_j+1}<\lambda_j\leq 2, \quad \gamma_j>0. \\ 1, & 1<p<2, \quad 0<\lambda_j\leq\frac{1}{\gamma_j+1}, \end{cases} \tag{5.37}$$

From (5.29-5.32) and (5.5), for $\gamma_j>0, 1<p<2, 0<\lambda_j\leq 2, m\geq 1$, we have:

$$A_n(z) \preceq M_n^{(+)} \|P_n\|_p \ , \tag{5.38}$$

where

$$M_n^{(+)} := \begin{cases} M_{n,1}^{(+)}, & m=1, \quad z \in \Omega(\delta), \\ M_{n,2}^{(+)}, & m \geq 2, \quad z \in \Omega(\delta), \\ M_{n,3}^{(+)}, & m \geq 1, \quad z \in \widehat{\Omega}(\delta), \end{cases}$$

$$M_{n,1}^{(+)} \cup M_{n,2}^{(+)} := \begin{cases} n^{\left(\frac{\gamma_j+2}{p}+m-1\right)\lambda_j}, & \gamma_j \geq \left(\frac{1}{\lambda_j}-1\right)[2+p(m-1)], \\ n^{m-1+\frac{2}{p}}, & 0<\gamma_j \leq \left(\frac{1}{\lambda_j}-1\right)[2+p(m-1)]; \end{cases}$$

$$M_{n,3}^{(+)} := \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j}, & 1<p<2, \quad \frac{1}{\gamma_j+1}<\lambda_j \leq 2, \\ n^{\frac{1}{p}}, & 1<p<2 \quad 0<\lambda_j \leq \frac{1}{\gamma_j+1}. \end{cases}$$

2.2. Let $\gamma_j < 0$. For $z \in \Omega(\delta)$ and $m \geq 1$, according to Lemma 4.1, we have:

$$\begin{aligned} \left[I(E_{R_1,1}^{11}(w_j))\right]^q &\preceq n^{(q-2)(\lambda_j-1)+\gamma_j(q-1)\lambda_j+qm\lambda_j} \int\limits_{E_{R_1,1}^{11}(w_j)} |d\tau| \\ &\preceq n^{(q-2)(\lambda_j-1)+\gamma_j(q-1)\lambda_j+qm\lambda_j} mes E_{R_1,1}^{11}(w_j) \\ &\preceq n^{(q-2)(\lambda_j-1)+\gamma_j(q-1)\lambda_j+qm\lambda_j-1}; \\ I(E_{R_1,1}^{11}(w_j)) &\preceq n^{\left(\frac{\gamma_j+2}{p}+m-1\right)\lambda_j-\frac{1}{p}}. \end{aligned} \tag{5.39}$$

$$\begin{aligned} \left(I(E_{R_1,2}^{11}(w_j))\right)^q &\preceq n^{(q-2)(\lambda_j-1)+\gamma_j(q-1)\lambda_j+qm\lambda_j} \int\limits_{E_{R_1,2}^{11}(w_j)} |d\tau| \\ &\preceq n^{(q-2)(\lambda_j-1)+\gamma_j(q-1)\lambda_j+qm\lambda_j-1}; \\ I(E_{R_1,2}^{11}(w_j)) &\preceq n^{\left(\frac{\gamma_j+2}{p}+m-1\right)\lambda_j-\frac{1}{p}}. \end{aligned} \tag{5.40}$$

$$\begin{aligned} \left[I(E_{R_1,1}^{12})(w_j)\right]^q &\preceq n^{2-q} \int\limits_{E_{R_1,1}^{12}} \frac{|d\tau|}{|\tau-w_j|^{(q-2)\lambda_j}|\tau-w|^{qm\lambda_j}} \\ &\preceq \begin{cases} n^{[qm+(q-2)]\lambda_j+1-q}, & [qm+(q-2)]\lambda_j>1, \\ n^{2-q}\ln n, & [qm+(q-2))]\lambda_j=1, \\ n^{2-q}, & [qm+(q-2)]\lambda_j<1; \end{cases} =: N_7; \end{aligned} \tag{5.41}$$

$$I(E_{R_1,1}^{12}(w_j)) \preceq \begin{cases} n^{\left(\frac{2}{p}+m-1\right)\lambda_j-\frac{1}{p}}, & 1<p<p_4, & 0<\lambda_j \leq \frac{1}{2m}, \\ n^{1-\frac{2}{p}}(\ln n)^{1-\frac{1}{p}}, & p=p_4, & 0<\lambda_j \leq \frac{1}{2m} \\ n^{1-\frac{2}{p}}, & 2>p>p_4. & 0<\lambda_j \leq \frac{1}{2m}, \\ n^{1-\frac{2}{p}}, & 1<p<2, & \frac{1}{m-1} \leq \lambda_j \leq 2, \\ n^{\left(\frac{2}{p}+m-1\right)\lambda_j-\frac{1}{p}} & 1<p<2, & \frac{1}{2m}<\lambda_j \leq \frac{1}{m-1}. \end{cases} =: N_8.$$

$$\left(I(E^{12}_{R_1,2}(w_j))\right)^q \preceq n^{q-2} \int\limits_{E^{12}_{R_1,2}(w_j)} \frac{|d\tau|}{|\tau - w|^{[qm-(2-q)]\lambda_j}} \preceq N_7; \tag{5.42}$$

$$I(E^{12}_{R_1,2}(w_j)) \preceq N_8.$$

$$\left(J^3_2(z)\right)^q \preceq \int\limits_{E^{13}_{R_1(w_j)}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w)|^{qm}} \preceq \int\limits_{E^{13}_{R_1}(w_j)} \frac{|d\tau|}{|\tau - w|^{qm}} \preceq n^{qm-1}; \tag{5.43}$$

$$J^3_2(z) \preceq n^{\frac{1}{p}+m-1}.$$

Then, from (5.39)-(5.43), for $\gamma_j < 0$ and $z \in \Omega(\delta)$, we get:

$$\sum_{k=1}^{3} J^{k,j}_2(z) \preceq n^{\left(\frac{\gamma_j+2}{p}+m-1\right)\lambda_j - \frac{1}{p}} + n^{\frac{1}{p}+m-1}$$

$$+ \begin{cases} n^{\left(\frac{2}{p}+m-1\right)\lambda_j-\frac{1}{p}}, & 1<p<p_4,\ 0<\lambda_j\le \frac{1}{2m}, \\ n^{1-\frac{2}{p}}\left(\ln n\right)^{1-\frac{1}{p}}, & p=p_4,\ 0<\lambda_j\le\frac{1}{2m} \\ n^{1-\frac{2}{p}}, & 2>p>p_4,\ 0<\lambda_j\le\frac{1}{2m}, \\ n^{1-\frac{2}{p}}, & 1<p<2,\ \frac{1}{m-1}\le\lambda_j\le 2, \\ n^{\left(\frac{2}{p}+m-1\right)\lambda_j-\frac{1}{p}} & 1<p<2,\ \frac{1}{2m}<\lambda_j\le\frac{1}{m-1}. \end{cases}$$

Combining the three inequalities and after a simple calculation, for $1 < p < 2$, $\gamma_j < 0$, $m \geq 1$ and $z \in \Omega(\delta)$, we have:

$$\sum_{k=1}^{3} J^{k,j}_2(z) \preceq \begin{cases} n^{\frac{1}{p}}, & 0<\lambda_j<1, \quad -2<\gamma_j<-2+\frac{2}{\lambda_j}, \quad m=1, \\ n^{\left(\frac{\gamma_j+2}{p}\right)\lambda_j-\frac{1}{p}}, & 0<\lambda_j<1, \quad -2+\frac{2}{\lambda_j}\le\gamma_j\le 0, \quad m=1, \\ n^{\left(\frac{\gamma_j+2}{p}\right)\lambda_j-\frac{1}{p}}, & 1\le\lambda_j\le 2, \quad -2+\frac{1}{\lambda_j}\le\gamma_j\le 0, \quad m=1, \\ n^{\left(\frac{2}{p}\right)\lambda_j-\frac{1}{p}}, & 1\le\lambda_j\le 2, \quad -2<\gamma_j<-2+\frac{1}{\lambda_j}, \quad m=1, \\ n^{\left(\frac{\gamma_j+2}{p}+m-1\right)\lambda_j-\frac{1}{p}}, & 1\le\lambda_j\le 2, \quad -2<\gamma_j\le 0, \quad m\ge 2, \\ n^{\frac{1}{p}+m-1}, & 0<\lambda_j<1, \quad -2<\gamma_j\le 0, \quad m\ge 2. \end{cases}$$

For the $z \in \widehat{\Omega}(\delta)$, analogously we obtain:

$$\left[I(E^{11}_{R_1,1}(w_j))\right]^q \preceq \int\limits_{E^{11}_{R_1,1}(w_j)} \frac{|\tau - w_j|^{-\gamma_j(q-1)\lambda_j}\,|d\tau|}{|\tau-w_j|^{(q-2)(\lambda_j-1)}} \tag{5.44}$$

$$\preceq n^{(q-2)(\lambda_j-1)+\gamma_j(q-1)\lambda_{jj}} mes E^{11}_{R_1,1}(w_j)$$

$$\preceq n^{(q-2)(\lambda_j-1)+\gamma_j(q-1)\lambda_j-1};$$

$$I(E^{11}_{R_1,1}(w_j)) \preceq n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j-\frac{1}{p}}.$$

$$\left(I(E^{11}_{R_1,2}(w_j))\right)^q \preceq n^{\gamma_j(q-1)\lambda_j} \int\limits_{E^{11}_{R_1,2}(w_j)} \frac{|d\tau|}{|\tau - w_j|^{(q-2)(\lambda_j-1)}} \tag{5.45}$$

$$\preceq n^{(q-2)(\lambda_j-1)+\gamma_j(q-1)\lambda_j} mesE^{11}_{R_1,2}(w_j)$$

$$\preceq n^{(q-2)(\lambda_j-1)+\gamma_j(q-1)\lambda_j-1};$$

$$I(E^{11}_{R_1,2}(w_j)) \preceq n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j-\frac{1}{p}}.$$

$$\left[I(E^{12}_{R_1,1})(w_j)\right]^q \preceq \int\limits_{E^{12}_{R_1,1}} \frac{|\Psi(\tau)-\Psi(w_1)|^{(-\gamma_j)(q-1)}|d\tau|}{|\Psi'(\tau)|^{q-2}} \preceq \int\limits_{E^{12}_{R_1,1}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}}$$

$$\preceq n^{2-q} \int\limits_{E^{12}_{R_1,1}} \frac{|d\tau|}{|\tau - w_j|^{(q-2)\lambda_j}} \preceq n^{2-q} \int\limits_{E^{12}_{R_1,1}} \frac{|d\tau|}{|\tau - w|^{(q-2)\lambda_j}}$$

$$\preceq \begin{cases} n^{(q-2)\lambda_j+1-q}, & (q-2)\lambda_j > 1, \\ n^{2-q}\ln n, & (q-2))\lambda_j = 1, \\ n^{2-q}, & (q-2)\lambda_j < 1; \end{cases}$$

$$\left[I(E^{12}_{R_1,1})(w_j)\right]^q \preceq \int\limits_{E^{12}_{R_1,1}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}} \preceq n^{2-q} \int\limits_{E^{12}_{R_1,1}} \frac{|d\tau|}{|\tau - w_j|^{(q-2)\lambda_j}} \tag{5.46}$$

$$\preceq n^{2-q} \int\limits_{E^{12}_{R_1,1}} \frac{|d\tau|}{|\tau - w|^{(q-2)\lambda_j}} \preceq \begin{cases} n^{(q-2)\lambda_j+1-q}, & (q-2)\lambda_j > 1, \\ n^{2-q}\ln n, & (q-2))\lambda_j = 1, \\ n^{2-q}, & (q-2)\lambda_j < 1; \end{cases}$$

$$I(E^{12}_{R_1,1}(w_j)) \preceq \begin{cases} n^{\left(\frac{2}{p}-1\right)\lambda_j-\frac{1}{p}}, & 1 < p < p_3, \quad m \geq 1, \\ n^{1-\frac{2}{p}}(\ln n)^{1-\frac{1}{p}}, & p = p_3, \quad m \geq 1, \\ n^{1-\frac{2}{p}}, & 2 > p > p_3, \quad m \geq 1, \end{cases}$$

$$\left(I(E^{12}_{R_1,2}(w_j))\right)^q \preceq \int\limits_{E^{12}_{R_1,2}(w_j)} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}} \preceq \begin{cases} n^{(2-q)\lambda_j+1-q}, & (2-q)\lambda_j > 1, \\ n^{2-q}\ln n, & (2-q)\lambda_j = 1, \\ n^{2-q}, & (2-q)\lambda_j < 1; \end{cases} \tag{5.47}$$

$$I(E^{12}_{R_1,2}(w_j)) \preceq \begin{cases} n^{\left(\frac{2}{p}-1\right)\lambda_j-\frac{1}{p}}, & 1 < p < p_3, \quad m \geq 1, \\ n^{1-\frac{2}{p}}(\ln n)^{1-\frac{1}{p}}, & p = p_3, \quad m \geq 1, \\ n^{1-\frac{2}{p}}, & 2 > p > p_3. \quad m \geq 1, \end{cases}$$

$$\left(J^3_2(z)\right)^q \preceq \int\limits_{E^{13}_{R_1}} \frac{|\Psi(\tau)-\Psi(w_1)|^{(-\gamma_j)(q-1)}|d\tau|}{|\Psi'(\tau)|^{q-2}|\Psi(\tau)-\Psi(w)|^{qm}} \preceq \int\limits_{E^{13}_{R_1}(w_j)} |d\tau| \preceq 1; \tag{5.48}$$

$$J^3_2(z) \preceq 1.$$

From (5.44)-(5.48), for $m \geq 1$, we get:

$$\sum_{k=1}^{3} J_2^{k,j}(z) \preceq n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j-\frac{1}{p}} + \begin{cases} n^{\left(\frac{2}{p}-1\right)\lambda_j-\frac{1}{p}}, & 1 < p < p_3, \\ n^{1-\frac{2}{p}} (\ln n)^{1-\frac{1}{p}}, & p = p_3, \\ n^{1-\frac{2}{p}}, & 2 > p > p_3. \end{cases}, \tag{5.49}$$

and, after the calculation, we have:

$$\sum_{k=1}^{3} J_2^{k,j}(z) \preceq \begin{cases} n^{\left(\frac{2}{p}-1\right)\lambda_j-\frac{1}{p}}, & 1 < p \leq 2-\frac{1}{\lambda_j}, & 1 \leq \lambda_j \leq 2, \\ 1, & 2-\frac{1}{\lambda_j} < p < 2, & 1 \leq \lambda_j \leq 2, \\ 1, & 1 < p < 2, & 0 < \lambda_j < 1. \end{cases}$$

So, for $\gamma_j < 0$, from (5.5), we have:

$$A_n(z) \preceq M_n^{(-)} \left\| P_n \right\|_p, \tag{5.50}$$

where

$$M_n^{(-)} := \begin{cases} M_{n,1}^{(-)}, & m = 1, & z \in \Omega(\delta), \\ M_{n,2}^{(-)}, & m \geq 2, & z \in \Omega(\delta), \\ M_{n,3}^{(-)}, & m \geq 1, & z \in \widehat{\Omega}(\delta), \end{cases}$$

and $M_{n,k}^{(-)}, k = 1, 2, 3$, defined as in (2.2)

B) Now, we begin to estimate the $\left|\left(\frac{1}{\Phi^{n+1}(z)}\right)^{(k)}\right|$.

Since $\Phi(\infty) = \infty$, then Cauchy integral representation for the region Ω_R gives:

$$\left(\frac{1}{\Phi^{n+1}(z)}\right)^{(k)} = -\frac{1}{2\pi i} \int\limits_{L_{R_1}} \frac{1}{\Phi^{n+1}(\zeta)} \frac{d\zeta}{(\zeta - z)^{m+1}}, \ z \in \Omega_R.$$

Then, we get:

$$\left|\left(\frac{1}{\Phi^{n+1}(z)}\right)^{(k)}\right| \leq \frac{1}{2\pi} \int\limits_{L_{R_1}} \frac{1}{|\Phi^{n+1}(\zeta)|} \frac{|d\zeta|}{|\zeta - z|^{k+1}} \leq \frac{1}{2\pi} \int\limits_{L_{R_1}} \frac{|d\zeta|}{|\zeta - z|^{k+1}}.$$

Replacing the variable $\tau = \Phi(\zeta)$, we obtain:

$$\left|\left(\frac{1}{\Phi^{n+1}(z)}\right)^{(k)}\right| \preceq \int\limits_{|\tau|=R_1} \frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau) - \Psi(w)|^{k+1}} = \sum_{k=1}^{3} \int\limits_{E_{R_1}^{1k}} \frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau) - \Psi(w)|^{k+1}}.$$

Let us estimate the integral for each $k = 1, 2, 3$ separately. Using Lemma 4.2 and according to (4.1, we have:

$$\int_{E_{R_1}^{11}} \frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau)-\Psi(w)|^{k+1}} \preceq n \int_{E_{R_1}^{11}} \frac{|d\tau|}{|\tau-w_j|^{k\lambda_j}} \preceq n^{k\lambda_j}, \text{ if } z\in\Omega(\delta),$$

$$\int_{E_{R_1}^{11}} \left|\Psi'(\tau)\right| |d\tau| \preceq \left(\frac{1}{n}\right)^{\lambda_j-1} \int_{E_{R_1}^{11}} |d\tau| \preceq \left(\frac{1}{n}\right)^{\lambda_j}, \text{ if } z\in\widehat{\Omega}(\delta);$$

$$\int_{E_{R_1}^{12}} \frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau)-\Psi(w)|^{k+1}} \preceq n \int_{E_{R_1}^{11}} \frac{|d\tau|}{|\tau-w_j|^{k\lambda_j}} \preceq n^{k\lambda_j}, \text{ if } z\in\Omega(\delta),$$

$$\int_{E_{R_1}^{12}} \left|\Psi'(\tau)\right| |d\tau| \preceq \int_{E_{R_1}^{12}} |\tau-w_j|^{\lambda_j-1} |d\tau| \preceq \left(\frac{1}{n}\right)^{\lambda_j} \preceq 1, \text{ if } z\in\widehat{\Omega}(\delta);$$

$$\int_{E_{R_1}^{13}} \frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau)-\Psi(w)|^{k+1}} \preceq \int_{E_{R_1}^{13}} \frac{|d\tau|}{|\tau-w|^{k+1}} \preceq n^{k}, \text{ if } z\in\Omega(\delta),$$

$$\int_{E_{R_1}^{13}} \left|\Psi'(\tau)\right| |d\tau| \preceq \int_{E_{R_1}^{13}} |d\tau| \preceq 1, \text{ if } z\in\widehat{\Omega}(\delta);$$

Therefore,

$$\left|\left(\frac{1}{\Phi^{n+1}(z)}\right)^{(k)}\right| \preceq \begin{cases} n^{k\widetilde{\lambda}_j}, & \text{if } z\in\Omega(\delta), \\ 1, & \text{if } z\in\widehat{\Omega}(\delta). \end{cases} ,k=\overline{1,m},\ \widetilde{\lambda}_j := \max\{1;\lambda_j\},\ j=1,2. \tag{5.51}$$

Combining estimates (5.2)-(5.6), (5.19), (5.27), (5.38), (5.50) and (5.51), we get:

$$\left|P_n^{(m)}(z)\right| \leq \left|\Phi^{n+1}(z)\right| \left\{\frac{A_n(z)}{d(z,L_{R_1})} + \sum_{k=1}^{m} C_m^j \left|P_n^{(m-k)}(z)\right| \left\{\begin{array}{c} n^{k\widetilde{\lambda}_j},\ \text{if } z\in\Omega(\delta), \\ 1,\ \text{if } z\in\widehat{\Omega}(\delta), \end{array}\right\}\right\}, \tag{5.52}$$

where for any $\gamma_j > -2$, $p \geq 2$, $m \geq 1$:

$A_n(z) \preceq D_n \cdot \|P_n\|_p$

$$D_n := \begin{cases} D_n^{(+)}, & \gamma_j > 0, \\ D_n^{(-)}, & -2<\gamma_j\leq 0, \end{cases} \quad \text{and } D_n^{(\pm)} := \begin{cases} D_{n,1}^{(\pm)}, & m=1, \quad z\in\Omega(\delta), \\ D_{n,2}^{(\pm)}, & m\geq 2, \quad z\in\Omega(\delta), \\ D_{n,3}^{(\pm)}, & m\geq 1, \quad z\in\widehat{\Omega}(\delta), \end{cases}$$

if $p \geq 2$ and

$A_n(z) \preceq \|P_n\|_p M_n,$

$$M_n := \begin{cases} M_n^{(+)}, & \gamma_j > 0, \\ M_n^{(-)}, & -2<\gamma_j\leq 0, \end{cases} \quad \text{and } M_n^{(\pm)} := \begin{cases} M_{n,1}^{(\pm)}, & m=1, \quad z\in\Omega(\delta), \\ M_{n,2}^{(\pm)}, & m\geq 2, \quad z\in\Omega(\delta), \\ M_{n,3}^{(\pm)}, & m\geq 1, \quad z\in\widehat{\Omega}(\delta), \end{cases}$$

if $1 < p < 2$, and $\widetilde{\lambda}_j := \max\{0; \lambda_j\}, j = 1, 2$. Therefore, the proof of Theorems 3.1 and 3.2 is completed. □

Proof of Theorems 3.3 and 3.4. Now let's start the evaluations of $|P_n(z)|$. For this, we will make the necessary evaluations by writing the above proof for $m = 0$. Since the function $H_n(z) := \frac{P_n(z)}{\Phi^{n+1}(z)}$, $H_n(\infty) = 0$, is analytic in Ω, continuous on $\overline{\Omega}$, then Cauchy integral representation for the region Ω_{R_1} gives

$$H_n(z) = -\frac{1}{2\pi i} \int\limits_{L_{R_1}} H_n(\zeta) \frac{d\zeta}{\zeta - z}, z \in \Omega_R.$$

Then,

$$\left|\frac{P_n(z)}{\Phi^{n+1}(z)}\right| \le \frac{1}{2\pi} \int\limits_{L_{R_1}} \left|\frac{P_n(\zeta)}{\Phi^{n+1}(\zeta)}\right| \frac{|d\zeta|}{|\zeta - z|} \le \frac{1}{2\pi d(z, L_{R_1})} \int\limits_{L_{R_1}} |P_n(\zeta)| |d\zeta| \tag{5.53}$$

and so,

$$|P_n(z)| \preceq \frac{\left|\Phi^{n+1}(z)\right|}{d(z, L_{R_1})} \int\limits_{L_{R_1}} |P_n(\zeta)| |d\zeta|.$$

Denote by

$$A_n := \int\limits_{L_{R_1}} |P_n(\zeta)| |d\zeta| \tag{5.54}$$

Proceeding in the same way as in estimating $A_n(z)$ in (5.3), for estimating A_n, we we find:

$$A_n \preceq n^{\frac{1}{p}} \|P_n\|_p \cdot (J_2^1 + J_2^2 + J_2^3), \tag{5.55}$$

where

$$\left(J_2^k\right)^q := \int\limits_{E_{R_1}^{1k}} \frac{|\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w_j)|^{\gamma_j(q-1)}},\ k = 1, 2, 3.$$

So, for any $k = 1, 2, 3$ we will estimate the integrals J_2^k .

Given the possible values q ($q > 2$ and $q < 2$) and γ_j ($-2 < \gamma_j < 0$ and $\gamma_j \ge 0$), we will consider the cases separately.

Case 1. Let $1 < q \le 2$ ($p \ge 2$).

1.1. Let $\gamma_j \ge 0$. Applying Lemma 4.2, we get:

$$\begin{aligned}\left(J_2^1\right)^q &\preceq \int\limits_{E_{R_1}^{11}} \frac{|\tau - w_j|^{(2-q)(\lambda_j - 1)} |d\tau|}{|\tau - w_j|^{\gamma_j(q-1)\lambda_j}} \preceq n^{\gamma_j(q-1)\lambda_j - (2-q)(\lambda_j - 1)} mes E_{R_1}^{11} \\ &\preceq n^{\gamma_j(q-1)\lambda_j - (2-q)(\lambda_j-1)-1}; \\ J_2^1 &\preceq n^{\left(\frac{\gamma_j+2}{p} - 1\right)\lambda_j - \frac{1}{p}}.\end{aligned} \tag{5.56}$$

$$\left(J_2^2\right)^q \le n^{(2-q)} \int\limits_{E_{R_1}^{12}} \frac{|d\tau|}{|\tau-w_j|^{[\gamma_j(q-1)-(2-q)]\lambda_j}} \tag{5.57}$$

$$\preceq \begin{cases} n^{[\gamma_j(q-1)-(2-q)]\lambda_j+1-q}, & [\gamma_j(q-1)-(2-q)]\,\lambda_j>1, \\ n^{(2-q)}\ln n, & [\gamma_j(q-1)-(2-q)]\,\lambda_j=1, \\ n^{(2-q)}, & [\gamma_j(q-1)-(2-q)]\,\lambda_j<1; \end{cases}$$

$$J_2^2 \preceq \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j-\frac{1}{p}}, & 2\le p<p_2, \quad 0<\lambda_j\le 2 \quad \gamma_j>\frac{1}{\lambda_j}, \\ n^{1-\frac{2}{p}}\left(\ln n\right)^{1-\frac{1}{p}}, & p=p_2, \quad 0<\lambda_j\le 2 \quad \gamma_j>\frac{1}{\lambda_j}, \\ n^{1-\frac{2}{p}}, & p>p_2, \quad 0<\lambda_j\le 2 \quad \gamma_j>\frac{1}{\lambda_j}, \\ n^{1-\frac{2}{p}}, & p\ge 2, \quad 0<\lambda_j\le 2 \quad 0<\gamma_j\le\frac{1}{\lambda_j}. \end{cases}$$

For $\tau \in E_{R_1}^{13}$ we see that $\eta<|\tau-w_1|<2\pi R_1$. Therefore, $|\Psi(\tau)-\Psi(w_1)|\succeq 1$, from Lemma 4.1 and applying (4.2), we get:

$$\left(J_2^3\right)^q \preceq \int\limits_{E_{R_1}^{13}} \left|\Psi'(\tau)\right|^{2-q}|d\tau| \preceq \int\limits_{E_{R_1}^{13}} |d\tau| \preceq 1; \quad J_2^3 \preceq 1. \tag{5.58}$$

Combining (5.56-5.58), for $p\ge 2, \gamma_j>0$, $0<\lambda_j\le 2$ and $z\in\Omega_R$, we get:

$$\sum_{k=1}^{3} J_2^k \preceq n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j-\frac{1}{p}} + \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j-\frac{1}{p}}, & 2\le p<p_2, \quad \gamma_j>\frac{1}{\lambda_j}, \\ n^{1-\frac{2}{p}}\left(\ln n\right)^{1-\frac{1}{p}}, & p=p_2, \quad \gamma_j>\frac{1}{\lambda_j}, \\ n^{1-\frac{2}{p}}, & p>p_2, \quad \gamma_j>\frac{1}{\lambda_j}, \\ n^{1-\frac{2}{p}}, & p\ge 2, \quad 0<\gamma_j\le\frac{1}{\lambda_j}. \end{cases} \tag{5.59}$$

$$\preceq \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j-\frac{1}{p}}, & 2\le p<p_2, \quad 0<\lambda_j\le 2 \quad \gamma_j>\frac{1}{\lambda_j}, \\ n^{1-\frac{2}{p}}\left(\ln n\right)^{1-\frac{1}{p}}, & p=p_2, \quad 0<\lambda_j\le 2 \quad \gamma_j>\frac{1}{\lambda_j}, \\ n^{1-\frac{2}{p}}, & p>p_2, \quad 0<\lambda_j\le 2 \quad \gamma_j>\frac{1}{\lambda_j}, \\ n^{1-\frac{2}{p}}, & p\ge 2, \quad 0<\lambda_j\le 2 \quad 0<\gamma_j\le\frac{1}{\lambda_j}. \end{cases}$$

From (5.55)- (5.59), we obtain:

$$A_n \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j}, & 2\le p<p_2, \quad 0<\lambda_j\le 2 \quad \gamma_j>\frac{1}{\lambda_j}, \\ n^{1-\frac{1}{p}}\left(\ln n\right)^{1-\frac{1}{p}}, & p=p_2, \quad 0<\lambda_j\le 2 \quad \gamma_j>\frac{1}{\lambda_j}, \\ n^{1-\frac{1}{p}}, & p>p_2, \quad 0<\lambda_j\le 2 \quad \gamma_j>\frac{1}{\lambda_j}, \\ n^{1-\frac{1}{p}}, & p\ge 2, \quad 0<\lambda_j\le 2 \quad 0<\gamma_j\le\frac{1}{\lambda_j}. \end{cases} \tag{5.60}$$

1.2. If $\gamma_j < 0$, analogously we have:

$$\left(J_2^1\right)^q \preceq \int\limits_{E_{R_1}^{11}} |\tau - w_j|^{(-\gamma_j)(q-1)\lambda_j} |\tau - w_j|^{(2-q)(\lambda_j-1)} |d\tau| \tag{5.61}$$

$$\preceq \left(\frac{1}{n}\right)^{(2-q)(\lambda_j-1)+(-\gamma_j)(q-1)\lambda_j} mesE_{R_1}^{11} \preceq 1;$$

$$J_2^1 \preceq 1.$$

For $\tau \in E_{R_1}^{12}$, we get:

$$\left(J_2^2\right)^q \preceq \int\limits_{E_{R_1}^{12}} |\tau - w_j|^{(-\gamma_j)(q-1)\lambda_j} |\tau - w_j|^{(2-q)(\lambda_j-1)} |d\tau| \preceq 1; \tag{5.62}$$

$$J_2^2 \preceq 1.$$

$$\left(J_2^3\right)^q \preceq \int\limits_{E_{R_1}^{12}} |\Psi(\tau) - \Psi(w_j)|^{(-\gamma_j)(q-1)} \left|\Psi'(\tau)\right|^{2-q} |d\tau| \preceq 1; \quad J_2^3 \preceq 1. \tag{5.63}$$

Therefore, combining (5.61)-(5.63) in case of $\gamma_j \le 0$ for $z \in \Omega_R$, we have:

$$\sum_{k=1}^{3} J_2^k \preceq 1.$$

and, consequently, in this case from (5.55), we have:

$$A_n \preceq n^{\frac{1}{p}} \cdot \|P_n\|_p \,, \; z \in \Omega_R. \tag{5.64}$$

Therefore, combining (5.59) and (5.64), for any $\gamma_j > -2, p \ge 2, z \in \Omega_R$, we obtain:

$$A_n \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j}, & 2 \le p < p_2, \quad 0 < \lambda_j \le 2, \quad \gamma_j > \frac{1}{\lambda_j}, \\ n^{1-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = p_2, \quad 0 < \lambda_j \le 2, \quad \gamma_j > \frac{1}{\lambda_j}, \\ n^{1-\frac{1}{p}}, & p > p_2, \quad 0 < \lambda_j \le 2, \quad \gamma_j > \frac{1}{\lambda_j}, \\ n^{1-\frac{1}{p}}, & p \ge 2, \quad 0 < \lambda_j \le 2, \quad 0 < \gamma_j \le \frac{1}{\lambda_j}, \\ n^{\frac{1}{p}}, & p \ge 2, \quad 0 < \lambda_j \le 2, \quad -2 < \gamma_j \le 0. \end{cases} \tag{5.65}$$

Case 2. Let $q > 2$ $(p < 2)$. Then, $2 - q < 0$, and so

$$\left(J_2^k(z)\right)^q := \int\limits_{E_{R_1}^{1k}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2} |\Psi(\tau) - \Psi(w_j)|^{\gamma_j(q-1)}}, \; k = 1,2,3. \tag{5.66}$$

2.1. If $\gamma_j > 0$, applying Lemma's 4.1, 4.2 and (4.2), we obtain:

$$\left(J_2^1\right)^q \preceq \int\limits_{E_{R_1}^{11}} \frac{|d\tau|}{|\tau - w_J|^{\gamma_j(q-1)\lambda_j + (q-2)(\lambda_j - 1)}} \preceq n^{\gamma_j(q-1)\lambda_j + (q-2)(\lambda_j-1)-1}; \tag{5.67}$$

$$J_2^1 \preceq n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j - \frac{1}{p}}.$$

$$\left(J_2^2\right)^q \preceq n^{(2-q)} \int\limits_{E_{R_1}^{12}} \frac{|d\tau|}{|\tau - w_j|^{[\gamma_j(q-1)-(q-2)]\lambda_j}} \tag{5.68}$$

$$\preceq \begin{cases} n^{[\gamma_j(q-1)-(q-2)]\lambda_j+1-q}, & [\gamma_j(q-1)-(q-2)]\,\lambda_j > 1, \\ n^{(2-q)}\ln n, & [\gamma_j(q-1)-(q-2)]\,\lambda_j = 1, \\ n^{(2-q)}, & [\gamma_j(q-1)-(q-2)]\,\lambda_j < 1; \end{cases}$$

$$J_2^2 \preceq \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j-\frac{1}{p}}, & 1 < p < p_2, \quad 0 < \lambda_j \le 2, \quad 0 < \gamma_j < \frac{1}{\lambda_j}, \\ n^{1-\frac{2}{p}}\left(\ln n\right)^{1-\frac{1}{p}}, & p = p_2, \quad 0 < \lambda_j \le 2, \quad \gamma_j < \frac{1}{\lambda_j}, \\ n^{1-\frac{2}{p}}, & 2 > p > p_2, \quad 0 < \lambda_j \le 2, \quad \gamma_j < \frac{1}{\lambda_j}, \\ n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j-\frac{1}{p}}, & 1 < p < 2, \quad 0 < \lambda_j \le 2, \quad \gamma_j \ge \frac{1}{\lambda_j}. \end{cases}$$

For $\tau \in E_{R_1}^{13}$, $\eta < |\tau - w_1| < 2\pi R_1$, then Lemma 4.1, we get:

$$\left(J_2^3\right)^q = \int\limits_{E_{R_1}^{13}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}\,|\Psi(\tau)-\Psi(w_j)|^{\gamma_j(q-1)}} \preceq \int\limits_{E_{R_1}^{13}} |d\tau| \preceq 1; \tag{5.69}$$

$$J_2^3 \preceq 1.$$

Therefore, for $0 < \lambda_j \le 2$, we find:

$$\sum_{k=1}^{3} J_2^k \preceq n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j-\frac{1}{p}} + \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j-\frac{1}{p}}, & 1 < p < p_2, \quad 0 < \gamma_j < \frac{1}{\lambda_j}, \\ n^{1-\frac{2}{p}}\left(\ln n\right)^{1-\frac{1}{p}}, & p = p_2, \quad \gamma_j < \frac{1}{\lambda_j}, \\ n^{1-\frac{2}{p}}, & 2 > p > p_2, \quad \gamma_j < \frac{1}{\lambda_j}, \\ n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j-\frac{1}{p}}, & 1 < p < 2, \quad \gamma_j \ge \frac{1}{\lambda_j}, \end{cases}$$

$$\preceq \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j-\frac{1}{p}}, & 1 < p < \gamma_j + 2 - \frac{1}{\lambda_j}, \quad \gamma_j > 0, \\ 1, & \gamma_j + 2 - \frac{1}{\lambda_j} \le p < 2, \quad \gamma_j > 0. \end{cases}$$

From (5.66-5.69) and (5.55), for $\gamma_j > 0, 1 < p < 2, z \in \Omega_R$, we have:

$$A_n \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j}, & 1 < p < \gamma_j + 2 - \frac{1}{\lambda_j}, \quad \begin{cases} 0 < \lambda_j < 1, \\ \frac{1}{\lambda_j} - 1 < \gamma_j < \frac{1}{\lambda_j}, \end{cases} \\ n^{\frac{1}{p}}, & \gamma_j + 2 - \frac{1}{\lambda_j} \le p < 2, \quad \begin{cases} 0 < \lambda_j < 1, \\ \frac{1}{\lambda_j} - 1 < \gamma_j < \frac{1}{\lambda_j}, \end{cases} \\ n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j}, & 1 < p < \gamma_j + 2 - \frac{1}{\lambda_j}, \quad \begin{cases} 1 \le \lambda_j \le 2, \\ 0 < \gamma_j < \frac{1}{\lambda_j}, \end{cases} \\ n^{\frac{1}{p}}, & 1 < p < 2, \quad \begin{cases} 1 \le \lambda_j \le 2, \\ 0 < \gamma_j < \frac{1}{\lambda_j}, \end{cases} \\ n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j}, & 1 < p < 2, \quad \begin{cases} 0 < \lambda_j \le 2, \\ \gamma_j \ge \frac{1}{\lambda_j}. \end{cases} \end{cases} \tag{5.70}$$

2.2. Let $\gamma_j \le 0$. For $z \in \Omega_R$, $0 < \lambda_j \le 2$,according to Lemma 4.1, we have:

$$\left(J_2^1\right)^q \preceq \int\limits_{E_{R_1}^{11}} |\tau - w_j|^{(-\gamma_j)(q-1)\lambda_j-(q-2)(\lambda_j-1)} |d\tau| \tag{5.71}$$

$$\preceq \left(\frac{1}{n}\right)^{(-\gamma_j)(q-1)\lambda_j-(q-2)(\lambda_j-1)} mesE_{R_1}^{11} \preceq \left(\frac{1}{n}\right)^{(-\gamma_j)(q-1)\lambda_j-(q-2)(\lambda_j-1)+1};$$

$$J_2^1 \preceq n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j-\frac{1}{p}}.$$

$$\left(J_2^2\right)^q \preceq \left(\frac{1}{n}\right)^{(q-2)} \int\limits_{E_{R_1}^{12}} \frac{|d\tau|}{|\tau - w_j|^{(q-2)\lambda_j}} \preceq \begin{cases} n^{(q-2)\lambda_j+1-q}, & (q-2)\lambda_j > 1, \\ n^{2-q}\ln n, & (q-2)\lambda_j = 1, \\ n^{2-q}, & (q-2)\lambda_j < 1; \end{cases} \tag{5.72}$$

$$J_2^2 \preceq \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j-\frac{1}{p}}, & 1 < p < p_3, \quad -2 < \gamma_j \le 0, \\ n^{1-\frac{2}{p}} (\ln n)^{1-\frac{1}{p}}, & p = p_3, \quad -2 < \gamma_j \le 0, \\ n^{1-\frac{2}{p}}, & p > p_3, \quad -2 < \gamma_j \le 0. \end{cases}$$

$$\left(J_2^3\right)^q \preceq \int\limits_{E_{R_1}^{13}} \frac{|\Psi(\tau) - \Psi(w_j)|^{(-\gamma_j)(q-1)} |d\tau|}{|\Psi'(\tau)|^{q-2}} \preceq 1; \quad J_2^3 \preceq 1. \tag{5.73}$$

$$\sum_{k=1}^{3} J_2^k \preceq n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j-\frac{1}{p}} + \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j-\frac{1}{p}}, & 1 < p < p_3, \quad -2 < \gamma_j \le 0, \\ n^{1-\frac{2}{p}} (\ln n)^{1-\frac{1}{p}}, & p = p_3, \quad -2 < \gamma_j \le 0, \\ n^{1-\frac{2}{p}}, & p > p_3, \quad -2 < \gamma_j \le 0, \end{cases}$$

$$\preceq \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j-\frac{1}{p}}, & 1 < p < \gamma_j + 2 - \frac{1}{\lambda_j}, 1 \le \lambda_j \le 2, \frac{1}{\lambda_j} - 1 < \gamma_j \le 0, \\ 1, & \gamma_j + 2 - \frac{1}{\lambda_j} \le p < 2, 1 \le \lambda_j \le 2, \frac{1}{\lambda_j} - 1 < \gamma_j \le 0, \\ 1, & 1 < p < 2, 1 \le \lambda_j \le 2, -2 < \gamma_j \le \frac{1}{\lambda_j} - 1, \\ 1, & 1 < p < 2, 0 < \lambda_j < 1, -2 < \gamma_j \le 0, \end{cases}$$

So, for $\gamma_j \leq 0$, $z \in \Omega_R$, from (5.55), we have:

$$A_n \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j}, & 1 < p < \gamma_j + 2 - \frac{1}{\lambda_j}, 1 \leq \lambda_j \leq 2, \frac{1}{\lambda_j} - 1 < \gamma_j \leq 0, \\ n^{\frac{1}{p}}, & \gamma_j + 2 - \frac{1}{\lambda_j} \leq p < 2, 1 \leq \lambda_j \leq 2, \frac{1}{\lambda_j} - 1 < \gamma_j \leq 0, \\ n^{\frac{1}{p}}, & 1 < p < 2, 1 \leq \lambda_j \leq 2, -2 < \gamma_j \leq \frac{1}{\lambda_j} - 1, \\ n^{\frac{1}{p}}, & 1 < p < 2, 0 < \lambda_j < 1, -2 < \gamma_j \leq 0. \end{cases} \tag{5.74}$$

Combining estimates (5.2)-(5.55), (5.65), (5.65), (5.74), we get:

$$|P_n(z)| \preceq \frac{\left|\Phi^{n+1}(z)\right|}{d(z, L_{R_1})} A_n, \tag{5.75}$$

where

$$A_n \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j}, & 2 \leq p < p_2, & 0 < \lambda_j \leq 2, & \gamma_j > \frac{1}{\lambda_j}, \\ n^{1-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = p_2, & 0 < \lambda_j \leq 2, & \gamma_j > \frac{1}{\lambda_j}, \\ n^{1-\frac{1}{p}}, & p > p_2, & 0 < \lambda_j \leq 2, & \gamma_j > \frac{1}{\lambda_j}, \\ n^{1-\frac{1}{p}}, & p \geq 2, & 0 < \lambda_j \leq 2, & 0 < \gamma_j \leq \frac{1}{\lambda_j}. \\ n^{\frac{1}{p}}, & p \geq 2, & 0 < \lambda_j \leq 2, & -2 < \gamma_j \leq 0, \end{cases}$$

for any $\gamma_j > -2$, $p \geq 2$, $z \in \Omega_R$;

$$A_n \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j}, 1 < p < \gamma_j + 2 - \frac{1}{\lambda_j}, 0 < \lambda_j < 1, \frac{1}{\lambda_j} - 1 < \gamma_j < \frac{1}{\lambda_j}, \\ n^{\frac{1}{p}}, \gamma_j + 2 - \frac{1}{\lambda_j} \leq p < 2, 0 < \lambda_j < 1, \frac{1}{\lambda_j} - 1 < \gamma_j < \frac{1}{\lambda_j}, \\ n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j}, 1 < p < \gamma_j + 2 - \frac{1}{\lambda_j}, 1 \leq \lambda_j \leq 2, 0 < \gamma_j < \frac{1}{\lambda_j}, \\ n^{\frac{1}{p}}, 1 < p < 2, 1 \leq \lambda_j \leq 2, 0 < \gamma_j < \frac{1}{\lambda_j}, \\ n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j}, 1 < p < 2, 0 < \lambda_j \leq 2, \gamma_j \geq \frac{1}{\lambda_j}, \end{cases}$$

for any $\gamma_j > 0$, $1 < p < 2$, $z \in \Omega_R$, and

$$A_n \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j}, 1 < p < \gamma_j + 2 - \frac{1}{\lambda_j}, 1 \leq \lambda_j \leq 2, \frac{1}{\lambda_j} - 1 < \gamma_j \leq 0, \\ n^{\frac{1}{p}}, \gamma_j + 2 - \frac{1}{\lambda_j} \leq p < 2, 1 \leq \lambda_j \leq 2, \frac{1}{\lambda_j} - 1 < \gamma_j \leq 0, \\ n^{\frac{1}{p}}, 1 < p < 2, 1 \leq \lambda_j \leq 2, -2 < \gamma_j \leq \frac{1}{\lambda_j} - 1, \\ n^{\frac{1}{p}}, 1 < p < 2, 0 < \lambda_j < 1, -2 < \gamma_j \leq 0, \end{cases}$$

for any $\gamma_j \leq 0$, $1 < p < 2$, $z \in \Omega_R$. Therefore, the proof of Theorems 3.3 and 3.4 is completed. □

Proof of Theorems 3.5 and 3.6. For $p \geq 2$, $m = 1$, from (5.75), (5.51) , (5.52) and (2.1), we have:

$$\left|P_n'(z)\right| \preceq \left|\Phi^{n+1}(z)\right| \left\{ \frac{\|P_n\|_p}{d(z, L_{R_1})} A^1_{n,p}(z,1) + |P_n(z)| \left\{ \begin{array}{ll} n^{\widetilde{\lambda}_j}, & \text{if } z \in \Omega(\delta), \\ 1, & \text{if } z \in \widehat{\Omega}(\delta), \end{array} \right\} \right.$$

where

$$A^1_{n,p}(z,1) := \begin{cases} D_n^{(+)} & \gamma_j > 0, \\ D_n^{(-)} & -2 < \gamma_j \le 0; \end{cases} \qquad D_n^{(\pm)} := \begin{cases} D_{n,1}^{(\pm)} & m = 1, \quad z \in \Omega(\delta), \\ D_{n,2}^{(\pm)} & m \ge 2, \quad z \in \Omega(\delta), \\ D_{n,3}^{(\pm)} & m \ge 1, \quad z \in \widehat{\Omega}(\delta). \end{cases}$$

Hense, after a simple calculation, we obtain:

a) For $\gamma_j > 0,\ z \in \Omega(\delta)$:

$$\left|P_n'(z)\right| \preceq \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z, L_{R_1})} \left\|P_n\right\|_p \times$$

$$\left\{ \begin{array}{l} \begin{cases} n^{\left(\frac{\gamma_j+2}{p}\right)\lambda_j}, 2 \le p < p_1(1), 0 < \lambda_j \le 2, \gamma_j > \max\left\{0; 2\left(\frac{1}{\lambda_j} - 1\right)\right\}, \\ n^{1-\frac{1}{p}}, p > p_1(1), 0 < \lambda_j \le 2, \gamma_j > \max\left\{0; 2\left(\frac{1}{\lambda_j} - 1\right)\right\}, \\ n^{\frac{2}{p}}, 2 \le p < 3, 0 < \lambda_j < 1, 0 < \gamma_j \le 2\left(\frac{1}{\lambda_j} - 1\right), \\ n^{1-\frac{1}{p}}, p \ge 3, 0 < \lambda_j < 1, 0 < \gamma_j \le 2\left(\frac{1}{\lambda_j} - 1\right), \\ (n \ln n)^{1-\frac{1}{p}}, p = p_1(1), 0 < \lambda_j \le 2, \gamma_j > 0, \end{cases} \\ \\ \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j + \widetilde{\lambda}_j}, & 2 \le p < p_2, & 0 < \lambda_j \le 2, & \gamma_j > \frac{1}{\lambda_j}, \\ n^{\widetilde{\lambda}_j + 1 - \frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = p_2, & 0 < \lambda_j \le 2, & \gamma_j > \frac{1}{\lambda_j}, \\ n^{\widetilde{\lambda}_j + 1 - \frac{1}{p}}, & p > p_2, & 0 < \lambda_j \le 2, & \gamma_j > \frac{1}{\lambda_j}, \\ n^{\widetilde{\lambda}_j + 1 - \frac{1}{p}}, & p \ge 2, & 0 < \lambda_j \le 2, & 0 < \gamma_j \le \frac{1}{\lambda_j}. \end{cases} \end{array} \right\}$$

$$\preceq N_9 := \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j + \widetilde{\lambda}_j}, 2 \le p < p_2, 0 < \lambda_j \le 2, \gamma_j > \max\left\{\frac{1}{\lambda_j}, 2\left(\frac{1}{\lambda_j} - 1\right)\right\}, \\ n^{\frac{2}{p}}, 2 \le p < p_2, 0 < \lambda_j < \frac{1}{2}, \frac{1}{\lambda_j} < \gamma_j \le 2\left(\frac{1}{\lambda_j} - 1\right) \\ n^{1-\frac{1}{p}+\lambda_j} (\ln n)^{1-\frac{1}{p}}, p = p_2, 1 \le \lambda_j \le 2, \gamma_j > \frac{1}{\lambda_j}, \\ n^{2-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, p = p_2, 0 < \lambda_j < \frac{1}{2}, \frac{1}{\lambda_j} < \gamma_j \le 2\left(\frac{1}{\lambda_j} - 1\right), \\ n^{2-\frac{1}{p}}, p_2 < p < 3, 0 < \lambda_j < 1, \gamma_j > \frac{1}{\lambda_j}, \\ n^{\left(\frac{\gamma_j+2}{p}\right)\lambda_j}, 2 \le p < \frac{\lambda_j(\gamma_j+2)+1}{2}, \frac{1}{2} \le \lambda_j \le 2, 0 < \gamma_j \le \frac{1}{\lambda_j}, \\ n^{2-\frac{1}{p}}, p \ge p_1(1), 0 < \lambda_j < 1, 0 < \gamma_j \le \frac{1}{\lambda_j}, \\ n^{1-\frac{1}{p}+\lambda_j}, p \ge \frac{\lambda_j(\gamma_j+2)+1}{2}, 1 \le \lambda_j \le 2, \gamma_j > 0. \end{cases}$$

b) For $-2 < \gamma_j \le 0,\ z \in \Omega(\delta)$:

$$\preceq \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z, L_{R_1})} \left\|P_n\right\|_p \cdot N_9$$

$$\left|P_n'(z)\right| \preceq \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z, L_{R_1})} \left\|P_n\right\|_p$$

$$\preceq \begin{cases} n^{\left(\frac{2}{p}\right)\lambda_j}, & 2 \le p < 2\lambda_j + 1, \quad 1 \le \lambda_j \le 2, \\ (n \ln n)^{1-\frac{1}{p}}, & p = 2\lambda_j + 1, \quad 1 \le \lambda_j \le 2, \\ n^{1-\frac{1}{p}}, & p > 2\lambda_j + 1, \quad 1 \le \lambda_j \le 2, \\ n^{\frac{2}{p}}, & 2 \le p < 3, \quad 0 < \lambda_j < 1, \end{cases} + \left\{ n^{\frac{1}{p}+\widetilde{\lambda}_j}, p \ge 2, 0 < \lambda_j \le 2, \right.$$

$$\preceq \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z, L_{R_1})} \|P_n\|_p \begin{cases} n^{\frac{1}{p}+\lambda_j}, & 2 \le p \quad 1 \le \lambda_j \le 2, \quad -2 < \gamma_j \le 0, \\ n^{1+\frac{1}{p}}, & 2 \le p < 3, \quad 0 < \lambda_j < 1, \quad -2 < \gamma_j \le 0. \end{cases}$$

c) For $\gamma_j > 0,\ z \in \widehat{\Omega}(\delta)$:

$$\left|P_n'(z)\right| \preceq \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z, L_{R_1})} \|P_n\|_p$$

$$\times \left\{ \begin{array}{l} \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j}, & 2 \le p < p_2, \quad 0 < \lambda_j \le 2, \\ n^{(1-\frac{1}{p})} (\ln n)^{1-\frac{1}{p}}, & p = p_2, \quad \frac{1}{\gamma_j} < \lambda_j \le 2, \\ n^{1-\frac{1}{p}}, & p > p_2, \quad 0 < \lambda_j \le 2, \end{cases} \\ \\ +n^{\widetilde{\lambda}_j} \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j}, \ 2 \le p < p_2, & 0 < \lambda_j \le 2, \ \gamma_j > \frac{1}{\lambda_j}, \\ n^{1-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, \ p = p_2, & 0 < \lambda_j \le 2, \ \gamma_j > \frac{1}{\lambda_j}, \\ n^{1-\frac{1}{p}}, \ p > p_2, & 0 < \lambda_j \le 2, \ \gamma_j > \frac{1}{\lambda_j}, \\ n^{1-\frac{1}{p}}, \ p \ge 2, & 0 < \lambda_j \le 2, \ 0 < \gamma_j \le \frac{1}{\lambda_j}, \end{cases} \end{array} \right\}$$

$$\preceq \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z, L_{R_1})} \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j}, 2 \le p < p_2, 0 < \lambda_j \le 2, \gamma_j > \frac{1}{\lambda_j}, \\ n^{1-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, p = p_2, 0 < \lambda_j \le 2, \gamma_j > \frac{1}{\lambda_j}, \\ n^{1-\frac{1}{p}}, p > p_2, 0 < \lambda_j \le 2, \gamma_j > 0. \end{cases}$$

d) For$-2 < \gamma_j \le 0,\ z \in \widehat{\Omega}(\delta)$:

$$\left|P_n'(z)\right| \preceq \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z, L_{R_1})} \|P_n\|_p \left\{ \begin{array}{l} \left\{ n^{1-\frac{1}{p}} \quad p \ge 2, \quad 0 < \lambda_j \le 2, \right. \\ +n^{\widetilde{\lambda}_j} \left\{ n^{\frac{1}{p}}, \quad p \ge 2, \quad 0 < \lambda_j \le 2, \right. \end{array} \right\}$$

$$\preceq \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z, L_{R_1})} \|P_n\|_p \left\{ n^{1-\frac{1}{p}} \quad p \ge 2, \quad 0 < \lambda_j \le 2, \right.$$

Now, let $1 < p < 2$. Analogously, in this case for $m = 1$, from (5.75), (5.51) , (5.52) and (2.2), we get:

$$\left|P_n'(z)\right| \preceq \left|\Phi^{n+1}(z)\right| \left\{ \frac{\|P_n\|_p}{d(z, L_{R_1})} A^1_{n,p}(z,1) + |P_n(z)| \begin{cases} n^{\widetilde{\lambda}_j}, & \text{if } z \in \Omega(\delta), \\ 1, & \text{if } z \in \widehat{\Omega}(\delta). \end{cases} \right\}$$

where

$$A^1_{n,p}(z,1) := \begin{cases} M_n^{(+)} & \gamma_j > 0, \\ M_n^{(-)} & -2 < \gamma_j \le 0; \end{cases} \qquad M_n^{(\pm)} := \begin{cases} M_{n,1}^{(\pm)} & m = 1, \quad z \in \Omega(\delta), \\ M_{n,2}^{(\pm)} & m \ge 2, \quad z \in \Omega(\delta), \\ M_{n,3}^{(\pm)} & m \ge 1, \quad z \in \widehat{\Omega}(\delta). \end{cases}$$

Hence, after a simple calculation, we obtain:

a) For $\gamma_j > 0$, $z \in \Omega(\delta)$:

$$\left|P_n'(z)\right| \preceq \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z, L_{R_1})} \|P_n\|_p$$

$$\times \left\{ \begin{array}{l} \left\{ \begin{array}{lll} n^{\left(\frac{\gamma_j+2}{p}\right)\lambda_j}, & 1<p<2, \quad 0<\lambda_j \le 2, & \gamma_j \ge 2\left(\frac{1}{\lambda_j}-1\right), \\ n^{\frac{2}{p}}, & 1<p<2, \quad 0<\lambda_j \le 2, & 0<\gamma_j \le 2\left(\frac{1}{\lambda_j}-1\right), \end{array} \right. + \\ \left\{ \begin{array}{lll} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j+\widetilde{\lambda}_j}, & 1<p<\gamma_j+2-\frac{1}{\lambda_j}, & \left\{ \begin{array}{c} 0<\lambda_j<1, \\ \frac{1}{\lambda_j}-1<\gamma_j<\frac{1}{\lambda_j}, \end{array} \right. \\ n^{\frac{1}{p}+\widetilde{\lambda}_j}, & \gamma_j+2-\frac{1}{\lambda_j} \le p<2, & \left\{ \begin{array}{c} 0<\lambda_j<1, \\ \frac{1}{\lambda_j}-1<\gamma_j<\frac{1}{\lambda_j}, \end{array} \right. \\ n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j+\widetilde{\lambda}_j}, & 1<p<\gamma_j+2-\frac{1}{\lambda_j}, & 1 \le \lambda_j \le 2, 0<\gamma_j<\frac{1}{\lambda_j}, \\ n^{\frac{1}{p}+\widetilde{\lambda}_j}, & 1<p<2, & 1 \le \lambda_j \le 2, 0<\gamma_j<\frac{1}{\lambda_j}, \\ n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j+\widetilde{\lambda}_j}, & 1<p<2, & 0<\lambda_j \le 2, \gamma_j \ge \frac{1}{\lambda_j}. \end{array} \right. \end{array} \right\}$$

$$\preceq \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z, L_{R_1})} \|P_n\|_p$$

$$\times \left\{ \begin{array}{lll} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j+1}, & 1<p<\gamma_j+2-\frac{1}{\lambda_j}, & \left\{ \begin{array}{c} 0<\lambda_j<1, \\ \frac{1}{\lambda_j}-1<\gamma_j \le 2\left(\frac{1}{\lambda_j}-1\right), \end{array} \right. \\ n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j+1}, & 1<p<2, & \frac{1}{2}<\lambda_j<1, \gamma_j \ge \frac{1}{\lambda_j}, \\ n^{\frac{1}{p}+1}, & \gamma_j+2-\frac{1}{\lambda_j} \le p<2, & \frac{1}{2}<\lambda_j<1, \frac{1}{\lambda_j}-1<\gamma_j<\frac{1}{\lambda_j}, \\ n^{\left(\frac{\gamma_j+2}{p}\right)\lambda_j}, & 1<p<2, & 1 \le \lambda_j \le 2, \gamma_j \ge \frac{1}{\lambda_j}, \\ n^{\frac{1}{p}+\lambda_j}, & 1<\gamma_j+2-\frac{1}{\lambda_j} \le p, & 1 \le \lambda_j \le 2, 0<\gamma_j<\frac{1}{\lambda_j}. \end{array} \right.$$

b) For $-2 < \gamma_j \le 0$, $z \in \Omega(\delta)$:

$$\left|P_n'(z)\right| \preceq \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z, L_{R_1})} \|P_n\|_p$$

$$\times\left\{\begin{array}{c}\left\{\begin{array}{llll} n^{\frac{2}{p}}, & 1<p<2, & 0<\lambda_j<1, & -2<\gamma_j<-2+\frac{2}{\lambda_j}, \\ n^{\left(\frac{\gamma_j+2}{p}\right)\lambda_j}, & 1<p<2, & 0<\lambda_j<1, & -2+\frac{2}{\lambda_j}\le\gamma_j\le 0, \\ n^{\left(\frac{\gamma_j+2}{p}\right)\lambda_j}, & 1<p<2, & 1\le\lambda_j\le 2, & -2+\frac{1}{\lambda_j}\le\gamma_j\le 0, \\ n^{\left(\frac{2}{p}\right)\lambda_j}, & 1<p<2, & 1\le\lambda_j\le 2, & -2<\gamma_j<-2+\frac{1}{\lambda_j}. \end{array}\right. + \\ \left\{\begin{array}{l} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j+\widetilde{\lambda}_j}, 1<p<\gamma_j+2-\frac{1}{\lambda_j}, 1\le\lambda_j\le 2, \frac{1}{\lambda_j}-1<\gamma_j\le 0, \\ n^{\frac{1}{p}+\widetilde{\lambda}_j}, \gamma_j+2-\frac{1}{\lambda_j}\le p<2, 1\le\lambda_j\le 2, \frac{1}{\lambda_j}-1<\gamma_j\le 0, \\ n^{\frac{1}{p}+\widetilde{\lambda}_j}, 1<p<2, 1\le\lambda_j\le 2, -2<\gamma_j\le\frac{1}{\lambda_j}-1, \\ n^{\frac{1}{p}+\widetilde{\lambda}_j}, 1<p<2, 0<\lambda_j<1, -2<\gamma_j\le 0. \end{array}\right. \end{array}\right\}$$

$$\preceq \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,L_{R_1})}\left\|P_n\right\|_p \left\{\begin{array}{l} n^{\frac{1}{p}+1}, 1<p<2, 0<\lambda_j<1, -2<\gamma_j<-2+\frac{2}{\lambda_j}, \\ n^{\left(\frac{\gamma_j+2}{p}\right)\lambda_j}, 1<p<2, 0<\lambda_j<1, -2+\frac{2}{\lambda_j}\le\gamma_j\le 0, \\ n^{\frac{1}{p}+\lambda_j}, 2-\frac{1}{\lambda_j}<p<2, 1\le\lambda_j\le 2, -2<\gamma_j<0, \\ n^{\left(\frac{2}{p}\right)\lambda_j}, 1<p\le 2-\frac{1}{\lambda_j}, 1\le\lambda_j\le 2, -2<\gamma_j<-2+\frac{1}{\lambda_j}, \end{array}\right.$$

c) For $\gamma_j>0,\ z\in\widehat{\Omega}(\delta)$:

$$\left|P_n'(z)\right| \preceq \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,L_{R_1})}\left\|P_n\right\|_p$$

$$\times\left\{\begin{array}{c}\left\{\begin{array}{lll} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j}, & 1<p<2, & \frac{1}{\gamma_j+1}<\lambda_j\le 2, \\ n^{\frac{1}{p}}, & 1<p<2, & 0<\lambda_j\le\frac{1}{\gamma_j+1}. \end{array}\right. + \\ \left\{\begin{array}{l} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j}, 1<p<\gamma_j+2-\frac{1}{\lambda_j}, 0<\lambda_j<1, \frac{1}{\lambda_j}-1<\gamma_j<\frac{1}{\lambda_j}, \\ n^{\frac{1}{p}}, \gamma_j+2-\frac{1}{\lambda_j}\le p<2, 0<\lambda_j<1, \frac{1}{\lambda_j}-1<\gamma_j<\frac{1}{\lambda_j}, \\ n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j}, 1<p<\gamma_j+2-\frac{1}{\lambda_j}, 1\le\lambda_j\le 2, 0<\gamma_j<\frac{1}{\lambda_j}, \\ n^{\frac{1}{p}}, 1<p<2, 1\le\lambda_j\le 2, 0<\gamma_j<\frac{1}{\lambda_j}, \\ n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j}, 1<p<2, 0<\lambda_j\le 2, \gamma_j\ge\frac{1}{\lambda_j}, \end{array}\right. \end{array}\right\}$$

$$\preceq \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,L_{R_1})}\left\|P_n\right\|_p$$

$$\times\left\{\begin{array}{l} n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j}, 1<p<\gamma_j+2-\frac{1}{\lambda_j}, 0<\lambda_j<1, \frac{1}{\lambda_j}-1<\gamma_j<\frac{1}{\lambda_j}, \\ n^{\frac{1}{p}}, \gamma_j+2-\frac{1}{\lambda_j}\le p<2, 0<\lambda_j<1, \frac{1}{\lambda_j}-1<\gamma_j<\frac{1}{\lambda_j}, \\ n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j}, 1<p<\gamma_j+2-\frac{1}{\lambda_j}, 1\le\lambda_j\le 2, 0<\gamma_j<\frac{1}{\lambda_j}, \\ n^{\frac{1}{p}}, \gamma_j+2-\frac{1}{\lambda_j}\le p<2, 1\le\lambda_j\le 2, 0<\gamma_j<\frac{1}{\lambda_j}. \end{array}\right.$$

d) For $-2<\gamma_j\leq 0,\ z\in\widehat{\Omega}(\delta)$:

$$\left|P_n'(z)\right|\preceq\frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,L_{R_1})}\left\|P_n\right\|_p\times$$

$$\left\{\begin{array}{l}\left\{\begin{array}{l}n^{\left(\frac{2}{p}-1\right)\lambda_j},1<p\leq 2-\frac{1}{\lambda_j},1\leq\lambda_j\leq 2,-2<\gamma_j\leq 0,\\ n^{\frac{1}{p}},2-\frac{1}{\lambda_j}<p<2,1\leq\lambda_j\leq 2,-2<\gamma_j\leq 0,\\ n^{\frac{1}{p}},1<p<2,0<\lambda_j<1,-2<\gamma_j\leq 0,\end{array}\right. +\\ \left\{\begin{array}{l}n^{\left(\frac{\gamma_j+2}{p}-1\right)\lambda_j},1<p<\gamma_j+2-\frac{1}{\lambda_j},1\leq\lambda_j\leq 2,\frac{1}{\lambda_j}-1<\gamma_j\leq 0,\\ n^{\frac{1}{p}},\gamma_j+2-\frac{1}{\lambda_j}\leq p<2,1\leq\lambda_j\leq 2,\frac{1}{\lambda_j}-1<\gamma_j\leq 0,\\ n^{\frac{1}{p}},1<p<2,1\leq\lambda_j\leq 2,-2<\gamma_j\leq\frac{1}{\lambda_j}-1,\\ n^{\frac{1}{p}},1<p<2,0<\lambda_j<1,-2<\gamma_j\leq 0,\end{array}\right.\end{array}\right\}$$

$$\preceq\frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,L_{R_1})}\left\|P_n\right\|_p$$

$$\times\left\{\begin{array}{llll}n^{\left(\frac{2}{p}-1\right)\lambda_j}, & 1<p\leq 2-\frac{1}{\lambda_j}, & 1\leq\lambda_j\leq 2, & -2<\gamma_j\leq 0,\\ n^{\frac{1}{p}}, & 2-\frac{1}{\lambda_j}<p<2, & 1\leq\lambda_j\leq 2, & -2<\gamma_j\leq 0,\\ n^{\frac{1}{p}}, & 1<p<2, & 0<\lambda_j<1, & -2<\gamma_j\leq 0.\end{array}\right.$$

Combining estimates for cases a) ($\gamma_j>0,\ z\in\Omega(\delta)$), b) ($-2<\gamma_j\leq 0, z\in\Omega(\delta)$) and c) ($\gamma_j>0,\ z\in\widehat{\Omega}(\delta)$), d) ($-2<\gamma_j\leq 0,\ z\in\widehat{\Omega}(\delta)$), we complete the proof of Theorem 3.5, for $p\geq 2$ and 3.6, for $0<p<1$. □

Proof of Theorems 3.7 and 3.8. The proof of Theorem 3.7 and Theorem 3.8 is obtained by combining Theorems 3.1, 3.2 with Thorems 3.5, 3.6, respectively. □

Finally, throughout the proof of Theorems 3.1-3.6 involves the quantity $d(z,L_{R_1})$, let us show that $d(z,L_{R_1})\succeq d(z,L)$ holds for all $z\in\Omega_R$. For the points $z\notin\Omega(L_{R_1},d(L_{R_1},L_R))$, we have: $d(z,L_{R_1})\succeq\delta\succeq d(z,L)$.

Now, let $z\in\Omega(L_{R_1},d(L_{R_1},L_R))$. Denote by $\xi_1\in L_{R_1}$ the point such that $d(z,L_{R_1})=|z-\xi_1|$, and point $\xi_2\in L$,such that $d(z,L)=|z-\xi_2|$, and for $w=\Phi(z)$, $t_1=\Phi(\xi_1),t_2=\Phi(\xi_2)$, we have: $|w-w_1|\geq||w-w_2|-|w_2-w_1||\geq\left||w-w_2|-\frac{1}{2}|w-w_2|\right|\geq\frac{1}{2}|w-w_2|$. Then, according to Lemma 4.1, we obtain: $d(z,L_{R_1})\succeq d(z,L)$.

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